Thermodynamics of (scalar-tensor) gravity: a new approach

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1. Scalar-tensor gravity as an effective fluid (dissipative, irrotational)
2. GR as an equilibrium state; temperature and shear viscosity of ST gravity
3. Approach to the GR equilibrium state, or deviations from it
4. Conclusions & open problems
Motivation

Two key ideas:

1. It is possible that gravity emerges as a sort of fluid-mechanical or thermodynamical limit ("Einstein equation as an effective equation of state")

2. On a landscape of theories of gravity, GR could be the state of equilibrium and modified gravity an excited state (both ideas advanced in Jacobson’s thermodynamics of spacetime).

Scalar-tensor gravity is the prototypical alternative to GR; $f(R)$ gravity, a subclass, is extremely popular to explain the current acceleration of the universe without an ad hoc dark energy. The field equations are recast as effective Einstein equations by moving geometric terms $\neq G_{ab}$ to the r.h.s., regarding them as an effective $T_{ab}^{(eff)}$ (a fluid).
The (Jordan frame) action is

\[ S_{ST} = \frac{1}{16\pi} \int d^4x \sqrt{-g} \left[ \phi R - \frac{\omega(\phi)}{\phi} \nabla^c \phi \nabla_c \phi - V(\phi) \right] + S^{(m)} \]

where \( \phi \sim G_{\text{eff}}^{-1} > 0 \) is the Brans-Dicke scalar.

Field equations:

\[ R_{ab} - \frac{1}{2} g_{ab} R = \frac{8\pi}{\phi} T^{(m)}_{ab} + \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla_c \phi \nabla^c \phi \right) + \frac{1}{\phi} (\nabla_a \nabla_b \phi - g_{ab} \Box \phi) - \frac{V}{2\phi} g_{ab}, \]

\[ \Box \phi = \frac{1}{2\omega + 3} \left( \frac{8\pi T^{(m)}}{\phi} + \phi \frac{dV}{d\phi} - 2V - \frac{d\omega}{d\phi} \nabla^c \phi \nabla_c \phi \right) \]

Effective fluid description?
The correspondence is possible if gradient $\nabla^a \phi$ is timelike; fluid 4-velocity is

$$u^a = \frac{\nabla^a \phi}{\sqrt{-\nabla^e \phi \nabla_e \phi}},$$

($u^c u_c = -1$)

3-D space “seen” by the comoving observers of the fluid with time direction $u^a$ has 3-metric

$$h_{ab} \equiv g_{ab} + u_a u_b,$$

while $h^b_a$ is the usual projection operator on this 3-space,

$$h_{ab} u^a = h_{ab} u^b = 0,$$

$$h^b_a h^c_b = h^c_a, \quad h^a_a = 3.$$

Fluid 4-acceleration is

$$\dot{u}^a = u^b \nabla_b u^a$$

(of course, orthogonal to 4-velocity, $\dot{u}^c u_c = 0$).
The (double) projection of the velocity gradient onto the 3-space orthogonal to \( u^c \) is the purely spatial tensor

\[
V_{ab} \equiv h^c_a h^d_b \nabla_d u_c
\]

which decomposes as

\[
V_{ab} = \theta_{ab} + \omega_{ab} = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab}
\]

where \( \theta \equiv \theta^c_c = \nabla^c u_c \) = expansion scalar, and these tensors are purely spatial. In general (Ellis ’71)

\[
\nabla_b u_a = \sigma_{ab} + \frac{\theta}{3} h_{ab} + \omega_{ab} - \dot{u}_a u_b = V_{ab} - \dot{u}_a u_b.
\]
Let’s specialize these general definitions to our particular case. Kinematic quantities (not given in Pimentel ’89):

\[ h_{ab} = g_{ab} - \frac{\nabla a \phi \nabla b \phi}{\nabla e \phi \nabla e \phi} \]

\[ \nabla_b u_a = \frac{1}{\sqrt{-\nabla e \phi \nabla e \phi}} \left( \nabla_a \nabla b \phi - \frac{\nabla a \phi \nabla c \phi \nabla b \nabla c \phi}{\nabla e \phi \nabla e \phi} \right). \]

4-acceleration is

\[ \dot{u}_a = (\nabla^e \phi \nabla e \phi)^{-2} \nabla^b \phi \left[ (\nabla^e \phi \nabla e \phi) \nabla_a \nabla b \phi + \nabla^c \phi \nabla b \nabla c \phi \nabla a \phi \right] \]
\[ V_{ab} = \frac{\nabla_a \nabla_b \phi}{(-\nabla^e \phi \nabla_e \phi)^{1/2}} \]
\[ + \frac{(\nabla_a \phi \nabla_b \nabla_c \phi + \nabla_b \phi \nabla_a \nabla_c \phi) \nabla^c \phi}{(-\nabla^e \phi \nabla_e \phi)^{3/2}} \]
\[ + \frac{\nabla_d \nabla_c \phi \nabla^c \phi \nabla^d \phi}{(-\nabla^e \phi \nabla_e \phi)^{5/2}} \nabla_a \phi \nabla_b \phi. \]

Vorticity \( \omega_{ab} \equiv V_{[ab]} \) vanishes identically, because 4-velocity originates from a gradient, so

\[ V_{ab} = \theta_{ab}, \quad \nabla_b u_a = \theta_{ab} - \dot{u}_a u_b, \]

\( u^a \) is hypersurface-orthogonal.
Expansion scalar:

\[ \theta = \nabla_a u^a = \frac{\Box \phi}{(-\nabla^e \phi \nabla_e \phi)^{1/2}} + \frac{\nabla_a \nabla_b \phi \nabla^a \phi \nabla^b \phi}{(-\nabla^e \phi \nabla_e \phi)^{3/2}} , \]

Shear tensor:

\[ \sigma_{ab} = ( -\nabla^e \phi \nabla_e \phi )^{-3/2} \left[ - ( \nabla^e \phi \nabla_e \phi ) \nabla_a \nabla_b \phi - \frac{1}{3} ( \nabla_a \phi \nabla_b \phi - g_{ab} \nabla^c \phi \nabla_c \phi ) \Box \phi - \frac{1}{3} \left( g_{ab} + \frac{2 \nabla_a \phi \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \right) \nabla_c \nabla_d \phi \nabla^c \phi \nabla^d \phi \nabla^e \phi \nabla^e \phi \right] , \]
In the vacuum field eqs written as effective Einstein eqs.

\[ G_{ab} = 8\pi T_{ab}^{(\phi)} = \frac{\omega}{\phi^2} \left( \nabla_a \phi \nabla_b \phi - \frac{1}{2} g_{ab} \nabla^c \phi \nabla_c \phi \right) \]

\[ + \frac{1}{\phi} \left( \nabla_a \nabla_b \phi - g_{ab} \nabla^c \phi \nabla_c \phi \right) - \frac{V}{2\phi} g_{ab} \]

the r.h.s. decomposes as

\[ T_{ab}^{(\phi)} = \rho^{(\phi)} u_a u_b + q_a^{(\phi)} u_b + q_b^{(\phi)} u_a + \Pi_{ab}^{(\phi)}, \]

(dissipative fluid), where
\[ \rho^{(\phi)} = T^{(\phi)}_{ab} u^a u^b , \]

\[ q_a^{(\phi)} = - T^{(\phi)}_{cd} u^c h_a^d , \]

\[ \Pi_{ab}^{(\phi)} \equiv P^{(\phi)} h_{ab} + \pi_{ab}^{(\phi)} = T^{(\phi)}_{cd} h_a^c h_b^d , \]

\[ P^{(\phi)} = \frac{1}{3} g^{ab} \Pi_{ab}^{(\phi)} = \frac{1}{3} h^{ab} T^{(\phi)}_{ab} , \]

\[ \pi_{ab}^{(\phi)} = \Pi_{ab}^{(\phi)} - P^{(\phi)} h_{ab} , \]

with

\[ q_c^{(\phi)} u^c = \Pi_{ab}^{(\phi)} u^b = \pi_{ab}^{(\phi)} u^b = \Pi_{ab}^{(\phi)} u^a = \pi_{ab}^{(\phi)} u^a = 0 , \quad \pi^a a = 0 . \]
Calculating these quantities explicitly,

\[
8\pi \rho^{(\phi)} = -\frac{\omega}{2\phi^2} \nabla^e \phi \nabla_e \phi + \frac{V}{2\phi} + \frac{1}{\phi} \left( \square \phi - \frac{\nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \right),
\]

\[
8\pi q_a^{(\phi)} = -\frac{\nabla^c \phi \nabla_a \nabla_c \phi}{\phi \left(-\nabla^e \phi \nabla_e \phi\right)^{1/2}} - \frac{\nabla^c \phi \nabla^d \phi \nabla_c \nabla_d \phi}{\phi \left(-\nabla^e \phi \nabla_e \phi\right)^{3/2}} \nabla_a \phi,
\]

\[
8\pi \Pi_{ab}^{(\phi)} = \left( -\frac{\omega}{2\phi^2} \nabla^c \phi \nabla_c \phi - \frac{\square \phi}{\phi} - \frac{V}{2\phi} \right) h_{ab} + \frac{1}{\phi} h_a^c h_b^d \nabla_c \nabla_d \phi,
\]

\[
8\pi P^{(\phi)} = -\frac{\omega}{2\phi^2} \nabla^e \phi \nabla_e \phi - \frac{V}{2\phi} - \frac{1}{3\phi} \left( 2 \square \phi + \frac{\nabla^a \phi \nabla^b \phi \nabla_a \nabla_b \phi}{\nabla^e \phi \nabla_e \phi} \right),
\]
\[ 8\pi \pi_{ab}^{(\phi)} = \frac{1}{\phi \nabla e \phi \nabla e \phi} \left[ \frac{1}{3} \left( \nabla_{a\phi} \nabla_{b\phi} - g_{ab} \nabla^c \phi \nabla_c \phi \right) \left( \square \phi - \frac{\nabla^c \phi \nabla^d \phi \nabla_d \nabla_c \phi}{\nabla^e \phi \nabla_e \phi} \right) \right. \\
\left. + \nabla^d \phi \left( \nabla_d \phi \nabla_a \nabla_b \phi - \nabla_b \phi \nabla_a \nabla_d \phi - \nabla_a \phi \nabla_d \nabla_b \phi \right) \\
+ \frac{\nabla_a \phi \nabla_b \phi \nabla^c \phi \nabla_c \nabla_d \phi}{\nabla^e \phi \nabla_e \phi} \right] \]

The heat flux density

\[ q_a^{(\phi)} = -\frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{8\pi \phi} \dot{u}_a \]

and the anisotropic stresses \( \pi_{ab}^{(\phi)} \) do not vanish.
Take the effective dissipative fluid seriously: what do we know about dissipation in GR? Dissipative fluids in GR are described by Eckart’s 1st order thermodynamics (Eckart ’40), notoriously plagued by non-causality and instabilities but still the most widely used approximation.

Constitutive relations in Eckart’s theory:

\[ P_{\text{viscous}} = -\zeta \theta \]
\[ q_a = -K \left( h_{ab} \nabla^b T + T u_a \right) \]
\[ \pi_{ab} = -2\eta \sigma_{ab} \]

where

\[ \zeta = \text{bulk viscosity} \]
\[ K = \text{thermal conductivity} \]
\[ \eta = \text{shear viscosity} \]
Compare with the expressions of $P^{(\phi)}$, $q_{a}^{(\phi)}$, $\pi^{(\phi)}_{ab} \rightarrow$

\[ P_{\text{viscous}} = 0 , \]

\[ KT = \frac{\sqrt{-\nabla^{c} \phi \nabla_{c} \phi}}{8 \pi \phi} > 0 , \]

\[ \eta = -\frac{KT}{2} < 0 \]

for a ST spacetime.
Negative viscosities appear in systems that exchange energy with their surroundings (atmosphere, ocean currents, liquid crystals, ...) and the non-minimally coupled $\phi$-fluid is not isolated. Entropy density can decrease.
Approach to the GR equilibrium state

\[ K \mathcal{T} = \frac{\sqrt{-\nabla^c \phi \nabla_c \phi}}{8 \pi \phi} \]

\[ \phi = \text{const.} \iff \mathcal{T} = 0 \quad \text{GR equilibrium state} \]

Differentiate →

\[ \frac{d (K \mathcal{T})}{d \tau} = 8 \pi (K \mathcal{T})^2 - \theta K \mathcal{T} + \frac{\Box \phi}{\sqrt{-\nabla^e \phi \nabla_e \phi}} \]

Physical interpretation in simplified scenarios
Electrovacuum, $\omega = \text{const.}$, $V(\phi) = 0 \rightarrow \Box \phi = 0$.

Then,

$$\theta < 0 \rightarrow \frac{d(KT)}{dT} > 8\pi (KT)^2$$

or, $KT$ diverges away from the GR equilibrium extremely fast.

Deviations of ST gravity from GR will be extreme near spacetime singularities.

Electrovacuum, $\theta > 0$:

$-\theta KT$ can dominate $(KT)^2$ then the solution $KT$ can approach 0: diffusion to GR equilibrium, expansion cools gravity.

But, if $KT$ is large, the positive term dominates r.h.s. and drives solution away from GR: approach to GR equilibrium state not always expected.

Several analytic solutions of BD/ST gravity corroborate these ideas.
Minimal assumptions: used only ST field equations and constitutive relations in Eckart’s theory (not the full theory) → \( \mathcal{T}, \eta \), approach to GR equilibrium.

Open problems:
- Cosmology; situations with non-timelike \( \nabla^c \phi \nabla_c \phi \)
- Other theories of gravity
- Alternative approach: trade temperature with chemical potential, assign \( \mathcal{T} = 0, S = 0 \) but \( \mu \neq 0 \) to the effective fluid (as in Vikman at al. 2011-16).

Stay tuned on arXiv!