Effective field theory from Relativistic Generalized Uncertainty Principle

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Outline

1. Minimum length and Quantum Gravity
2. Relativistic Generalized Uncertainty Principle [1]
3. Quantum Field Theory with Minimum Length [2 3]
4. Summary and outlook

1. V. Todorinov, P. Bosso, S. Das, Ann. Phys. **405**, 92-100 (2019) [arXiv:1810.11761].
2. P. Bosso, S. Das, V. Todorinov [arXiv:2005.03771 [gr-qc]]
3. P. Bosso, S. Das, V. Todorinov [arXiv:2005.03772 [gr-qc]].

Generalized Uncertainty Principle

- Uncertainty relation for a Heisenberg microscope with Newtonian potntial

$$\Delta x \Delta p \geq \left(h + \underbrace{\frac{Gh}{c^3}}_{=l_{Pl}^2} \frac{\Delta p^2}{h} \right)$$

- *"Hilbert space representation of the minimal length uncertainty relation" published in 1995*[4]

$$[x, p] = i\hbar(1 + \beta p^2)$$

- This amounts to the following minimum uncertainty in position

$$\Delta x_{\text{min}} = \hbar\sqrt{\beta}$$

4. A. Kempf, G. Mangano, and R. B. Mann, Phys. Rev. **D52**, 1108 (1995) [arXiv:hep-th/9412167]

Problems

- Frame-Dependance of Minimum length
  - Length is **not** a Lorentz invariant
  - Generalized Uncertainty Principle adopt this form

$$[x_i, p_j] = i\hbar\delta_{ij}(1 + \beta_1\vec{p}^2) + i\hbar\beta_2 p_i p_j$$

  where $i, j \in \{1, 2, 3\}$.

- Composition Law Problem
  - Modified dispersion relation

$$E^2 = p^2c^2 + m^2c^4 \;\rightarrow\; E^2 = p^2c^2 + m^2c^4 + \mathcal{O}(p^4)$$

  - It can be shown that for a composite system, moving at the same speed

$$\vec{P}_3 \neq \vec{P}_2 + \vec{P}_1 \qquad E_3 \neq E_1 + E_3$$

Relativistic Generalized Uncertainty Principle (RGUP)

- Modification of the Heisenberg unceratinty relation
  $\Rightarrow$ Position and Momentum not canonically conjugate
- Defining auxiliary canonically conjugate variables

$$p_0^\mu = -i\frac{\partial}{\partial x_{0\,\mu}}, \quad [x_0^\mu, p_0^\nu] = i\eta^{\mu\nu}$$

- Expression of the position and momentum operators

$$x^\mu = x_0^\mu - \alpha\gamma p_0^\rho p_{0\rho} x_0^\mu + \underbrace{\beta\gamma p_0^\mu p_0^\rho x_{0\rho} + \xi\hbar\gamma p_0^\mu}_{\textit{breaks isotropy}},$$

$$p^\mu = p_0^\mu \left(1 + \varepsilon\gamma p_0^\rho p_{0\rho}\right)$$

- Where $\gamma = \frac{1}{(M_{\text{Pl}}c)^2}$ and $\alpha, \beta, \varepsilon$ and ξ are numerical coefficients used to fix the model

Relativistic Generalized Uncertainty Principle (RGUP)

- Position-Momentum commutator

$$[x^\mu, p^\nu] = i\hbar\,(1 + (\varepsilon - \alpha)\gamma p^\rho p_\rho)\,\eta^{\mu\nu} + i\,\hbar(\beta + 2\varepsilon)\gamma p^\mu p^\nu$$ [5]

- Non-commutative spacetime

$$[x^\mu, x^\nu] = i\hbar\gamma \frac{-2\alpha + \beta}{1 + (\varepsilon - \alpha)\gamma p^\rho p_\rho}\,(x^\mu p^\nu - x^\nu p^\mu)$$

5. C. Quesne and V. M. Tkachuk, Czech. J. Phys. **56**, 1269 (2006) [quant-ph/0612093].

Poincaré group

- Lorentz generators

$$M^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu = [1 + (\varepsilon - \alpha)\gamma p_0^\rho p_{0\,\rho}]\,(x_0^\mu p_0^\nu - x_0^\nu p_0^\mu)$$

- Poincaré algebra

$$\begin{aligned}
[x^\mu, M^{\nu\rho}] &= i\hbar[1 + (\varepsilon - \alpha)\gamma p^\alpha p_\alpha]\,(x^\nu \delta^{\mu\rho} - x^\rho \delta^{\mu\nu}) + i\hbar 2(\varepsilon - \alpha)\gamma p^\mu M^{\nu\rho} \\
[p^\mu, M^{\nu\rho}] &= i\hbar[1 + (\varepsilon - \alpha)\gamma p^\alpha p_\alpha]\,(p^\nu \delta^{\mu\rho} - p^\rho \delta^{\mu\nu}) \\
[M^{\mu\nu}, M^{\rho\sigma}] &= i\hbar\,(1 + (\varepsilon - \alpha)\gamma p^\alpha p_\alpha)\,(\eta^{\mu\rho} M^{\nu\sigma} \\
&\quad - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho})
\end{aligned}$$

- Special line in parameter space, which preserves the Poincaré algebra

$$\alpha = \varepsilon$$

- Position and momentum operators and their commutation relations

$$[x^\mu, p^\nu] = i\hbar\,(\eta^{\mu\nu} + 2\alpha\gamma p^\mu p^\nu) \qquad \text{RGUP!}$$
$$[x^\mu, x^\nu] = -2i\hbar\alpha\gamma\,(x^\mu p^\nu - x^\nu p^\mu) \quad \text{NC spacetime}$$

Frame dependence and Composition Law Problems

- Form of the RGUP commutator is the same in every frame

$$U[x^\mu, p^\nu]U^{-1} = U\left\{i\hbar\left(1+(\varepsilon-\alpha)\gamma p^\rho p_\rho\right)\eta^{\mu\nu} + i\,\hbar(\beta+2\varepsilon)\gamma p^\mu p^\nu\right\}U^{-1}$$
$$[x'^\mu, p'^\nu] = i\hbar\left(1+(\varepsilon-\alpha)\gamma p'^\rho p'_\rho\right)\eta^{\mu\nu} + i\,\hbar(\beta+2\varepsilon)\gamma p'^\mu p'^\nu\,.$$

- The Casimir invariants and therefore the Dispersion relation of the Modified Poincaré group are quadratic in the physical moentum

$$p^\mu p_\mu \text{ and } W^\mu W_\mu$$

Fixing the model

- Expression of the position and momentum operators

$$x^\mu = x_0^\mu,$$
$$p^\mu = p_0^\mu \left(1 + \gamma p_0^\rho p_{0\rho}\right)$$

- We find the following expression for Relativistic GUP $\gamma = \gamma_0/(M_{Pl}c)^2$

$$[x^\mu, p^\nu] = i\, \left(1 + \gamma p^\rho p_\rho\right) \eta^{\mu\nu} + i\, 2\gamma p^\mu p^\nu$$

Particular Poincaré group

- Lorentz generators

$$M^{\mu\nu} = p^\mu x^\nu - p^\nu x^\mu = [1 + \gamma p_0^\rho p_{0\,\rho}]\,(x_0^\mu p_0^\nu - x_0^\nu p_0^\mu)$$

- Poincaré algebra

$$[x^\mu, M^{\nu\rho}] = i[1 + \gamma p^\alpha p_\alpha]\,(x^\nu \delta^{\mu\rho} - x^\rho \delta^{\mu\nu}) + i2\gamma p^\mu M^{\nu\rho}$$
$$[p^\mu, M^{\nu\rho}] = i[1 + \gamma p^\alpha p_\alpha]\,(p^\nu \delta^{\mu\rho} - p^\rho \delta^{\mu\nu})$$
$$[M^{\mu\nu}, M^{\rho\sigma}] = i\,(1 + \gamma p^\alpha p_\alpha)\,(\eta^{\mu\rho} M^{\nu\sigma} - \eta^{\mu\sigma} M^{\nu\rho} - \eta^{\nu\rho} M^{\mu\sigma} + \eta^{\nu\sigma} M^{\mu\rho})$$

- Casimir operator and Dispersion relation

$E^2 = (pc)^2 + (mc^2)^2 \Rightarrow$ $p^\rho p_\rho = -m^2 \Longrightarrow p_0^\rho p_{0\rho}(1 + 2\gamma p_0^\sigma p_{0\sigma}) = -m^2$

- Fourth order differential equation using $p_0^2 = -\partial^\rho \partial_\rho = \Box_0$

$$\Box_0(1 + 2\gamma\Box_0) = -m^2 \quad \text{where} \quad \Box_0 = \frac{\partial}{\partial x_0^\mu}\frac{\partial}{\partial x_{0\mu}}$$

Recovering the Lagrangian for scalar and spinor fields

- We use the Ostrogradsky method for recovering the Euler-Lagrange equations for high derivative theories

$$\frac{\partial \mathcal{L}}{\partial \phi} - \partial_{\mu_1} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1}\phi)} + \ldots + (-1)^m \partial_{\mu_1} \ldots \partial_{\mu_m} \frac{\partial \mathcal{L}}{\partial(\partial_{\mu_1} \ldots \partial_{\mu_m}\phi)} = 0$$

- The Equations of Motion for the Lagrangians we are interested in are

$$\underbrace{\Box_0(1 + 2\gamma\Box_0) = -m^2}_{\text{RGUP modified Klein-Gordon equation}}$$

$$\underbrace{[i\tau^\mu \partial_\mu(1 + \gamma\partial_\rho\partial^\rho) - m]\,\psi = 0}_{\text{RGUP modified Dirac equation}}$$

Recovered Lagrangian

- Scalar field Lagrangian

$$\mathcal{L}_{\phi,\mathbb{C}} = \frac{1}{2}\left(\partial_\mu\phi\right)^\dagger \partial^\mu\phi - \frac{1}{2}m^2\phi^\dagger\phi + \underbrace{\gamma\left[\left(\partial_\nu\partial^\nu\partial^\mu\phi\right)^\dagger \partial_\mu\phi + \partial_\nu\partial^\nu\partial^\mu\phi\left(\partial_\mu\phi\right)^\dagger\right]}_{RGUPcorrections}$$

- Dirac field Lagrangian

$$\mathcal{L}_\psi = \bar{\psi}\left[i\tau^\mu\partial_\mu\underbrace{\left(1+\gamma\,\partial_\rho\partial^\rho\right)}_{RGUPcorrections} - m\right]\psi$$

- Gauge field Lagrangian

$$\mathcal{L}_A = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu} = -\frac{1}{4}F_0^{\mu\nu}F_{\mu\nu 0} - \underbrace{\frac{\gamma}{2}F_{\mu\nu 0}\partial_\rho\partial^\rho F_0^{\mu\nu}}_{RGUPcorrections}$$

Feynman rules : Propagators

- Scalar field propagator

$$G(x-x') = \int \frac{d^4 p_0}{(2\pi)^4} \frac{-i}{-p_0^2 \underbrace{(1+\gamma p_0^2)^2}_{\text{RGUP}} + m^2} e^{-ip_0\cdot(x-x')} .$$

- Dirac propagator

$$G(x-x') = \int \frac{d^4 p_0}{(2\pi)^4} \frac{-i\left(\tau^\mu p_{0\,\mu}(1-\gamma p_{0\rho} p_0^\rho) + m\right)}{\left(p_0^\mu p_{0\,\mu}(1-\gamma p_{0\rho} p_0^\rho)^2 - m^2\right)} e^{-ip_0\cdot(x-x')} .$$

- Gauge field propagator

$$G(x-x') = \int \frac{d^4 q_0}{(2\pi)^4} \frac{-i}{-q_0^2 + 2\gamma q_0^4} e^{-iq_0\cdot(x-x')} ,$$

Feynman rules : Vertices

- Scalar field : Vertices involving two scalar and up to four gauge bosons

	Powers of γ			
Powers of α	$\alpha^{1/2}$	$\alpha^{1/2}\gamma$	$\alpha^{1/2}\gamma^2$	$\alpha^{1/2}\gamma^3$
	α	$\alpha\gamma$	$\alpha\gamma^2$	$\alpha\gamma^3$
	N/A	$\alpha^{3/2}\gamma$	$\alpha^{3/2}\gamma^2$	$\alpha^{3/2}\gamma^3$
	N/A	$\alpha^2\gamma$	$\alpha^2\gamma^2$	$\alpha^2\gamma^3$
	N/A	$\alpha^{5/2}\gamma$	$\alpha^{5/2}\gamma^2$	$\alpha^{5/2}\gamma^3$
	N/A	$\alpha^3\gamma$	$\alpha^3\gamma^2$	$\alpha^3\gamma^3$

- Dirac field : Vertices involving two fremion and up to three gauge bosons

	Powers of γ	
Powers of α	$\alpha^{1/2}$	$\alpha^{1/2}\gamma$
	N/A	$\alpha\gamma$
	N/A	$\alpha^{3/2}\gamma$

Differential cross sections

- Scalar QED electron-muon scattering

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{1}{4\,s}\alpha^2 \left(\frac{3+\cos\theta}{1-\cos\theta}\right)^2 \left[1 + 16\gamma E^2(1-\cos\theta)\right]$$

- QED electron-muon scattering

$$\left.\frac{d\sigma}{d\Omega}\right|_{CM} = \frac{\alpha^2}{4\,s}\left[\frac{1}{2}\left(1+\cos^2\theta\right) + \frac{1}{4}\gamma(m_e^2 + m_{\text{muon}}^2)\left(\cos\theta + \cos^2\theta\right)\right].$$

$$\sigma\left(e^+e^- \to \mu^+\mu^-\right) = \frac{4\pi\alpha^2}{3\,s}\left[1 + \frac{\gamma}{8}(m_e^2 + m_{\text{muon}}^2)\right].$$

- Comparing the corrected amplitude for Xe-Xe scattering measured by ATLAS in CERN [6] we can put bouhnds on the paremeters

$$\gamma_0 \lesssim 10^{32}$$

6. P. Balek. Nucl. Phys. A, 982:571574, (2019)

Summary

- We have formulated RGUP leading to frame independent minimum length and bypasses the composition law problem
- We formulated and Effective quantum field theory with minimum length
- We have calculated the QG corrected cross sections for electron-muon scattering and found correlation between the magnitude of the corrections and the mass of the participating particles.
- Applied to existing Quantum experiments give prediction for the scale of Quantum Gravity.

Thank you for your attention !