Inflationary supersymmetric FRLW quantum cosmology

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Introduction

- Cosmology describes the general laws of the universe.

- In general classical formulation, good agreement with observational bounds.

- First stages of very early universe:
  ▶ Quantum gravity phase, quantum space-time fluctuations.
  ▶ Homogeneous phase.
  ▶ Inflationary phase, inhomogeneous perturbations originate structure seeds.

- Inhomogeneities follow from quantum fluctuations of QFT of metric perturbations in a homogeneous background.

- Suggests quantum description. Quantum cosmology

- Classical physics follows from quantum theory.
- Inflationary phase below Planck scale, very little is known.

- Supersymmetry, well known facts.
  - Unified framework for bosons and fermions.
  - Improves significantly behaviour of divergences in quantum field theory.
  - Ingredient of candidates for UV completion above Planck energy: string theory, supergravity.
  - Broken at low energies, above standard model scale.

- Supersymmetric quantum cosmology.
  - First order instead second order equations instead of the Wheeler-DeWitt equation.
  - Does not require boundary (initial) conditions.
Quantum cosmology

- A canonical quantization of general relativity. Hamiltonian function, linear combination of Hamiltonian $H$ and momentum constraints $H_i$.

- Constraints to operator equations on wave function of the universe $\psi(3g(x), \phi(x))$, defined on the fields on space slices.

- Momentum constraints $H_i\psi = 0$, invariance under space diffeomorphisms.

- Hamiltonian constraint $H\psi = 0$, Wheeler-DeWitt equation. There is no time dependent Schrödinger equation.

- Wave function gives probabilities to find the universe in given configurations $|\psi(3g(x), \phi(x))|^2$.

- WdW second order. Boundary conditions required.
Time

- Space-slice are parametrized by arbitrary time.

- A continuous sequence of most probable configurations, in subsequent slices, gives a trajectory (supertime of Misner\(^1\)).

- To a trajectory could be associated an uncertainty in superspace and in time.

- Internal time can be given, in one to one correspondence to certain function of observables.

Supergravity

- Supersymmetry relates one bosonic to one fermionic degree of freedom, gauge or matter. $\phi(x) \rightarrow \text{supermultiplet} (\phi(x), \psi(x), \ldots)$

- Supergravity: $e_m^a \rightarrow (e_m^a, \psi^\alpha_m)$, $\psi_m^\alpha$ $3 \over 2$-spin Rarita-Schwinger field.

- Extension of space-time to superspace $x \rightarrow z = (x, \theta, \bar{\theta})$, superfields functions of superspace, $\theta$ $1 \over 2$-spin variables.

- Supersymmetry transformations $x'^\mu = x^\mu + i(\theta^\mu \bar{\xi} - \xi^\mu \bar{\theta})$

- Supergravity: general relativity on superspace, fields $\rightarrow$ superfields, $\phi(x) \rightarrow \Phi(z)$, invariance under local superspace transformations $z' = z'(z)$.

- Differential, tensor and integral calculus on superspace.

- Supergravity from invariant action $\int dz \mathcal{E} L$, $\mathcal{E}$ invariant density.
Supersymmetric quantum cosmology

- Hamiltonian formulation of supergravity, constraints: Hamiltonian and momentum constraints $H_a$, Lorentz constraints $L^{ab}$, supersymmetric spinorial constraints $S$ and $\bar{S}$.

- Supersymmetric quantum cosmology has been developed for homogeneous spaces. In this case $\{S, \bar{S}\} = -2H$.

- Quantization: $S\psi = \bar{S}\psi = 0$, first order (square root) equations, instead of second order Wheeler-DeWitt equation.

- Here we consider supergravity of the FRLW model with a scalar field $\phi(t)$.

- The corresponding superfields are

$$N(t) \rightarrow \mathcal{E} = -N(t) - \frac{i}{2}(\Theta \bar{\psi}(t) + \bar{\Theta}\psi(t))$$

$$a(t) \rightarrow A = a(t) + \Theta \lambda(t) - \bar{\Theta} \bar{\lambda}(t) + \Theta \bar{\Theta} B(t)$$

$$\phi(t) \rightarrow \Phi = \phi(t) + \Theta \eta(t) - \bar{\Theta} \bar{\eta}(t) + \Theta \bar{\Theta} G(t)$$
- Action\(^2\) \( I = I_G + I_M \)

\[
I_G = \frac{3}{\kappa^2} \int \mathcal{E} \left( \mathcal{A} \nabla \bar{\Theta} \mathcal{A} \nabla \Theta \mathcal{A} - \sqrt{k} \mathcal{A}^2 \right) \, d\Theta d\bar{\Theta} dt,
\]

\[
I_M = \int \mathcal{E} A^3 \left[ -\frac{1}{2} \nabla \bar{\Phi} \nabla \Theta \Phi + W(\Phi) \right] \, d\Theta d\bar{\Theta} dt,
\]

- Supersymmetric constraints

\[
\frac{1}{\sqrt{\hbar c}} S = \frac{c \kappa}{2 \sqrt{6}} \left( a^{-\frac{1}{2}} \pi_a + \pi_a a^{-\frac{1}{2}} \right) \alpha + c a^{-\frac{3}{2}} \pi_\phi \beta + \frac{3i \kappa}{\sqrt{6}} a^2 W \alpha
\]

\[
+ i a^2 \, W' \beta - i \frac{\sqrt{6} \kappa}{\kappa} a^{\frac{1}{2}} \alpha - \frac{i \sqrt{3}}{4 \sqrt{2}} \hbar c \kappa a^{-\frac{3}{2}} \alpha[\bar{\beta}, \beta] = 0,
\]

where \( \lambda = \sqrt{\frac{\hbar c \kappa^2}{6}} \alpha, \quad \bar{\lambda} = \sqrt{\frac{\hbar c \kappa^2}{6}} \bar{\alpha}, \quad \eta = \sqrt{\hbar c} \beta \) and \( \bar{\eta} = \sqrt{\hbar c} \bar{\beta} \).

\[
\{ \alpha, \bar{\alpha} \} = 1, \quad \{ \beta, \bar{\beta} \} = -1.
\]

Quantization

- Ground state $|1\rangle$
  \[ \alpha |1\rangle = \beta |1\rangle = 0. \]

- Four orthogonal states
  \[ |1\rangle, \quad |2\rangle = \bar{\alpha} |1\rangle, \quad |3\rangle = \bar{\beta} |1\rangle \quad \text{and} \quad |4\rangle = \bar{\alpha} \bar{\beta} |1\rangle. \]

- Norms
  \[ \langle 2|2 \rangle = \langle 1|1 \rangle, \quad \langle 3|3 \rangle = -\langle 1|1 \rangle \quad \text{and} \quad \langle 4|4 \rangle = -\langle 1|1 \rangle. \]

- General state
  \[ |\Psi\rangle = \psi_1(a, \phi) |1\rangle + \psi_2(a, \phi) |2\rangle + \psi_3(a, \phi) |3\rangle + \psi_4(a, \phi) |4\rangle. \]

- Constraints
  \[ S |\Psi\rangle = \bar{S} |\Psi\rangle = 0. \]
- Equations from $S \left| \Psi \right\rangle = 0$

\[
a \left( \partial_a - \frac{3}{\hbar c} a^2 W + \frac{6 \sqrt{k}}{\hbar c \kappa^2} a + \frac{1}{2} a^{-1} \right) \psi_2 - \sqrt{\frac{6}{\kappa}} \left( \partial_\phi - a^3 W' \right) \psi_3 = 0,
\]

\[
\left( \partial_a - \frac{3}{\hbar c} a^2 W + \frac{6 \sqrt{k}}{\hbar c \kappa^2} a - a^{-1} \right) \psi_4 = 0
\]

\[
\left( \partial_\phi - \frac{1}{\hbar c} a^3 W' \right) \psi_4 = 0,
\]

- Equations from $\bar{S} \left| \Psi \right\rangle = 0$

\[
a \left( \partial_a + \frac{3}{\hbar c} a^2 W - \frac{6 \sqrt{k}}{\hbar c \kappa^2} a + \frac{1}{2} a^{-1} \right) \psi_3 - \sqrt{\frac{6}{\kappa}} \left( \partial_\phi + a^3 W' \right) \psi_2 = 0,
\]

\[
\left( \partial_a + \frac{3}{\hbar c} a^2 W - \frac{6 \sqrt{k}}{\hbar c \kappa^2} a - a^{-1} \right) \psi_1 = 0
\]

\[
\left( \partial_\phi + \frac{1}{\hbar c} a^3 W' \right) \psi_1 = 0,
\]
Solutions

- For $\psi_2$ and $\psi_3$ we could not get analytic solutions. An inspection shows no normalizable solutions.

- For $\psi_2$ and $\psi_3$ unique solutions, no boundary conditions needed.

$$\psi_1(a, \phi) = \mathcal{N} a \exp \left[ -\frac{1}{\hbar c} \left( a^3 W(\phi) - \frac{3\sqrt{k}a^2}{\kappa^2} \right) \right],$$

$$\psi_4(a, \phi) = \mathcal{N} a \exp \left[ \frac{1}{\hbar c} \left( a^3 W(\phi) - \frac{3\sqrt{k}a^2}{\kappa^2} \right) \right].$$

- If $W(\phi) \neq 0$, one of these solutions normalizable. We take for the other solution the trivial solution.

- Normalizable state can be chosen with positive norm.

- For $k = 0$ we write for the solution

$$\psi(a, \phi) = \mathcal{N} a \exp \left[ -\frac{1}{\hbar c} \left( a^3 |W(\phi)| \right) \right].$$
Gaussian type form wave function for fixed $\phi$, vanishes at $a = 0$. Take $\phi$ as time, with normalized effective wave function

$$\Psi(a, t) = \frac{1}{\sqrt{\int_0^\infty da \lvert \psi(a, t) \rvert^2}} \psi(a, t) = \sqrt{\frac{6|W(t)|}{\hbar c}} a \exp \left[ -\frac{a^3|W(t)|}{\hbar c} \right].$$

Corresponds to conditional probability for $a$, for given value of $\phi = t$

$$\lvert \Psi(a, t) \rvert^2 = \frac{\lvert \psi(a, t) \rvert^2}{\int_0^\infty da \lvert \psi(a, t) \rvert^2}.$$

Time dependent mean value of $a$

$$a(t) = \int_0^\infty a \lvert \Psi(a, t) \rvert^2 da = \Gamma(4/3) \left( \frac{\hbar c}{2|W(t)|} \right)^{1/3}. $$
Inflationary scenarios

- Set the superpotential so that $a(0)$ has a minimum.

- Inflation sets on at $t_i > 0$. Before full quantum gravity expected.

- Consider positive superpotential $W(t) > 0$, has a maximum at $t = 0$. Accelerated expansion $\ddot{a}(t) > 0$ requires $\frac{W}{W^{3/4}}$ is an increasing function.

- Accelerated expansion up to $t_f$, inflation with $N$ e-folds

\[
N = \ln \frac{a_f}{a_i} = \ln \frac{a_f}{a_i} = \frac{1}{3} \ln \frac{W(t_i)}{W(t_f)}
\]

\[
W(t_f) = W(t_i)e^{-3N}
\]

- The exit of inflation can be achieved by adding to superpotential a constant $\lambda = W(t_f)$. Requires fine tuning.
Superpotentials

- Gaussian superpotentials

\[ W(t) = \frac{c^4 M_p^3}{\hbar^2} t^{-2\alpha} \left( e^{-t^2} + \lambda \right), \]  

(1)

- Step superpotentials

\[ W(t) = \frac{c^4 M_p^3}{\hbar^2} t^{-2\alpha} \left( \frac{1}{t^2 + 1} + \lambda \right), \]  

(2)

- For an exit at \( N \) e-folds, \( \lambda \lesssim e^{-3N} \). For \( N = 60 \), \( \lambda \lesssim 10^{-79} \), depends on the details of the superpotential.

- If \( \lambda = 0 \) eternal inflation.

- If \( \alpha = 0 \), \( a(0) = \frac{\Gamma(4/3)}{2^{1/3}} \ell_p \).

- If \( \alpha > 0 \), \( a(0) = 0 \).

- Set unities \( c = 1, \hbar = 1, \) and \( \kappa = 1 \).
Figure: Scale factor $a(\tau)$ for gaussian superpotentials.

Figure: Comoving Hubble radius $(aH)^{-1}$ for gaussian superpotentials.
**Figure:** Scale factor $a(\tau)$ for step superpotentials.

**Figure:** Comoving Hubble radius $(aH)^{-1}$ for step superpotentials.
Conclusions

- We introduce time in the wave function from the conditional probability for a given value of the scalar field, taken as clock.

- Supergravity for homogeneous models (FRLW)

- Supersymmetric constraints give first order equations equivalent to Wheeler-DeWitt equation.

- Initial conditions not required for unique solution, only normalizability.

- Analytic solution, positive norm.

- We give two types of superpotentials which generate inflationary cosmologies.

- Require fine tuning.

Thank you