Corrections to the Newtonian potential through $f(R)$ theories of gravity

A virtual presentation at the 16th Marcel Grossmann meeting.

Outline for the talk

The binary problem

- In the motion of extended objects in the Universe around each other, it has been empirically observed that there is an *equivalence* between the gravitational and inertial masses of the said extended but compact objects.
- From a mathematical standpoint it implies that smoothly varying observables like angular frequency of the binary depends, in the early inspiral stage even solely, on the monopole moments of the source constituting the objects in the binary.
- Such a relation, observed in the natural world, is known as the equivalence principle, and holds true as long as the stellar sizes, compared to the distances between them, is negligible compared to unity.

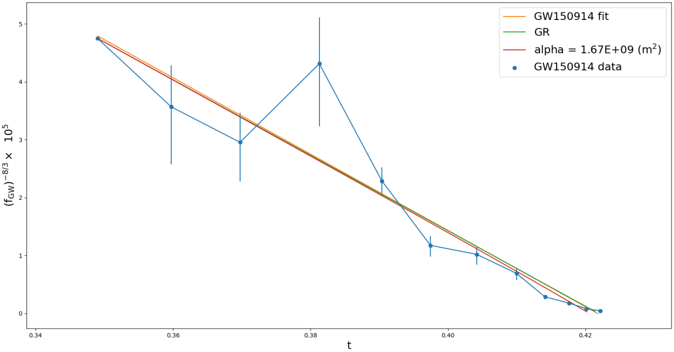
Can the predictions of the equivalence principle, as found through the various frameworks of general relativity, be found in Post-Einsteinian theories of gravitation as well? Or is GR the only theory to satisfy such a principle?

A naive delta function assumption

for the binary problem in $R + \alpha R^2$ gravity

Data source: "The basic physics of the binary black hole merger GW150914,"

arXiv:1608.01940



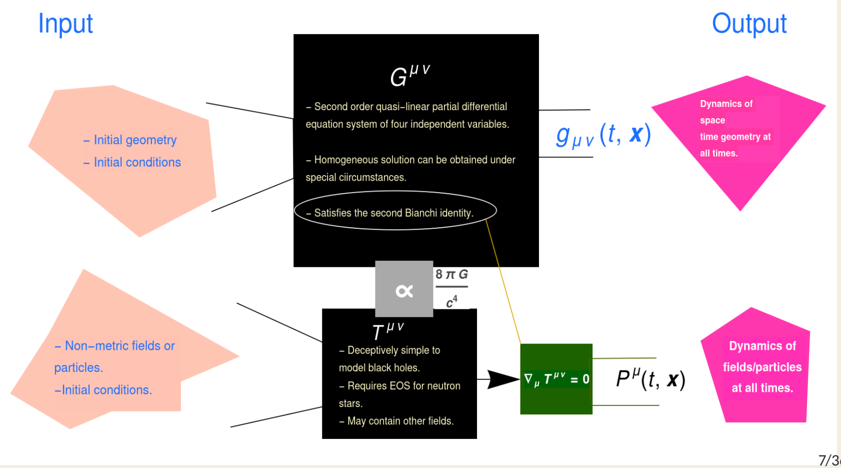
The equivalence principle

over the ages.

- For cavepeople, fall of boulder and fruit at the same rate might not have caused any surprise. Instinct of gravity inbuilt in all animals.
- Timeline of the equivalence principle: documented approximately 1500 years through Philoponus $\longrightarrow$ Galileo $\longrightarrow$ Newton $\longrightarrow$ Einstein models of celestial mechanics.
- General relativity: features built in equivalence principle.
- Loosely grouped as a weak or special version (weak equivalence principle), and a more strong or general version (strong equivalence principle).

General Relativity / Modified theories of gravity

as a black box system



- **Basic statement of GR:** The *trace reversed* Ricci tensor(Einstein tensor), itself being obtained by contracting the Riemann tensor, is directly proportional to the classical matter energy-momentum tensor.

$$G^{\mu\nu} \equiv R^{\mu\nu} - \frac{1}{2} g^{\mu\nu} R = \frac{8\pi G}{c^4} T_m^{\mu\nu}$$

- However, no such easy relation between the derivatives of the metric and the energy-momentum tensor; requires a transformation to obtain a dependent variable from the metric, such that the transformed variable has a more direct relation with $T_{\mu\nu}$.

Rewriting the $f(R)$ field equations

Given the Einstein tensor follows the covariant conservation law, all tensor terms that are related to the Einstein tensor, by virtue of the field equations of $f(R)$ theories of gravity, will therefore be conserved. Hence, one may rewrite the field equations in an Einsteinian form as

$$G^{\mu\nu} = \frac{8\pi}{f'}\left(T_m^{\mu\nu} + T_{eff}^{\mu\nu}\right)$$

$$T_{eff}^{\mu\nu} \equiv \nabla^\mu \nabla^\nu f' + \frac{g^{\mu\nu}}{2}\left(f - Rf'\right) - g^{\mu\nu} g^{\alpha\beta} \nabla_\alpha \nabla_\beta f'$$

$$3 g^{\mu\nu} \nabla_\mu \nabla_\nu f' + f' R - 2f = 8\pi T_m$$

The maximally symmetric spacetime

background to construct solutions around

- In a physical sense, a space that is homogeneous and isotropic everywhere is called a maximally symmetric space-time.
- In a mathematical sense, a maximally symmetric spacetime is a constant curvature spacetime possessing the maximum possible number of Killing vectors, which leads to the following relation between the Riemann tensor and the metric tensor of such space-times

$$R_{\lambda\rho\sigma\nu} = k\left(g_{\nu\rho}g_{\lambda\sigma} - g_{\sigma\rho}g_{\nu\lambda}\right)$$

- Such a space-time need not be flat.

The contravariant gothic metric

and $H^{\lambda\rho\sigma\nu}$

In the Landau-Lifshitz formulation of GR, a contravariant metric density $\mathfrak{g}^{\mu\nu}$ is used as the dynamical variable instead of the usual metric $g_{\mu\nu}$. One generally obtains a tensor with Riemann symmetry usually denoted as $H^{\lambda\rho\sigma\nu}$, and is proportional to the maximally symmetric space-time.

$$H^{\lambda\rho\sigma\nu} = \mathfrak{g}^{\nu\rho}\mathfrak{g}^{\lambda\sigma} - \mathfrak{g}^{\sigma\rho}\mathfrak{g}^{\nu\lambda}$$

where

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}\, g^{\mu\nu}$$

The Einstein tensor rewritten

In the field equations of GR the Einstein tensor can be algebraically split as a double partial derivative contracted $H^{\lambda\rho\sigma\nu}$ plus a tensor term that corresponds to a deviation from maximal symmetry $t_{LL}^{\mu\nu}$ (known as the Landau-Lifshitz energy-momentum pseudo-tensor in the literature), in an appropriate coordinate system or 'gauge', which are respectively given as

$$\begin{aligned} \partial_{\alpha\beta} H^{\mu\alpha\nu\beta} &= 16\pi\,(-g)\left(T_m^{\mu\nu} + t_{LL}^{\mu\nu} + t_{boundary}^{\mu\nu}\right) \\ \partial_\mu \mathfrak{g}^{\mu\nu} &= 0 \end{aligned}$$

A modified metric density

as the main variable on a Minkowski/flat background

Based on the fact that $f(R)$ theories of gravity have two *frames of expression* in the literature, namely the *Jordan* and the *Einstein* frame, the gothic metric density of such theories cannot be the usual metric density of GR but a modified one, defined as follows

$$\tilde{\mathfrak{g}}^{\mu\nu} = f'(R)\sqrt{-g}\,g^{\mu\nu}$$

The factor $f'(R)$ facilitates the alternate conformal expression of the field equations in the so called *Einstein* frame in literature. The above also leads to a transverse (and later traceless for GW calculations) tensor variable for a quadratic $f(R)$ theory in Berry & Gair 2011 once linearized.

Reformulating the field equations once more

Based on the modified gothic metric density redefinition with $\tilde{\mathfrak{g}}^{\mu\nu}$, one can rewrite the tensor field equations in a form that resembles the Landau-Lifshitz form of General Relativity, given by

$$\partial_{\alpha\beta} H^{\alpha\mu\beta\nu} \;=\; -16\,\pi\,(-g)\,f'(R)\,\left(T_m^{\mu\nu} + t_{eff}^{\mu\nu} + t_{LL}^{\mu\nu} + t_{boundary}^{\mu\nu}\right)$$

where the forms of $H^{\alpha\mu\beta\nu}$ and $t_{LL}^{\mu\nu}$ remain the same as in GR (with $\mathfrak{g}$ replaced by $\tilde{\mathfrak{g}}$). $t_{eff}^{\mu\nu}$ is a non-linear function of products of first derivatives of $f'(R)$ and $\tilde{\mathfrak{g}}^{\mu\nu}$, whose exact form becomes illuminating only after choosing a *weak-limit* form for the functions $f(R)$ and $f'(R)$.

Choosing a particular form

for $f(R)$ and $f'(R)$

- $f(R)$ theories of gravity have a tendency to harbor a massive scalar field, in contrast to its GR counterpart whose waves are tensorial in nature and massless. For this reason, a treatment on a general $f(R)$ theory of gravity is more difficult compared to choosing a particular polynomial form.
- To keep deviations from GR as less as possible, the quadratic form can be safely chosen as a low energy limit of a high energy physics model, given as

$$\begin{aligned} f(R) &= R + \frac{f''(0)}{2} R^2 \\ f'(R) &= 1 + f''(0)\, R \end{aligned}$$

The Post Newtonian Sequence

and an older representation of the trace reversed metric tensor in GW literature.

The post Newtonian sequence of solutions for $\mathfrak{g}^{\mu\nu}$ is a standard one and involves definition of a perturbed tensor potential $\mathfrak{h}^{\mu\nu}$ obeying the Lorenz gauge condition, given respectively as

$$\begin{aligned} \mathfrak{h}^{\mu\nu} &= \eta^{\mu\nu} - \mathfrak{g}^{\mu\nu} \\ \partial_\mu \mathfrak{h}^{\mu\nu} &= 0 \end{aligned}$$

leading to the following wave equation for $\mathfrak{h}^{\mu\nu}$ with standard solutions

$$\Box \mathfrak{h}^{\mu\nu} = -16\,\pi\,(-g)\,\Lambda^{\mu\nu}$$

Defining a scalar density

out of the usual Ricci scalar

- Since there is the Ricci scalar manifesting as a scalar field, one needs to define a dynamical variable that propagates on a Ricci flat background, and has a ’proper’ scaling relationship with the trace of the classical energy-momentum tensor (which may comprise of a pair of slowly spinning fluids, gravitational mass monopoles, or a combination of both).
- One may choose to structure the trace of the field equations in a manner that connects the d’Alembert operated scalar density, say $\Box \mathfrak{R}$, to the trace of $\underset{m}{T}^{\mu\nu}$ with an overall $(-g)$ factor. Such a redefinition was found to be as follows

$$\mathfrak{R} \;=\; (-g)\, R$$

Perturbing the new gothic metric density

- In order to obtain a PN sequence, the modified gothic metric density needs to be perturbed about a Minkowski/flat background, which can then be expanded as multipolar integrals of a (different from GR) EM pseudo-tensor.
- Similarly, a gauge/coordinate needs to be chosen on which to project the components of a perturbed metric density. That will be the Lorenz gauge.

The above conditions are mathematically defined as

$$\tilde{\mathfrak{h}}^{\mu\nu} = \eta^{\mu\nu} - \tilde{\mathfrak{g}}^{\mu\nu}$$
$$\tilde{\mathfrak{h}}^{\mu\nu}{}_{,\nu} = 0$$

Tensor field equations rewritten

with $\tilde{\mathfrak{h}}^{\mu\nu}$ and $\mathfrak{R}$

The tensor equations that needs to be solved for can now be expressed as follows

$$
\begin{aligned}
\Box\tilde{\mathfrak{h}}^{\mu\nu} &= -16\,\pi\,\tilde{\Lambda}^{\mu\nu} \\
\tilde{\Lambda}^{\mu\nu} &= (-g)\left(T^{\mu\nu}_{m} + t^{\mu\nu}_{LL} + t^{\mu\nu}_{H} + t^{\mu\nu}_{eff}\right) \\
t^{\mu\nu}_{eff} &= -\frac{3\,f''(0)}{(-g)}\,\mathfrak{R}\,T^{\mu\nu}_{m} - \frac{f''(0)}{64\,\pi\,(-g)^2}\,\Big[\mathfrak{R}^{,\rho}\;\tilde{\mathfrak{h}}_{,\rho}\;\eta^{\mu\nu} + \\
&\quad -2\,\tilde{\mathfrak{h}}^{,(\mu}\,\mathfrak{R}^{,\nu)} - 4\,(\mathfrak{R})^2\,\eta^{\mu\nu} + E^{\mu\nu}\Big]
\end{aligned}
$$

where $E^{\mu\nu}$, containing products of first derivatives of $\tilde{\mathfrak{h}}^{\mu\nu}$ and $\mathfrak{R}$ appear higher at the PN order under the transformation $s = \epsilon\, t$.

Scalar field equations rewritten

with T_m and $\mathfrak{R}$

The trace of the field equations, after ignoring total derivatives as boundary terms, reduces to the following

$$\begin{aligned} \Box\,\mathfrak{R} + \gamma^2\,\mathfrak{R} &= -8\,\pi\,(-g)\,\gamma^2 \left[T_m - f''(0) \left\{ \frac{2}{(-g)}\,T_m\,\mathfrak{R} \right.\right. \\ &\qquad \left.\left. - \frac{\mathfrak{R}^2}{4\,\pi\,(-g)^2} \right\} \right] + \mathcal{O}\left(\frac{[f''(0)]^2}{(-g)^3\,(\text{length})^4} \right) \\ \gamma^2 &\equiv -\frac{\sqrt{-g}}{3\,f''(0)} \end{aligned}$$

Thus the dynamics of the Ricci scalar density $\mathfrak{R}$ is independent of the dynamics of $\tilde{\mathfrak{h}}^{\mu\nu}$, and in a way, independent of T_m, the trace of the classical matter EM tensor as well, as will be seen next.

Modelling the Energy-Momentum tensor of extended objects

- Usual Post-Newtonian approximation assumes weak field, external and internal space-time wise.
- In GR, BH forms before point particle is reached. Density / compactness of stars increase to maximal neutron star situations before the *mass-radius ratio* becomes constant for all times.
- If a choice of coordinate system can be made where both mass and radius scale in the same manner, and is related to the PN parameter ϵ, one can mathematically take care of the strong internal fields inside stars and can proceed to apply the usual PN formalism.

Body zone coordinate

Spatial scaling :

$$(X^i)_L \equiv \frac{(x^i)_L - (z^i)_L}{\epsilon^2}$$

Temporal scaling :

$$s = \epsilon \cdot t$$

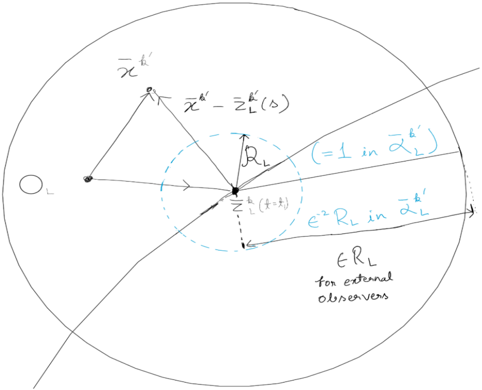


Figure: A single star, as measured in two different coordinate frames. In black, lengths as measured by an observer at asymptotic infinity. In blue: as measured in scaled coordinate. Black outer sphere: boundary of patched coordinate systems.

The Newtonian dynamical time

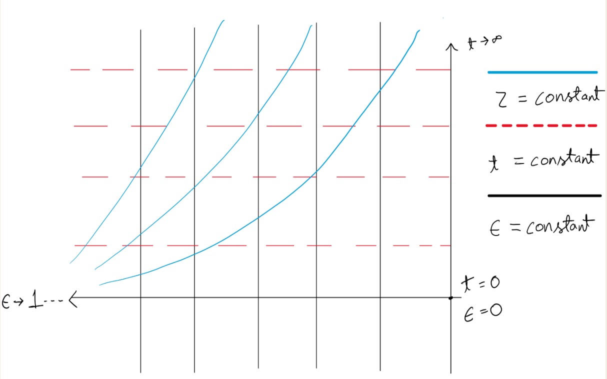
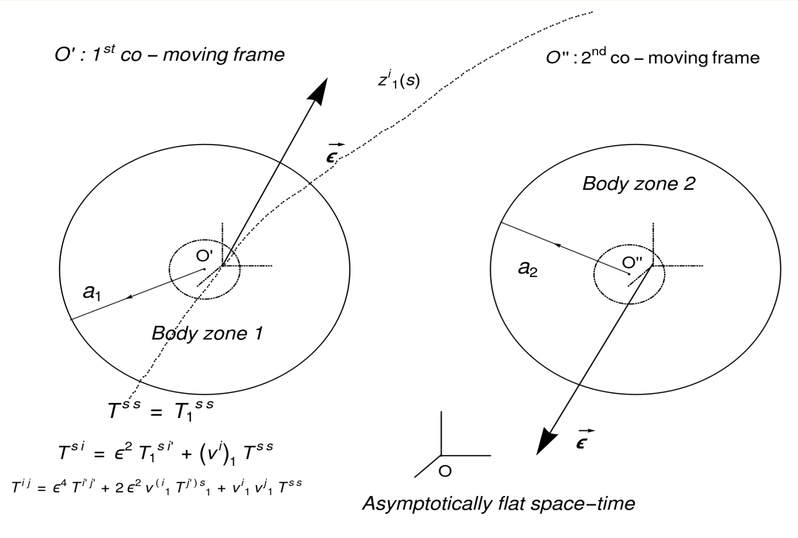


Figure: Constant $'s'$ curves which form hyperbolas. If for $\epsilon = 0.1$ the system completes one orbit after some dynamical time $'s_1'$, then it would have completed its first orbit in roughly the same dynamical time for $\epsilon = 0.01$. Implying the phase of the system along the $s = constant$ curve remains approximately the same as time $'t'$ varies.

Comoving frames

- In order to patch together two space-times (corresponding to each compact body and their neighborhoods) into a single asymptotically Minkowskian spacetime, two co-moving (but non-rotating with respect to the asymptotic observer) coordinate systems centered around the two compact objects are defined.
- Mathematically, this can be achieved by defining two Fermi-Normal coordinates along the world-line of the two compact objects. At the leading order, the two co-moving volumes can safely be approximated as spheres.
- The spherical co-moving zone boundary is not well motivated for higher than 2.5 PN calculations in GR due to the Lorentz contraction of a moving sphere leading to spurious multipole moments.

Comoving frames



Defining a measure of the four-momentum

The 4-momentum in the body zone of the L^{th} compact object is mathematically defined as (Futamase & Itoh 2007, Giulini 2018)

$$P_L^{\mu}(s) = \epsilon^2 \int_{\mathcal{B}_L} d^3X_L \, \Lambda^{s\,\mu}, \tag{1}$$

$$\Lambda^{\mu\nu} = T_m^{\mu\nu} + t_{LL}^{\mu\nu} \tag{2}$$

Given a post Newtonian sequence of solutions for $\mathfrak{g}^{\mu\nu}$ about $\epsilon \to 0$ can be found, the pseudo-tensor $\Lambda^{\mu\nu}$ can be expanded in a series over ϵ, leading to a quasi-conserved 4-momentum at each PN order.

Evolution of the 4-momentum

To obtain the equations of motion, one takes the dynamical time ($'s'$) derivative of the 4-momentum, and uses the conservation equation on $\Lambda^{\mu\nu}$ in order to obtain the 4-force on the L^{th} body as vector surface integrals of the net EM pseudo-tensor over the sphere which is the outer boundary of the body zone L (Futamase & Itoh 2007)

$$\frac{dP^{\mu}_{L}}{ds} = -\epsilon^{-4} \oint_{\partial \mathscr{B}_L} dS_k \, \Lambda^{k\mu} + \epsilon^{-4} v^k_L \oint_{\partial \mathscr{B}_L} dS_k \, \Lambda^{s\mu} \tag{3}$$

dS_k being an infinitesimal normal 1-form (or an infinitesimal covariant normal vector) on the spherical outer boundary of the body zone.

Momentum-velocity relation

in general

The 3-momentum vs 3-velocity relationship can be identified from the 4-momenta, and is given by (Futamase & Itoh 2007)

$$P_L^i = P_L^s v_L^i + Q_L^i + \mathcal{O}\left(\epsilon^2\right) \quad (4)$$

$$Q_L^i = \epsilon^{-4} \oint_{\partial \mathscr{B}_L} dS_k \left(\Lambda^{sk} - v_L^k \Lambda^{ss}\right) X_L^i \quad (5)$$

which reduces to the standard momentum-velocity relationship of classical mechanics at the leading order if P_L^s is identified as the gravitational mass contained in the body zone L.

3-velocity evolution

or the 3-acceleration

The momentum-velocity relationship at the leading order can be substituted in the spatial part of the 4-momentum to obtain the acceleration, or the equations of motion, of the L^{th} compact object

$$\begin{aligned} P^s_L \frac{dv^i_L}{ds} &= -\epsilon^{-4} \oint_{\partial \mathscr{B}_L} dS_k \, \Lambda^{ki} + \epsilon^{-4} v^k_L \oint_{\partial \mathscr{B}_L} dS_k \, \Lambda^{si} \\ &\quad + \epsilon^{-4} v^i_L \left(\oint_{\partial \mathscr{B}_L} dS_k \, \Lambda^{ks} - v^k_L \oint_{\partial \mathscr{B}_L} dS_k \, \Lambda^{ss} \right) \\ &\quad - \frac{dQ^i_L}{ds} \end{aligned} \tag{6}$$

The form of $\Lambda^{\mu\nu}$ at each PN order (constructed from PN solution $\mathfrak{h}^{\mu\nu}$ at an earlier order) determines the equations of motion at the corresponding order.

The one and half PN equations of motion

Using the surface integral approach to obtain the force on the first body zone using the surface intgral approach, the 1.5 post Newtonian equations were found to be as follows

$$
\begin{aligned}
M_1 \frac{dv_1^i}{ds} &= F^i_{Newton} + \epsilon^2 F^i_{1PN,GR} \\
&\quad + \epsilon^3 \frac{2\pi}{9} \frac{M_1 M_2}{r_{12}} \left(\frac{1}{r_{12}} + \gamma \right) e^{-\gamma r_{12}} n^i
\end{aligned}
$$

where n^i is the unit vector pointing from the COM of the first body zone to the COM of the second body zone, and the distance between the COM of the two bodies or the orbital separation is r_{12}.

Velocity profile

It is interesting to note how the velocities depend on the distance r_{12}. After canceling M_1 from both sides one can rewrite the equations of motion till 1.5 PN (ignoring the 1PN force) as the following

$$n_i v_1^i(r_{12}) = \pm\sqrt{\frac{2}{r_{12}}}\sqrt{1 - e^{-\gamma r_{12}} + C\gamma r_{12}}$$

Outline for the talk

A new treatment of $f(R)$ theories of gravity

- For a Landau Lifshitz like treatment of the field equations of $f(R)$ theories of gravity, a modified approach to defining a gothic metric density was formulated utilizing a conformal symmetry of metric $f(R)$ theories of gravity. The properties are connected to the dual frame expression (Einstein and Jordan) of $f(R)$ theories of gravity and the separation of the gravitational wave part of the space-time from the non-radiating part of the space-time.
- A scalar density, acting as the dynamical variable for the massive scalar field of $f(R)$ theories of gravity was defined, whose dynamics was related to the classical matter energy-momentum scalar in the same manner the perturbed modified metric density's dynamics relates to the classical EM tensor.

Post Newtonian equations of motion of a binary

till 1.5 PN in quadratic $f(R)$ theories of gravity

- Proper definition of tensor and scalar dynamical variables allow one to express the net energy-momentum tensor sourcing the dynamics of the tensor+scalar system as a polynomial series on the combined *deviation from GR parameter* and determimant of the metric $\frac{f''(0)}{\sqrt{-g}}$. This particular series of the net EM pseudo-tensor allows one to figure out which term/s become dominant at which PN order just by counting the power on $\frac{f''(0)}{\sqrt{-g}}$ in the same.
- Most of the highly non-linear terms are pushed to higher than first PN order. At the next-to-leading order from the Newtonian solution, the non-Einsteinian force is composed of an *inverse length density* γ which is a universal constant in $f(R)$ theories.

An extra misbehaving force

but only at large distances

- The extra force signaling a deviation from GR appearing at the 1.5 PN order is starkly different at large distances from any force that might result from consequences of Newtonian or Einsteinian gravity. Unlike the former, the new force can be repulsive at large distances from another object (which produces an external gravitational field).
- Asymptotic flatness is maintained as the total gravitational force at large distances become smaller in magnitude. However, the net gravitational force cancels out to zero at large distances, leading to a flat velocity profile.

Slightly wrong end result but correct tools in arXiv:2103.04627

Thank you for your attention...