Inflationary supersymmetric FRLW quantum cosmology

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Introduction

- Cosmology describes the general laws of the universe.

- In general classical formulation, good agreement with observational bounds.

- First stages of very early universe:
  - Quantum gravity phase, quantum space-time fluctuations.
  - Homogeneous phase.
  - Inflationary phase, inhomogeneous perturbations originate structure seeds.

- Inhomogeneities follow from quantum fluctuations of QFT of metric perturbations in a homogeneous background.

- Suggests quantum description. Quantum cosmology

- Classical physics follows from quantum theory.

- Inflationary phase below Planck scale, very little is known.

- Supersymmetry, well known facts.
  - Unified framework for bosons and fermions.
  - Improves significantly behaviour of divergences in quantum field theory.
  - Ingredient of candidates for UV completion above Planck energy: string theory, supergravity.
  - Broken at low energies, above standard model scale.

- Supersymmetric quantum cosmology.
  - First order instead second order equations instead of the Wheeler-DeWitt equation.
  - Does not require boundary (initial) conditions.

Quantum cosmology

- A canonical quantization of general relativity. Hamiltonian function, linear combination of Hamiltonian H and momentum constraints H_i.

- Constraints to operator equations on wave function of the universe $\psi({}^3g(x), \phi(x))$, defined on the fields on space slices.

- Momentum constraints $H_i\psi = 0$, invariance under space diffeomorphisms.

- Hamiltonian constraint $H\psi = 0$, Wheeler-DeWitt equation. There is no time dependent Schrödinger equation.

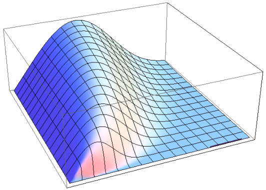
- Wave function gives probabilities to find the universe in given configurations $|\psi({}^3g(x), \phi(x))|^2$.

- WdW second order. Boundary conditions required.

Time

- Space-slice are parametrized by arbitrary time.

- A continuous sequence of most probable configurations, in subsequent slices, gives a trajectory (supertime of Misner[1]).



- To a trajectory could be associated an uncertainty in superspace and in time.

- Internal time can be given, in one to one correspondence to certain function of observables.

[1]C.W. Misner in Magic without Magic: John Archibald Wheeler, Ed. John R. Klauder, B.W.H. Freeman and Co. 1972

Supergravity

- Supersymmetry relates one bosonic to one fermionic degree of freedom, gauge or matter. $\phi(x) \to$ supermultiplet $(\phi(x), \psi(x), \dots)$

- Supergravity: $e_m^{\ a} \to (e_m^{\ a}, \psi_m^\alpha)$, ψ_m^α $\frac{3}{2}$-spin Rarita-Schwinger field.

- Extension of space-time to superspace $x \to z = (x, \theta, \bar{\theta})$, superfields functions of superspace, θ $\frac{1}{2}$-spin variables.

- Supersymmetry transformations $x'^\mu = x^\mu + i(\theta\sigma^\mu\bar{\xi} - \xi\sigma^\mu\bar{\theta})$

- Supergravity: general relativity on superspace, fields$\to$superfields, $\phi(x) \to \Phi(z)$, invariance under local superspace transformations $z' = z'(z)$.

- Differential, tensor and integral calculus on superspace.

- Supergravity from invariant action $\int dz\mathcal{E}L$, $\mathcal{E}$ invariant density.

Supersymmetric quantum cosmology

- Hamiltonian formulation of supergravity, constraints: Hamiltonian and momentum constraints H_a, Lorentz constraints L^{ab}, supersymmetric spinorial constraints S and $\bar{S}$.

- Supersymmetric quantum cosmology has been developed for homogeneous spaces. In this case $\{S, \bar{S}\} = -2H$.

- Quantization: $S\psi = \bar{S}\psi = 0$, first order (square root) equations, instead of second order Wheeler-DeWitt equation.

- Here we consider supergravity of the FRLW model with a scalar field $\phi(t)$.

- The corresponding superfields are

$$N(t) \to \mathcal{E} = -N(t) - \frac{i}{2}(\Theta\bar{\psi}(t) + \bar{\Theta}\psi(t))$$

$$a(t) \to \mathcal{A} = a(t) + \Theta\lambda(t) - \bar{\Theta}\bar{\lambda}(t) + \Theta\bar{\Theta}B(t)$$

$$\phi(t) \to \Phi = \phi(t) + \Theta\eta(t) - \bar{\Theta}\bar{\eta}(t) + \Theta\bar{\Theta}G(t)$$

- Action[2] $I = I_G + I_M$

$$I_G = \frac{3}{\kappa^2}\int \mathcal{E}\left(\mathcal{A}\nabla_{\bar\Theta}\mathcal{A}\nabla_{\Theta}\mathcal{A} - \sqrt{k}\mathcal{A}^2\right) d\Theta d\bar\Theta dt,$$

$$I_M = \int \mathcal{E}\mathcal{A}^3\left[-\frac{1}{2}\nabla_{\bar\Theta}\Phi\nabla_{\Theta}\Phi + W(\Phi)\right] d\Theta d\bar\Theta dt,$$

- Supersymmetric constraints

$$\frac{1}{\sqrt{\hbar c}}S = \frac{c\kappa}{2\sqrt{6}}\left(a^{-\frac{1}{2}}\pi_a + \pi_a a^{-\frac{1}{2}}\right)\alpha + ca^{-\frac{3}{2}}\pi_\phi\beta + \frac{3i\kappa}{\sqrt{6}}a^{\frac{3}{2}}W\alpha$$
$$+ ia^{\frac{3}{2}}W'\beta - i\frac{\sqrt{6k}}{\kappa}a^{\frac{1}{2}}\alpha - \frac{i\sqrt{3}}{4\sqrt{2}}\hbar c\kappa a^{-\frac{3}{2}}\alpha[\bar\beta,\beta] = 0,$$

where $\lambda = \sqrt{\frac{\hbar c\kappa^2}{6}}\alpha$, $\bar\lambda = \sqrt{\frac{\hbar c\kappa^2}{6}}\bar\alpha$, $\eta = \sqrt{\hbar c}\beta$ and $\bar\eta = \sqrt{\hbar c}\bar\beta$.

$$\{\alpha,\bar\alpha\} = 1, \qquad \{\beta,\bar\beta\} = -1.$$

[2]N.E. Martínez Pérez, C. Ramírez, V. Vázquez-Báez, arXiv:2104.12914

Quantization

- Ground state $|1\rangle$

$$\alpha |1\rangle = \beta |1\rangle = 0.$$

- Four orthogonal states

$$|1\rangle , \quad |2\rangle = \bar{\alpha} |1\rangle , \quad |3\rangle = \bar{\beta} |1\rangle \quad \text{and} \quad |4\rangle = \bar{\alpha}\bar{\beta} |1\rangle .$$

- Norms

$$\langle 2|2\rangle = \langle 1|1\rangle , \langle 3|3\rangle = - \langle 1|1\rangle \text{ and } \langle 4|4\rangle = - \langle 1|1\rangle .$$

- General state

$$|\Psi\rangle = \psi_1(a,\phi) |1\rangle + \psi_2(a,\phi) |2\rangle + \psi_3(a,\phi) |3\rangle + \psi_4(a,\phi) |4\rangle .$$

- Constraints

$$S |\Psi\rangle = \bar{S} |\Psi\rangle = 0.$$

- Equations from $S\,|\Psi\rangle = 0$

$$a\left(\partial_a - \frac{3}{\hbar c}a^2W + \frac{6\sqrt{k}}{\hbar c\kappa^2}a + \frac{1}{2}a^{-1}\right)\psi_2 - \frac{\sqrt{6}}{\kappa}\left(\partial_\phi - a^3W'\right)\psi_3 = 0,$$

$$\left(\partial_a - \frac{3}{\hbar c}a^2W + \frac{6\sqrt{k}}{\hbar c\kappa^2}a - a^{-1}\right)\psi_4 = 0$$

$$\left(\partial_\phi - \frac{1}{\hbar c}a^3W'\right)\psi_4 = 0,$$

- Equations from $\bar{S}\,|\Psi\rangle = 0$

$$a\left(\partial_a + \frac{3}{\hbar c}a^2W - \frac{6\sqrt{k}}{\hbar c\kappa^2}a + \frac{1}{2}a^{-1}\right)\psi_3 - \frac{\sqrt{6}}{\kappa}\left(\partial_\phi + a^3W'\right)\psi_2 = 0,$$

$$\left(\partial_a + \frac{3}{\hbar c}a^2W - \frac{6\sqrt{k}}{\hbar c\kappa^2}a - a^{-1}\right)\psi_1 = 0$$

$$\left(\partial_\phi + \frac{1}{\hbar c}a^3W'\right)\psi_1 = 0,$$

Solutions

- For ψ_2 and ψ_3 we could not get analytic solutions. An inspection shows no normalizable solutions.

- For ψ_2 and ψ_3 unique solutions, no boundary conditions needed.

$$\psi_1(a,\phi) = \mathcal{N} a \exp\left[-\frac{1}{\hbar c}\left(a^3 W(\phi) - \frac{3\sqrt{k}a^2}{\kappa^2}\right)\right],$$

$$\psi_4(a,\phi) = \mathcal{N} a \exp\left[\frac{1}{\hbar c}\left(a^3 W(\phi) - \frac{3\sqrt{k}a^2}{\kappa^2}\right)\right].$$

- If $W(\phi) \neq 0$, one of these solutions normalizable. We take for the other solution the trivial solution.

- Normalizable state can be chosen with positive norm.

- For $k = 0$ we write for the solution

$$\psi(a,\phi) = \mathcal{N} a \exp\left[-\frac{1}{\hbar c}\left(a^3 |W(\phi)|\right)\right].$$

Time

Gaussian type form wave function for fixed ϕ, vanishes at $a = 0$.
Take ϕ as time, with normalized effective wave function

$$\Psi(a,t) = \frac{1}{\sqrt{\int_0^\infty da\, |\psi(a,t)|^2}}\, \psi(a,t) = \sqrt{\frac{6|W(t)|}{\hbar c}}\, a \exp\left[-\frac{a^3|W(t)|}{\hbar c}\right].$$

Corresponds to conditional probability for a, for given value of $\phi = t$

$$|\Psi(a,t)|^2 = \frac{|\psi(a,t)|^2}{\int_0^\infty da\, |\psi(a,t)|^2}.$$

Time dependent mean value of a

$$a(t) = \int_0^\infty a|\Psi(a,t)|^2 da = \Gamma\left(4/3\right)\left[\frac{\hbar c}{2|W(t)|}\right]^{1/3}.$$

Inflationary scenarios

- Set the superpotential so that $a(0)$ has a minimum.

- Inflation sets on at $t_i > 0$. Before full quantum gravity expected.

- Consider positive superpotential $W(t) > 0$, has a maximum at $t = 0$. Accelerated expansion $\ddot{a}(t) > 0$ requires $\frac{W}{\dot{W}^{3/4}}$ is an increasing function.

- Accelerated expansion up to t_f, inflation with N e-folds

$$N = ln\frac{a_f}{a_i} = ln\frac{a_f}{a_i} = \frac{1}{3}ln\frac{W(t_i)}{W(t_f)}$$

$$W(t_f) = W(t_i)e^{-3N}$$

- The exit of inflation can be achieved by adding to superpotential a constant $\lambda = W(t_f)$. Requires fine tuning.

Superpotentials

- Gaussian superpotentials

$$W(t) = \frac{c^4 M_p^3}{\hbar^2} t^{-2\alpha} \left(e^{-t^2} + \lambda \right), \tag{1}$$

- Step superpotentials

$$W(t) = \frac{c^4 M_p^3}{\hbar^2} t^{-2\alpha} \left(\frac{1}{t^2+1} + \lambda \right), \tag{2}$$

- For an exit at N e-folds, $\lambda \lesssim e^{-3N}$. For $N = 60$, $\lambda \lesssim 10^{-79}$, depends on the details of the superpotential.

- If $\lambda = 0$ eternal inflation.

- If $\alpha = 0$, $a(0) = \frac{\Gamma(4/3)}{2^{1/3}} \ell_p$.

- If $\alpha > 0$, $a(0) = 0$.

-Set unities $c = 1$, $\hbar = 1$, and $\kappa = 1$.

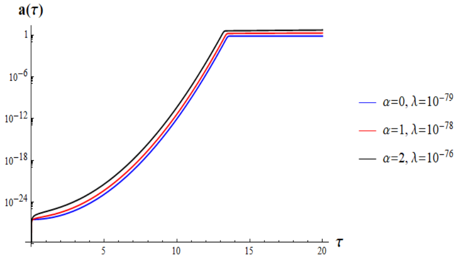


Figure: Scale factor $a(\tau)$ for gaussian superpotentials.

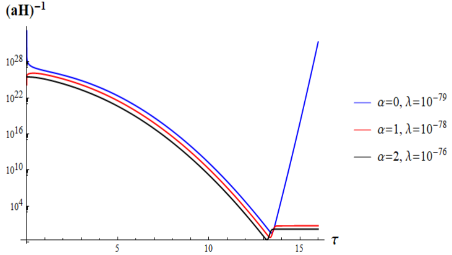


Figure: Comoving Hubble radius $(aH)^{-1}$ for gaussian superpotentials.

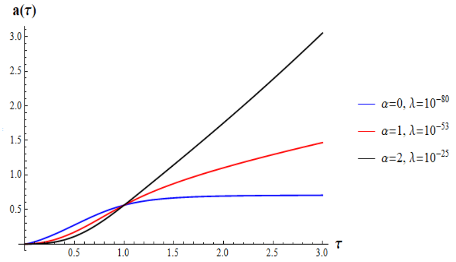


Figure: Scale factor $a(\tau)$ for step superpotentials.

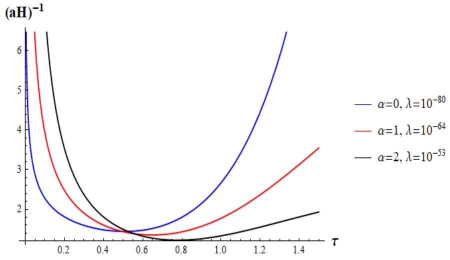


Figure: Comoving Hubble radius $(aH)^{-1}$ for step superpotentials.

Conclusions

- We introduce time in the wave function from the conditional probability for a given value of the scalar field, taken as clock.

- Supergravity for homogeneous models (FRLW)

- Supersymmetric constraints give first order equations equivalent to Wheeler-DeWitt equation.

- Initial conditions not required for unique solution, only normalizability.

- Analytic solution, positive norm.

- We give two types of superpotentials which generate inflationary cosmologies.

-Require fine tuning.

Reference: N.E. Martínez Pérez, C. Ramírez, V. Vázquez-Báez, arXiv:2104.12914

Thank you