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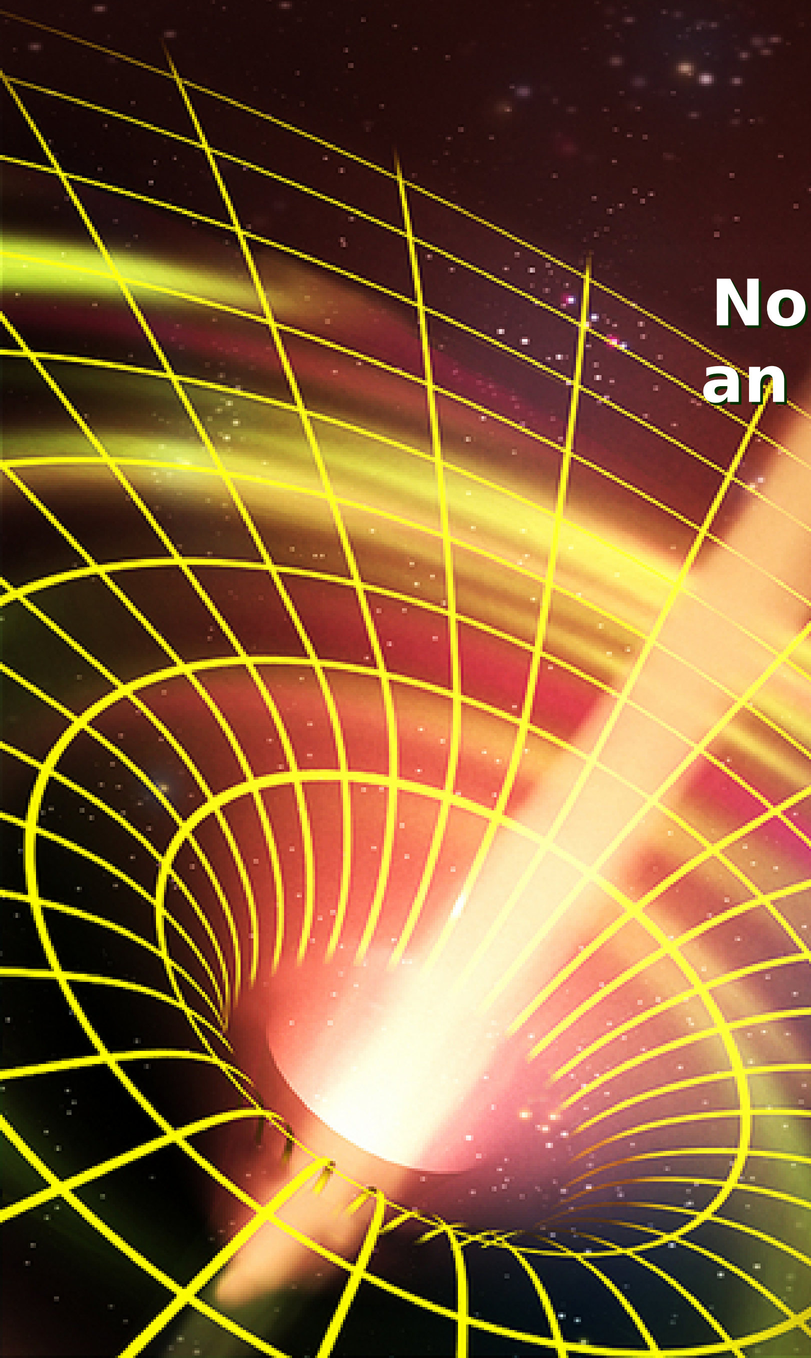

Nonsymmetric metric tensor as an approach to quantum gravity

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Abstract:
We argue that the minimal length uncertainty emerging from the generalized uncertainty principle, in which the gravitational impacts on the noncommutative space are thoughtfully taken into account, modifies the whole spacetime geometry. The resulting spacetime metric tensor consists of the symmetric GR compatible metric tensor $g_{\mu\nu}$ and another term comprising $g_{\mu\nu}$ multiplied by $\beta_0(\ell_p/\hbar)^2$ squared. Towards approaching quantized spacetime geometry, we analyze the line element and discuss on the resulting geodesic, which encompasses acceleration, jerk, and snap (jounce) of a test particle in discretized gravitational field.

Main Problem and Questions

- Gravity Quantization requires spacetime quantization or fundamental limits on length and momentum.

$$R_{\mu\nu} - \frac{1}{2} R\, g_{\mu\nu} + \Lambda\, g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}$$

- Various models predict minimum measurable length (space discretization).
- Minimum measurable length and/or maximal momentum => generalized uncertainty principle (GUP)

$$\Delta x\, \Delta p \geq \frac{\hbar}{2}\left(1 + \beta\langle \hat{p}^2\rangle\right) = \frac{\hbar}{2}\left[1 + \beta\,(\Delta p)^2 + \beta\langle \hat{p}\rangle^2\right]$$

- $T_{\mu\nu}$ in EFE (rhs) is quantizable!
- Would GUP suggest quantization for lhs of EFE?!
- How look like the metric tensor, line metric and geodesic equation?

QM and GR

- **QM which explains interactions through exchange of particles is inherently probabilistic, for instance,**

$$\left[-\frac{\hbar^2}{2m}\nabla^2 + V\right]\Psi = i\hbar\frac{\partial\Psi}{\partial t}$$

- **GR describes that the gravitational interactions as manifestation of spacetime geometry**

$$R_{\mu\nu} - \frac{1}{2}R\,g_{\mu\nu} + \Lambda\,g_{\mu\nu} = \frac{8\pi G}{c^4}T_{\mu\nu}$$

in which

1- $g_{\mu\nu}$ plays a central role,

2- free particle moves along geodesics $\frac{d^2x^\mu}{d\tau^2} = -\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}$

3- $g_{\mu\nu}$ governs this motion & emerges effects of gravity,

4- matter content $T_{\mu\nu}$ affects the spacetime curvature,

5- EFE describes how spacetime geometry is governed by matter

GUP Minimal Length

- Quantum gravity predicts quadratic-momenta-corrections to Heisenberg uncertainty principle. Kempf-Mangano-Mann suggested that

$$[x_i, p_j] = i\hbar\left(\delta_{ij} + \frac{\beta_0}{(M_P c)^2}\delta_{ij}p^2 + \frac{2\beta_0}{(M_P c)^2}p_i p_j\right)$$

where $p^2 := \vec{p}^2 = \sum_{j=1}^{3} p^j p_j$ and $M_P = \sqrt{\hbar c/G}$ is the Planck mass.

and

$$\Delta x_i \Delta p_i \geq \frac{\hbar}{2}\left(1 + \frac{\beta_0}{(M_P c)^2}\left((\Delta p)^2 + \langle p\rangle^2 + 2\Delta p_i^2 + 2\langle p_i\rangle^2\right)\right)$$

implying a minimum measurable length

$$\Delta x_{min} = \sqrt{3\beta_0}L_P, \text{ where } L_P = \sqrt{\hbar G/c^3}$$

- There are various GUP proposals, for example, Maggiore,

$$[x_i, p_j] = i\hbar\delta_{ij}\sqrt{1 + \frac{\gamma_0}{(M_P c)^2}(p^2 + m^2c^2)},$$

which is motivated by quantum deformation of the Pioncarre algebra. The minimal length associated with reads

$$\Delta x_{min} \simeq \sqrt{\gamma_0/2}L_P$$

Noncommutation relations

- Inspired by **GUP**, the canonical noncommutation relation of quantum operators can be expressed as

$$[\hat{x}_i\,,\hat{p}_j] \geq \delta_{ij} i\,\hbar\,\left(1+\beta p^2\right)\,,$$

where $p^2 = g_{ij} p^{0i} p^{0j}$ and g_{ij} is the Minkowski spacetime metric tensor, for instance $(-,+,+,+)$.

$$\hat{x}_i = \hat{x}_{0i}(1+\beta p^2), \qquad \hat{p}_j = \hat{p}_{0j},$$

and the operators $\hat{x}_{0i}$ and $\hat{p}_{0j}$ are similarly deduced from corresponding noncommunitation relation $[\hat{x}_{0i}, \hat{p}_{0j}] = \delta_{ij} i\,\hbar.$

- Such noncommutation relation can be included in the spacetime geometry through enlarging Minkowski manifold to an **8d-spacetime** tangent bundle like

$$x^A = (x^\mu(\zeta^a), \beta\dot{x}^\mu(\zeta^a)), \text{ where } \dot{x}^\mu = \frac{dx^\mu}{ds}$$

- Minkowskian line metric
- $$ds^2 = g_{\mu\nu} dx^\mu dx^\nu$$
-

Modified Minkowskian line metric

$$d\tilde{s}^2 = g_{AB}\,dx^A\,dx^B,$$

where $g_{AB} = g_{\mu_\nu} \otimes g_{\mu\nu}$

μ,ν,λ=0,1,2,3

A,B,a,b=0,1,…,7

4D Metric Tensor

- **8d-manifold can be reduced to 4d-spacetime geometry**

$$x^A = x^A(\zeta^\mu)$$

- **The modified 4d metric tensor is then given as**

$$\tilde{g}_{\mu\nu} = g_{AB}\frac{\partial x^A}{\partial \zeta^\mu}\frac{\partial x^B}{\partial \zeta^\nu} \simeq g_{ab}\left[\frac{\partial x^a}{\partial \zeta^\mu}\frac{\partial x^b}{\partial \zeta^\nu} + \beta\frac{\partial \dot{x}^a}{\partial \zeta^\mu}\frac{\partial \dot{x}^b}{\partial \zeta^\nu}\right] \simeq \left(1 + \beta \ddot{x}^\lambda \ddot{x}_\lambda\right) g_{\mu\nu},$$

where $\ddot{x}^\mu = \partial \dot{x}^\mu / \partial \zeta^\mu$ is the four-dimensional acceleration.

- **For flat spacetime, where $g_{\mu\nu} = \eta_{\mu\nu}$, the modified metric tensor reads**

$$\tilde{g}_{\mu\nu} = \left(1 + \beta \ddot{x}^\lambda \ddot{x}_\lambda\right)\eta_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu},$$

where $h_{\mu\nu} = \beta \ddot{x}^\lambda \ddot{x}_\lambda \eta_{\mu\nu}$ presses the quantum corrections due to GUP.

Modified Line Metric

- If h ->0, the GUP contributions diminish and EFE is restored.

- This means that the principle of the general covariance is satisfied even in absence of gravitational effects on modified Minkowski metric tensor.

- Modified 4d line metric reads

$$d\tilde{s}^2 = g_{\mu\nu}\left(dx^\mu dx^\nu + \beta^2 d\dot{x}^\mu\, d\dot{x}^\nu\right) = \left(1 + \beta^2 \ddot{x}^\lambda \ddot{x}_\lambda\right) ds^2.$$

where $\beta^2 \ddot{x}^\lambda \ddot{x}_\lambda$ expresses the quantum corrections due to GUP.

Modified Geodesic Equation

- **Using the variational principle and extremizing the path S_{AB}, SR and GR could be generalized through modifying geodesic equations,**
- **Flat Space:**

$$\beta\mathcal{L}\frac{d^2\dot{x}^\mu}{d\tau^2} - \frac{dx^\mu}{d\tau} + c = 0$$

- **Curved space:**

$$\begin{aligned}
\frac{d^2x^2}{d\tau^2} - \beta\frac{d}{d\tau}\left(\mathcal{L}\frac{d^3x^2}{d\tau^3}\right) &= -\Gamma^2_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \beta g^{2\alpha}g_{\mu\nu,\alpha}\frac{d^2x^\mu}{d\tau^2}\frac{d^2x^\nu}{d\tau^2} \\
&+ \beta\mathcal{L}g^{2\alpha}g_{\mu\alpha,\gamma}\left[\frac{dx^\gamma}{d\tau}\frac{d^2\dot{x}^\mu}{d\tau^2} + \frac{d}{d\tau}\left(\frac{dx^\gamma}{d\tau}\frac{d\dot{x}^\mu}{d\tau}\right)\right] \\
&+ \beta\mathcal{L}g^{2\alpha}g_{\mu\alpha,\gamma,\delta}\frac{dx^\delta}{d\tau}\frac{dx^\gamma}{d\tau}\frac{d\dot{x}^\mu}{d\tau}
\end{aligned}$$

$$\tau = \int \mathcal{L}(s,\dot{x},\ddot{x})ds, \quad \dot{x}^\mu = \frac{dx^\mu}{ds}, \quad g_{\mu\nu,\alpha} = \frac{\partial g_{\mu\nu}}{\partial x^\alpha}, \quad g_{\mu\alpha,\gamma,\delta} = \frac{\partial}{\partial x^\delta}\left(\frac{\partial g_{\mu\alpha}}{\partial x^\gamma}\right),$$

$$\Gamma^2_{\mu\nu} = \frac{1}{2}g^{2\alpha}\left[g_{\mu\alpha,\nu} - g_{\alpha\nu,\mu} + g_{\mu\nu,\alpha}\right], \qquad \mathcal{L} = \left[g_{\mu\nu}\left(\frac{dx^\mu}{ds}\frac{dx^\nu}{ds} + \beta\frac{d\dot{x}^\mu}{ds}\frac{d\dot{x}^\nu}{ds}\right)\right]^{1/2}$$

Modified Geodesic Equation

Our results mean that not only acceleration of a test particle in the gravitational field is included, but also higher derivatives as well, namely snap or jounce, $x^{(4)}$, which - in turn - is derived from jerk, $x^{(3)}$.

- Acceleration, as in Einstein's geodesic, without jerk is just a static load, i.e. neither vibration nor transition are allowed.

- Jerk gives the change in the force acting on that test particle, while snap is resulted from change in the jerk, itself.

- Both quantities are finite when vibration occurs, especially for multi-resonant modes and for sudden transitions between different radii of curvature.

Summary

GUP

$$[\hat{x}_i\,,\hat{p}_j] \geq \delta_{ij}\, i\,\hbar\,\left(1+\beta p^2\right)\,, \qquad \hat{x}_i = \hat{x}_{0i}(1+\beta p^2),\;\; \hat{p}_j = \hat{p}_{0j},$$

where $p^2 = g_{ij}p^{0i}p^{0j}$ and g_{ij} is the Minkowski spacetime metric tensor

Metric tensor

$$\tilde{g}_{\mu\nu} = g_{AB}\frac{\partial x^A}{\partial \zeta^\mu}\frac{\partial x^B}{\partial \zeta^\nu} \simeq g_{ab}\left[\frac{\partial x^a}{\partial \zeta^\mu}\frac{\partial x^b}{\partial \zeta^\nu} + \beta\frac{\partial \dot{x}^a}{\partial \zeta^\mu}\frac{\partial \dot{x}^b}{\partial \zeta^\nu}\right] \simeq \left(1+\beta\ddot{x}^\lambda\ddot{x}_\lambda\right)g_{\mu\nu},$$

Line metric

$$d\tilde{s}^2 = g_{\mu\nu}\left(dx^\mu dx^\nu + \beta^2 d\dot{x}^\mu\, d\dot{x}^\nu\right) = \left(1+\beta^2\ddot{x}^\lambda\ddot{x}_\lambda\right)ds^2$$

Geodesic eq.:

$$\beta\mathcal{L}\frac{d^2\dot{x}^\mu}{d\tau^2} - \frac{dx^\mu}{d\tau} + c = 0$$

flat space

$$\begin{aligned}\frac{d^2x^2}{d\tau^2} - \beta\frac{d}{d\tau}\left(\mathcal{L}\frac{d^3x^2}{d\tau^3}\right) &= -\Gamma^2_{\mu\nu}\frac{dx^\mu}{d\tau}\frac{dx^\nu}{d\tau} + \beta g^{2\alpha}g_{\mu\nu,\alpha}\frac{d^2x^\mu}{d\tau^2}\frac{d^2x^\nu}{d\tau^2} \\ &+ \beta\mathcal{L}g^{2\alpha}g_{\mu\alpha,\gamma}\left[\frac{dx^\gamma}{d\tau}\frac{d^2\dot{x}^\mu}{d\tau^2} + \frac{d}{d\tau}\left(\frac{dx^\gamma}{d\tau}\frac{d\dot{x}^\mu}{d\tau}\right)\right] \\ &+ \beta\mathcal{L}g^{2\alpha}g_{\mu\alpha,\gamma,\delta}\frac{dx^\delta}{d\tau}\frac{dx^\gamma}{d\tau}\frac{d\dot{x}^\mu}{d\tau}\end{aligned}$$

curved space