Multipole Moments in Asymptotically de Sitter Spacetime

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Outline

- Noether charge formalism for multipole moments.
- de Sitter spacetime in harmonic gauge.
- Residual gauge transformation and multipole symmetry vector field.
- Perturbation of de Sitter in cosmological coordinates.
- Multipole moments of Kerr-de Sitter spacetime.

References

- **SC**, Hoque and Oliveri, arXiv: 2105.09971
- Mukherjee and **SC**, PRD 102, 124058 (2020).

Multipole Moment in Asymptotically Flat Spacetimes

- **Geroch-Hansen Formalism**: This requires stationary situation, along with asymptotically flat 3-metric. [Backdahl and Herberthson, arXiv: gr-qc/0506086]

  [Hansen, JMP 15, 46 (1974)]

  [Geroch, JMP 11, 2580 (1970)]

- **Thorne formalism**: Expansion of metric coefficients in powers of (1/r) in the ACMC coordinate system. [Thorne, RMP 52, 299 (1980)]

- **Noether charge formalism**: Noether charges associated with vector fields preserving the harmonic gauge, connected with background asymptotically flat spacetime. [Compere, Oliveri and Seraj, arXiv: 1711.08806]

- Kerr mass and spin moments:

$$M_{2\ell} = (-1)^{\ell} M a^{2\ell} \ ; \qquad S_{2\ell+1} = (-1)^{\ell} M a^{2\ell+1} \ .$$

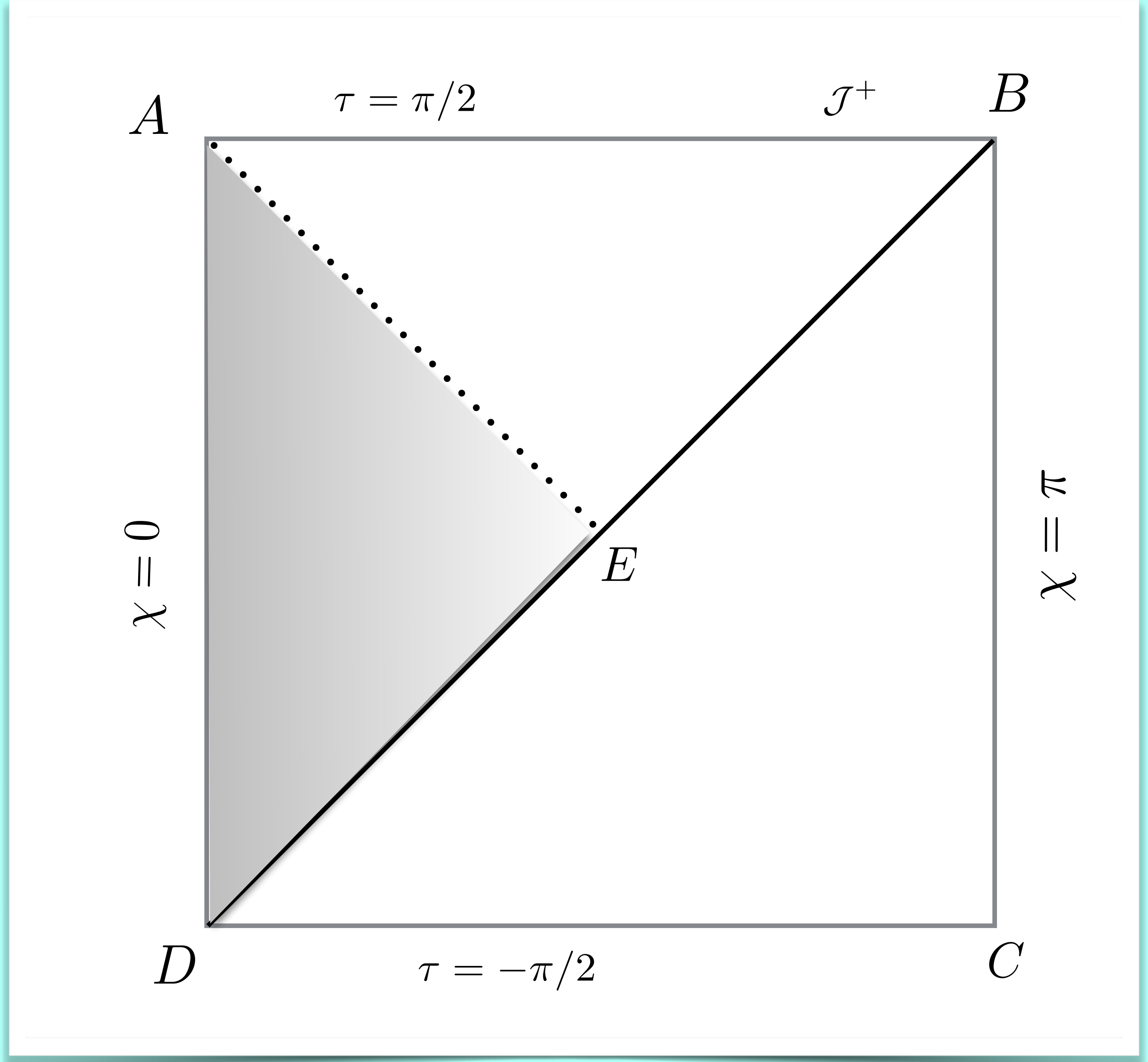
Multipole Symmetry and Noether Charge

- **Harmonic gauge**: The metric must satisfy $\partial_\mu(\sqrt{-g}g^{\mu\nu})=0$. For flat spacetimes, $\eta_{\mu\nu}$ satisfy harmonic gauge in Cartesian coordinates.
- **Residual gauge symmetry**: Diffeomorphism by ξ^μ will preserve the harmonic gauge, provided, $\Box_g \xi^\mu = 0$.
- **Multipole symmetry vector**: If this vector field ξ^μ also preserve the asymptotic fall-off condition $g_{0\mu}=\eta_{0\mu}+\mathcal{O}(1/r)$, then it is referred to as the multipole symmetry vector.
- **Multipole moments**: The Noether charge associated with the above vector field and a linearized solution $h_{\mu\nu}$ around background, will produce the multipole moments.

[Compere, Oliveri and Seraj, arXiv: 1711.08806]

Coordinate Charts of de Sitter

- **Global patch**: The full box ABCD described by $(\tau, \chi, \theta, \phi)$ as the coordinates.
- **Cosmological patch**: The triangle ABD, described by coordinates (t, r, θ, ϕ) or by the coordinates (η, r, θ, ϕ).
- **Static patch**: The triangle AED, described by the coordinates (T, R, θ, ϕ).



[Date and Hoque, arXiv: 1510.07856]

de Sitter in Harmonic Gauge

- We have to find out a new coordinate $\bar{x}^\mu = f^\mu(x^\alpha)$, which satisfies the following equation $\square_{\mathrm{dS}} f^\mu = 0$.
- In the cosmological coordinates, we took $\bar{x} = x, \bar{y} = y, \bar{z} = z$, while the time coordinate changes $\bar{t} = f(t)$.
- The function $f(t)$ can be solved, by using the above differential equation.

$$f(t) = \frac{1}{3H}\left(1 - e^{-3Ht}\right)$$

- de Sitter metric becomes,

[SC, Hoque and Oliveri, arXiv: 2105.09971]

$$\begin{aligned} ds^2 &= -dt^2 + e^{2Ht}(dx^2 + dy^2 + dz^2) \\ &= -\frac{d\bar{t}^2}{(1 - 3H\bar{t})^2} + (1 - 3H\bar{t})^{-2/3}(dx^2 + dy^2 + dz^2) \end{aligned}$$

Residual Gauge Transformation

- The asymptotic fall-off condition must be preserved, $\pounds_{\xi} g^{\mathrm{dS}}_{0\mu} = 0$.
- This vector field ξ^{μ} in the harmonic coordinate can be solved.

$$\xi^0 = (1 - 3H\bar{t})\,\epsilon(\mathbf{x})$$

$$\xi^i = \frac{1}{2H}\left[1 - (1 - 3H\bar{t})^{2/3}\right]\delta^{ij}\partial_j\epsilon(\mathbf{x}) + \zeta^{\mathbf{i}}(\mathbf{x})$$

- If this vector field has to satisfy residual gauge transformation, it follows that

[SC, Hoque and Oliveri, arXiv: 2105.09971]

$$\delta^{ij}\partial_i\partial_j\epsilon(\mathbf{x}) = 0$$

$$\delta^{kl}\partial_k\partial_l\zeta^i(\mathbf{x}) - H\delta^{ij}\partial_j\epsilon(\mathbf{x}) = \mathbf{0}$$

Multipole Symmetry Vectors

- The complete multipole symmetry vector is [SC, Hoque and Oliveri, arXiv: 2105.09971]

$$\xi^{\mu} = \epsilon(\mathbf{x})(\partial_t)^{\mu} + \left[\frac{1}{2H}\left(1 - e^{-2Ht}\right)\delta^{ij}\partial_j\epsilon(\mathbf{x}) + \zeta^i(\mathbf{x})\right](\partial_i)^{\mu}$$

- It can be decomposed into three parts,

$$K_{\epsilon} = \epsilon(\mathbf{x})(\partial_t) + \frac{1}{2H}(1 - e^{-2Ht})\nabla\epsilon(\mathbf{x}) - Hx^i\partial_i$$

(**Mass Multipole Symmetry**)

$$L_{\epsilon} = -\mathbf{r} \times \nabla\epsilon(\mathbf{x})$$

(**Spin Multipole Symmetry**)

$$P_{\epsilon} = \nabla\epsilon(\mathbf{x})$$

(**Momentum Multipole Symmetry**)

Linearized Perturbation of dS

- The Einstein's equations when perturbed about dS background, linearly, the following wave equation is obtained,

$$\Box_{\rm dS}\widetilde{h}_{\mu\nu} - \left[2\nabla^{\rm dS}_{(\mu}B_{\nu)} - g^{\rm dS}_{\mu\nu}\left(\nabla^{\rm dS}_{\alpha}B^{\alpha}\right)\right] - \frac{2\Lambda}{3}\left(\widetilde{h}_{\mu\nu} - \widetilde{h}g^{\rm dS}_{\mu\nu}\right) = -16\pi T_{\mu\nu}$$

- Here, $\widetilde{h}_{\mu\nu} = h_{\mu\nu} - (1/2)hg^{\rm dS}_{\mu\nu}$ and $B_{\mu} = \nabla^{\rm dS}_{\alpha}\widetilde{h}^{\alpha}_{\mu}$.
- Linearized equation simplifies, with gauge condition $\nabla^{\rm dS}_{\alpha}\widetilde{h}^{\alpha}_{\mu} = -2H\widetilde{h}_{0\mu}$
- Then the spatial part and the temporal-spatial part of the perturbation is decoupled. Defining, $\widetilde{\mathcal{H}} = \widetilde{h}_{00} + e^{-2Ht}\left(\delta^{ij}\widetilde{h}_{ij}\right)$, the temporal part is also decoupled.

[Ashtekar, Bonga and Kesavan, arXiv: 1510.05593]
[SC, Hoque and Oliveri, arXiv: 2105.09971]

Emergence of Multipole Symmetry

- We also want to preserve the gauge condition used to simplify the linearized Einstein's equations, under diffeomorphism.
- In addition it can be used to eliminate $\widetilde{\mathcal{H}}$ and $\widetilde{h}_{0i}$. This provides the following differential equations for the time and spatial components of the diffeomorphism vector field, [SC, Hoque and Oliveri, arXiv: 2105.09971]

$$\partial_0\xi_0 = 0$$

$$\partial_0\xi_i + \partial_i\xi_0 - 2H\xi_i = 0$$

- The solution leads to the same vector field we have derived in the context of harmonic gauge. Provides an alternative derivation of the multipole symmetry vectors.

Spin Moments of Kerr dS

- The vector field generating spin multipole moments can be decomposed into spherical harmonics

$$L_{\ell m} = N_\ell \; r^{\ell-1} \left({}^{\mathrm{B}}Y^{\theta}_{\ell m} \partial_\theta + \frac{1}{\sin\theta} \; {}^{\mathrm{B}}Y^{\phi}_{\ell m} \partial_\phi \right)$$

- All even spin moments vanish, while odd spin moments are non-zero.
- The first few spin moments become

$$S_1 = \frac{Ma}{(1+a^2H^2)^2}; \qquad S_3 = -Ma^3 \left[1 - \frac{28}{15} a^2 H^2 + \mathcal{O}(H^4) \right];$$

$$S_5 = Ma^5 \left[1 - \frac{118}{63} a^2 H^2 + \mathcal{O}(H^4) \right].$$

Mass Moments of Kerr dS

- The spherical harmonic decomposition of the mass multipole symmetry vector

$$K_{\ell m} = K_\ell \; r^\ell \left(Y_{\ell m}\partial_t + \frac{1}{r}\chi^r_{\ell m}\partial_r + \frac{1}{r^2}\chi^\theta_{\ell m}\partial_\theta + \frac{1}{r^2\sin\theta}\chi^\phi_{\ell m}\partial_\phi \right) - H\delta_{1\ell}\partial_r$$

- Here $\vec{\chi}_{\ell m} = \frac{1}{2H}\left(1 - e^{-2Ht}\right)\left(\sqrt{\ell(\ell+1)}\; {}^{\mathrm{E}}\vec{Y}_{\ell m} + \ell\; {}^{\mathrm{R}}\vec{Y}_{\ell m}\right)$.

- Only even mass moments are non-zero, while all odd mass moments identically vanishes. The first few mass moments become

$$M_0 = \frac{M}{1 + a^2H^2}\,; \qquad M_2 = -\frac{Ma^2}{1 + a^2H^2}\,;$$

$$M_4 = Ma^4\left[1 + \frac{4}{5}Ht + \mathcal{O}(H^2)\right].$$

Conclusion-I

- de Sitter spacetime has been expressed in harmonic coordinates.
- The residual gauge symmetry helped to determine the multipole symmetry vector.
- Identical vector appears from preserving gauge conditions of linearized perturbations around de Sitter.

Conclusion-II

- The Noether charge associated with these multipole symmetry vector for Kerr-de Sitter spacetime provides the mass and spin multipole moments.
- These differ from the Geroch-Hansen moments by factors $\mathcal{O}(Ht)$ and $\mathcal{O}(Ha)$.
- Reduces to the correct expressions for Kerr BH spacetime in the limit $H \to 0$.

Thank You