Chiral vortical effect for free fermions on anti-de Sitter space

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Outline

Motivation

Vortical effects relevant for:

- Rotating black holes (frame dragging)[1]
- Neutron stars (anomalous transport)[2]
- QGP (anomalous transport and spin polarisation)[3]

Why AdS?

- Relevant to QGP AdS/CFT duality[4,5]
- AdS has timelike boundary $\rightarrow$ no SLS for "mild" Ω.[6]

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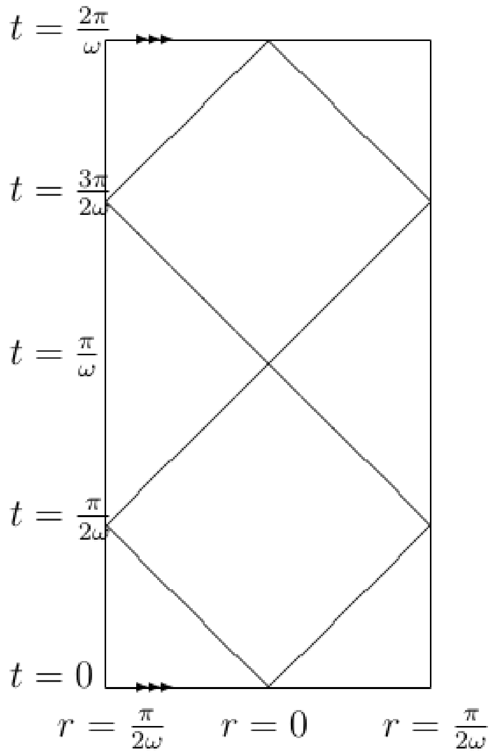
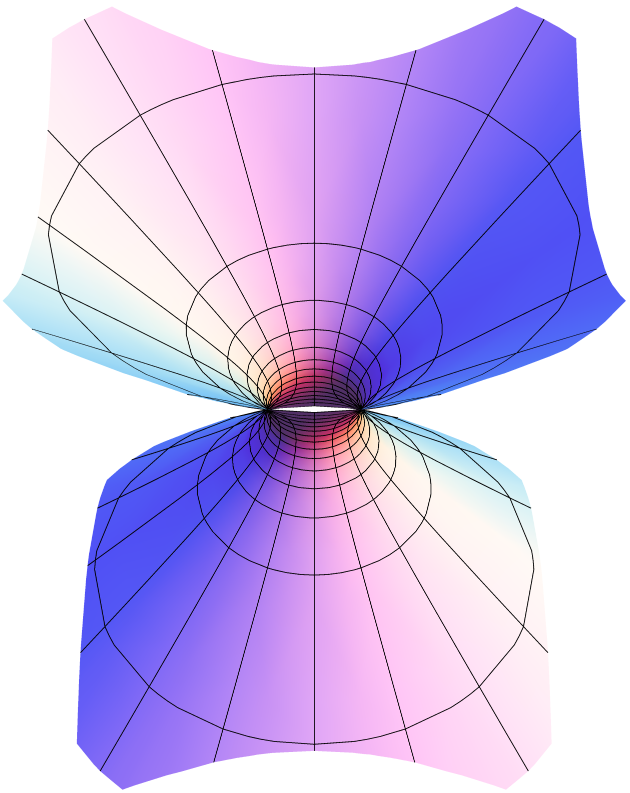
[1] M. Casals *et al.*, Phys. Rev. D **87** (2013) 064027.
[2] M. Kaminski *et al.*, Phys. Lett. B **760** (2016) 170–174.
[3] STAR Collaboration, Nature **548** (2017) 62–65.
[4] O. Aharony, S.S. Gubser, J.M. Maldacena, H. Ooguri, Y. Oz, Phys. Rep. **323** (2000) 183.
[5] D. T. Son, A. O. Starinets, JHEP**03** (2006) 052.
[6] R. Panerai, Phys. Rev. D **93** (2016) 104021.

Anti-de Sitter space

- Maximal symmetry: $R_{\mu\nu} = \frac{1}{4} R g_{\mu\nu}$, $R = 4\Lambda = -12\ell^{-2}$.





- Boundary at $r = \pi\ell/2$ reached in a finite time only by null geodesics.
- Closed timelike loops $\Rightarrow$ CAdS ($-\infty < t < \infty$).
- Line element:

$$ds^2 = \frac{\ell^2}{\cos^2 \overline{r}} \left[-d\overline{t}^2 + d\overline{r}^2 + \sin^2 \overline{r} \left(d\theta^2 + \sin^2 \theta d\varphi^2 \right) \right],$$

where $\overline{t} = t/\ell$, $\overline{r} = r/\ell$.

Rigid rotation

- Space-times with spherical symmetries can be parametrised using central charts:
$$ds^2 = W^2\left[-dt^2 + \frac{dr^2}{U^2} + \frac{r^2}{V^2}(d\theta^2 + \sin^2\theta d\varphi^2)\right]. \tag{1}$$
- $W = U = V = 1$ for Mink, while on adS,
$$W = \frac{1}{\cos\overline{r}}, \qquad U = 1, \qquad V = \frac{r}{\sin\overline{r}}. \tag{2}$$
- The relativistic Boltzmann equation, $k^\mu \partial_\mu f = C[f]$, is satisfied when
$$f = f^{\text{(eq)}}(-\beta\cdot k + \alpha), \qquad \nabla_\mu \beta_\nu + \nabla_\nu \beta_\mu = 0, \qquad \nabla_\mu \alpha = 0, \tag{3}$$
where $\beta^\mu = T^{-1} u^\mu$ and $\alpha = \mu/T$ ($= 0$ in this talk).
- Rigid rotation $\beta = \beta_0(\partial_t + \Omega\partial_\varphi)$ is a solution of the Killing equation, with
$$u = \frac{\Gamma}{W(r)}(\partial_t + \Omega\partial_\varphi), \qquad \begin{pmatrix} T \\ \mu \end{pmatrix} = \Gamma \begin{pmatrix} T_0 \\ \mu_0 \end{pmatrix}, \qquad \Gamma = (1-\rho^2\Omega^2)^{-1/2}, \tag{4}$$
where $\rho = r\sin\theta/V$ is $r\sin\theta$ on Minkowski and $\ell\sin\overline{r}\sin\theta$ on adS.

Kinematic tetrad[6]

Velocity :

$$u = \Gamma(\partial_t + \Omega\partial_\varphi),$$

Acceleration :

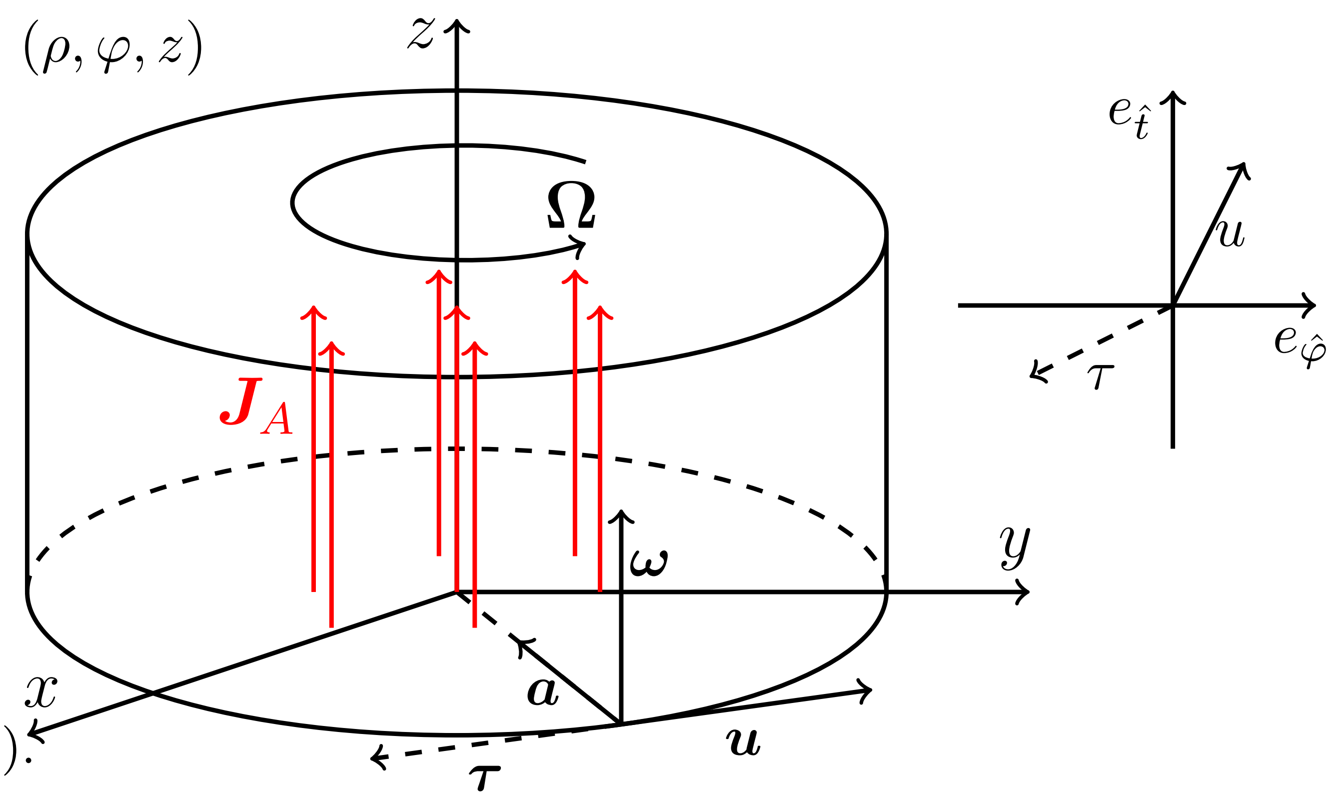
$$a = \nabla_u u = -\rho\Omega^2\Gamma^2\partial_\rho,$$

Vorticity :

$$\omega = \tfrac{1}{2}\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\sigma} e_{\hat\alpha} u_{\hat\beta} (\nabla_{\hat\gamma} u_{\hat\sigma}) = \Gamma^2\Omega\partial_z,$$

Fourth vector :

$$\tau = -\varepsilon^{\hat\alpha\hat\beta\hat\gamma\hat\sigma} e_{\hat\alpha}\omega_{\hat\beta} a_{\hat\gamma} u_{\hat\sigma} = -\Omega^3\Gamma^5(\rho^2\Omega\partial_t + \partial_\varphi).$$



Axial vortical effect:

$$J_A^\mu = \sigma_A^\omega \omega^\mu, \qquad \sigma_A^\omega = \frac{T^2}{6} + \frac{1}{24\pi^2}\left(\boldsymbol{\omega}^2 + 3\boldsymbol{a}^2 - 6M^2\right) + O(M^4). \tag{5}$$

[7] F. Becattini, E. Grossi, Phys. Rev. D **92** (2015) 045037.

Kinematic tetrad for adS.

$$\text{Velocity}: u = \Gamma(e_{\hat{t}} + \rho\Omega e_{\hat{\varphi}}),$$

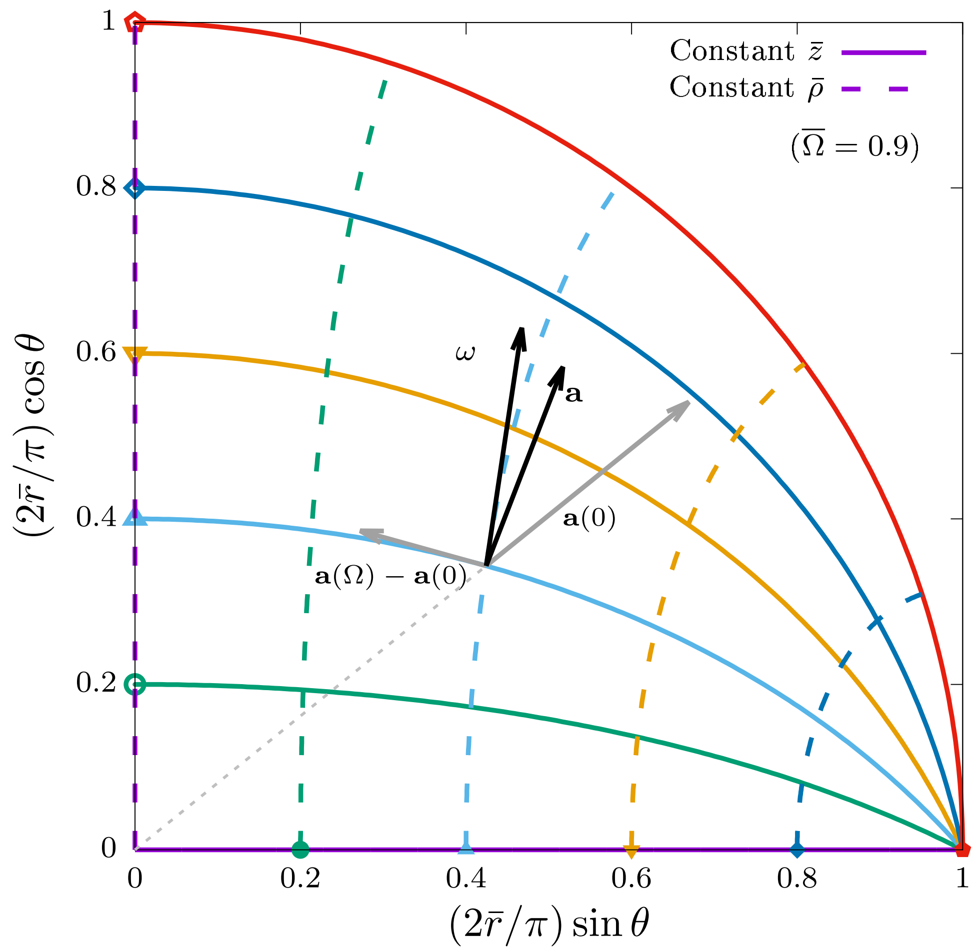
$$\text{Acceleration}: a = a(0) + [a(\Omega) - a(0)],$$

$$a(0) = \frac{\sin\overline{r}\cos\overline{r}}{\ell^2}\partial_{\overline{r}},$$

$$a(\Omega) - a(0) = -\rho\Omega^2\Gamma^2\frac{1-\overline{\rho}^2}{\sec^2\overline{r}}\partial_\rho,$$

$$\text{Vorticity}: \omega = \Omega\Gamma^2(1-\overline{\rho}^2)\partial_z,$$

$$\text{Fourth vector}: \tau = -\rho\Omega(1-\overline{\Omega}^2)\Gamma^5\cos^3\overline{r} \times (\rho\Omega\partial_t + \rho^{-1}\partial_\varphi).$$



Axial vortical effect correction:

$$J^\mu_A = \sigma^\omega_A \omega^\mu, \qquad \sigma^\omega_A = \frac{T^2}{6} + \frac{1}{24\pi^2}\left(\boldsymbol{\omega}^2 + 3\boldsymbol{a}^2 - 6M^2 + \frac{R}{4}\right) + O(M^4). \tag{6}$$

Geometric approach: point-splitting

- The t.e.v. of an operator $\widehat{A}$ is

$$\langle \widehat{A} \rangle_{\beta_0,\Omega} = Z^{-1} \mathrm{tr}(\hat{\varrho}\widehat{A}), \qquad \hat{\varrho} = e^{-\beta_0(\widehat{H} - \Omega \widehat{M}^z)}, \qquad Z = \mathrm{tr}(\hat{\varrho}). \tag{7}$$

- T.e.v.s can be obtained via point-splitting, e.g.

$$\langle \widehat{J}^\mu_A \rangle_{\beta_0,\Omega} = \lim_{x' \to x} \mathrm{tr}[\gamma^\mu \gamma^5 S^F_{\beta_0,\Omega}(x,x') \Lambda(x',x)], \tag{8}$$

where $\Lambda(x,x')$ is the bispinor of parallel transport (TBD).

- The Feynman propagator can be obtained from the Wightman functions,

$$\begin{aligned}
S^F_{\beta_0,\Omega}(x,x') =& \Theta_c(\tau - \tau') S^+_{\beta_0,\Omega}(x,x') + \Theta_c(\tau' - \tau) S^-_{\beta_0,\Omega}(x,x'), \\
i S^+_{\beta_0,\Omega}(x,x') =& \langle \widehat{\Psi}(x) \widehat{\overline{\Psi}}(x') \rangle_{\beta_0,\Omega}, \\
i S^-_{\beta_0,\Omega}(x,x') =& - \langle \widehat{\overline{\Psi}}(x') \widehat{\Psi}(x) \rangle_{\beta_0,\Omega},
\end{aligned} \tag{9}$$

where $\Theta_c(\tau - \tau')$ satisfies ($\varepsilon > 0$):[8]

$$\Theta_c(t - t' - i\varepsilon) = 1, \qquad \Theta_c(t - t' + i\varepsilon) = 0. \tag{10}$$

[8] S. Mallik, S. Sarkar, *Hadrons at finite temperature* (CUP, 2016).

Wightman functions and KMS relation

- Taking into account $\hat{\rho} = e^{-\beta_0(\widehat{H} - \Omega \widehat{M}^z)}$, $\widehat{\Psi}$ obeys

$$\hat{\rho}\widehat{\Psi}(t,\varphi)\hat{\rho}^{-1} = e^{-\beta_0 \Omega S^z} \widehat{\Psi}(t + i\beta_0, \varphi + i\beta_0\Omega). \tag{11}$$

- The KMS relation is readily derived,

$$\begin{aligned} S^-_{\beta_0,\Omega}(\tau,\varphi;x') =& iZ^{-1}\mathrm{tr}[\hat{\rho}\widehat{\overline{\Psi}}(\tau',\varphi')\widehat{\Psi}(\tau,\varphi)] \\ =& iZ^{-1}\mathrm{tr}[\hat{\rho}e^{\beta_0\Omega S^z}\widehat{\Psi}(\tau - i\beta_0, \varphi - i\beta_0\Omega)\widehat{\overline{\Psi}}(\tau',\varphi')] \\ =& -e^{\beta_0\Omega S^z} S^+_{\beta_0,\Omega}(\tau - i\beta_0, \varphi - i\beta_0\Omega; x'). \end{aligned} \tag{12}$$

- The KMS relation above allows $S^F_{\beta_0,\Omega}$ to be written in terms of $S^F_{\infty,\Omega}$:

$$\boxed{S^F_{\beta_0,\Omega}(x,x') = \sum_j (-1)^j e^{-j\beta_0\Omega S^z} S^F_{\infty,\Omega}(t + ij\beta_0, \varphi + ij\beta_0\Omega; x').} \tag{13}$$

- In bounded systems, $S^F_{\infty,\Omega} = S^F_{\infty,0}$ if no SLS forms inside the boundary.[9]
- A similar argument holds on adS, if $\Omega < \ell^{-1}$ (subcritical rotation).

[9] V. E. Ambruș, E. Winstanley, Phys. Rev. D **93** (2016) 104014.

Vacuum propagator

- For the maximally symmetric vacuum state, S_F can be written as:[10]

$$iS^F_{\rm vac}(x,x') = [\mathcal{A}(s) + \mathcal{B}(s)\not{n}]\Lambda(x,x').$$

- The geodesic interval $s = \ell\overline{s}$ can be given through:

$$\cos\overline{s} = \frac{\cos\Delta\overline{t}}{\cos\overline{r}\cos\omega\overline{r}'} - \cos\gamma\tan\overline{r}\tan\overline{r}',$$

where $\cos\gamma = \cos\theta\cos\theta' + \sin\theta\sin\theta'\cos\Delta\varphi$.

- $n_\mu = \nabla_\mu s(x,x')$ is the normalised tangent to the geodesic at x.
- $\mathcal{A}$ and $\mathcal{B}$ depend only on $\overline{s}$ and satisfy:

$$i\frac{d}{d\overline{s}}\begin{pmatrix}\mathcal{A}\\ \mathcal{B}\end{pmatrix} + \frac{3i}{2}\begin{pmatrix}-\mathcal{A}\tan(\overline{s}/2)\\ \mathcal{B}\cot(\overline{s}/2)\end{pmatrix} - \ell M\begin{pmatrix}\mathcal{B}\\ \mathcal{A}\end{pmatrix} = \begin{pmatrix}0\\ i(-g)^{-1/2}\delta(x,x')\end{pmatrix}.$$

- The equations can be solved exactly. When $M=0$, we have:

$$\mathcal{A}\rfloor_{M=0} = \frac{1}{16\pi^2\ell^3}\left(\cos\frac{\overline{s}}{2}\right)^{-3}, \qquad \mathcal{B}\rfloor_{M=0} = \frac{i}{16\pi^2\ell^3}\left(\sin\frac{\overline{s}}{2}\right)^{-3}.$$

[10] W. Mück, J. Phys. A **33** (2000) 2000 3021.

Bi-spinor of parallel transport

- The bi-spinor of parallel transport $\Lambda(x,x')$ satisfies:[10]

$$D_\mu \Lambda(x,x') = -i\omega S_{\mu\nu} n^\nu \Lambda(x,x') \tan\left(\frac{\overline{s}}{2}\right).$$

- Employing the Cartesian gauge for the tetrad:[11]

$$e_{\hat{t}} = \ell^{-1} \cos\overline{r}\, \partial_{\overline{t}}, \qquad e_{\hat{\imath}} = \cos\overline{r} \left[\frac{\overline{r}}{\sin\overline{r}} \left(\delta_{ij} - \frac{x^i x^j}{r^2}\right) + \frac{x^i x^j}{r^2}\right] \partial_j, \tag{14}$$

allows $\Lambda(x,x')$ to be expressed as:[12]

$$\Lambda(x,x') = \frac{\sec(\overline{s}/2)}{\sqrt{\cos\overline{r}\cos\overline{r}'}} \left[\cos\frac{\Delta\overline{t}}{2}\left(\cos\frac{\overline{r}}{2}\cos\frac{\overline{r}'}{2} + \sin\frac{\overline{r}}{2}\sin\frac{\overline{r}'}{2}\frac{\boldsymbol{x}\cdot\boldsymbol{\gamma}}{r}\frac{\boldsymbol{x}'\cdot\boldsymbol{\gamma}}{r'}\right)\right.$$
$$\left. + \sin\frac{\Delta\overline{t}}{2}\left(\sin\frac{\overline{r}}{2}\cos\frac{\overline{r}'}{2}\frac{\boldsymbol{x}\cdot\boldsymbol{\gamma}}{r}\gamma^{\hat{t}} + \sin\frac{\overline{r}'}{2}\cos\frac{\overline{r}}{2}\frac{\boldsymbol{x}'\cdot\boldsymbol{\gamma}}{r'}\gamma^{\hat{t}}\right)\right]. \tag{15}$$

[10] W. Mück, J. Phys. A **33** (2000) 2000 3021.
[11] I. I. Cotăescu, Rom. J. Phys. **52** (2007) 895.
[12] V. E. Ambruș, E. Winstanley, Class. Quant. Grav. **34** (2017) 145010.

Thermal expectation values

- Introducing the notation

$$\zeta_j = -\frac{1}{\sin^2 \frac{\overline{s}_j}{2}} = \frac{\cos^2 \overline{r}}{\sinh^2 \frac{j\beta_0}{2\ell} - \overline{\rho}^2 \sinh^2 \frac{\Omega j \beta_0}{2}}, \tag{16}$$

we find $J_A^{\hat{\alpha}} = \sigma_A^\omega \omega^{\hat{\alpha}}$, where $[\Gamma_k = \Gamma(2+k)\sqrt{\pi}/4^k \Gamma(\frac{1}{2}+k)]$

$$\begin{aligned}\sigma_A^\omega =& \frac{\Gamma_k}{2\pi^2 \ell^3 \Omega \Gamma^2 \cos^2 \overline{r}} \sum_{j=1}^\infty (-1)^{j+1} \zeta_j^{2+k} \sinh \frac{j\beta_0}{2\ell} \sinh \frac{\Omega j \beta_0}{2} \\ & \times {}_2F_1(k, 2+k; 1+2k; -\zeta_j) \\ =& \frac{T^2}{6} + \frac{1}{24\pi^2}\left(\boldsymbol{\omega}^2 + 3\boldsymbol{a}^2 - 6M^2 + \frac{R}{4}\right) + O(T^{-1}).\end{aligned} \tag{17}$$

- At high temperature, the Minkowski expression is reproduced, including the temperature-independent terms.
- When $T \to 0$, $\sigma_A^\omega \to 0$ and the $O(T^0)$ are also suppressed.
- The curvature correction appears in the $O(T^0)$ term and is proportional to $R = -12\ell^{-2}$.

Critical rotation: $\overline{\Omega} = 1$

- At critical rotation, $\Gamma = 1/\sqrt{1-\overline{\rho}^2}$ and

$$\zeta_j = \frac{\Gamma^2 \cos^2 \overline{r}}{\sinh^2 \frac{j\beta_0}{2\ell}}. \tag{18}$$

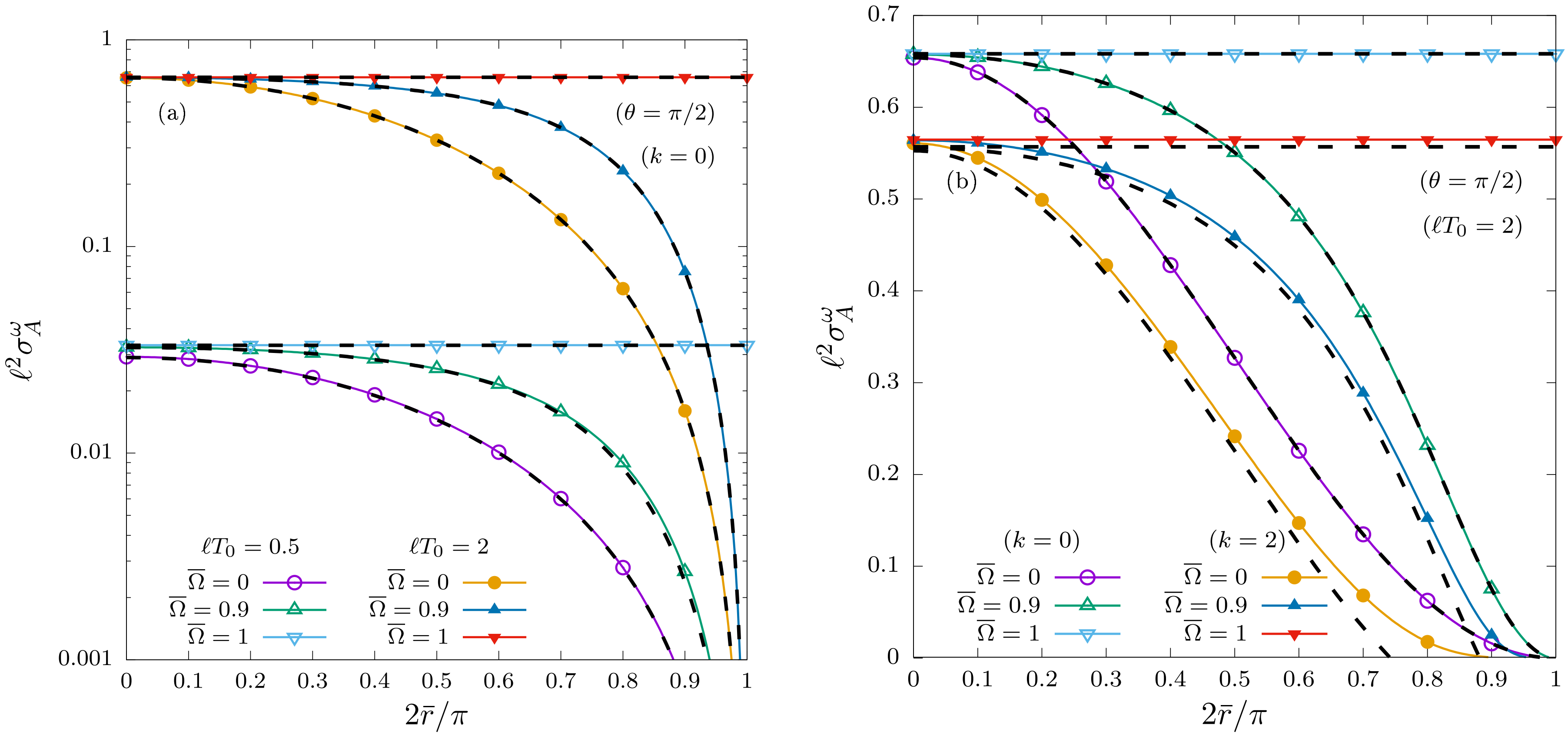
- In the equatorial plane ($\theta = \frac{\pi}{2}$),

$$\Gamma = \frac{1}{\cos \overline{r}} \qquad \Rightarrow \qquad T = T_0 \Gamma \cos \overline{r} = T_0, \qquad \zeta_j = \operatorname{cosech}^2 \frac{j\beta_0}{2\ell}. \tag{19}$$

- In particular, σ_A^ω is constant in the EP:

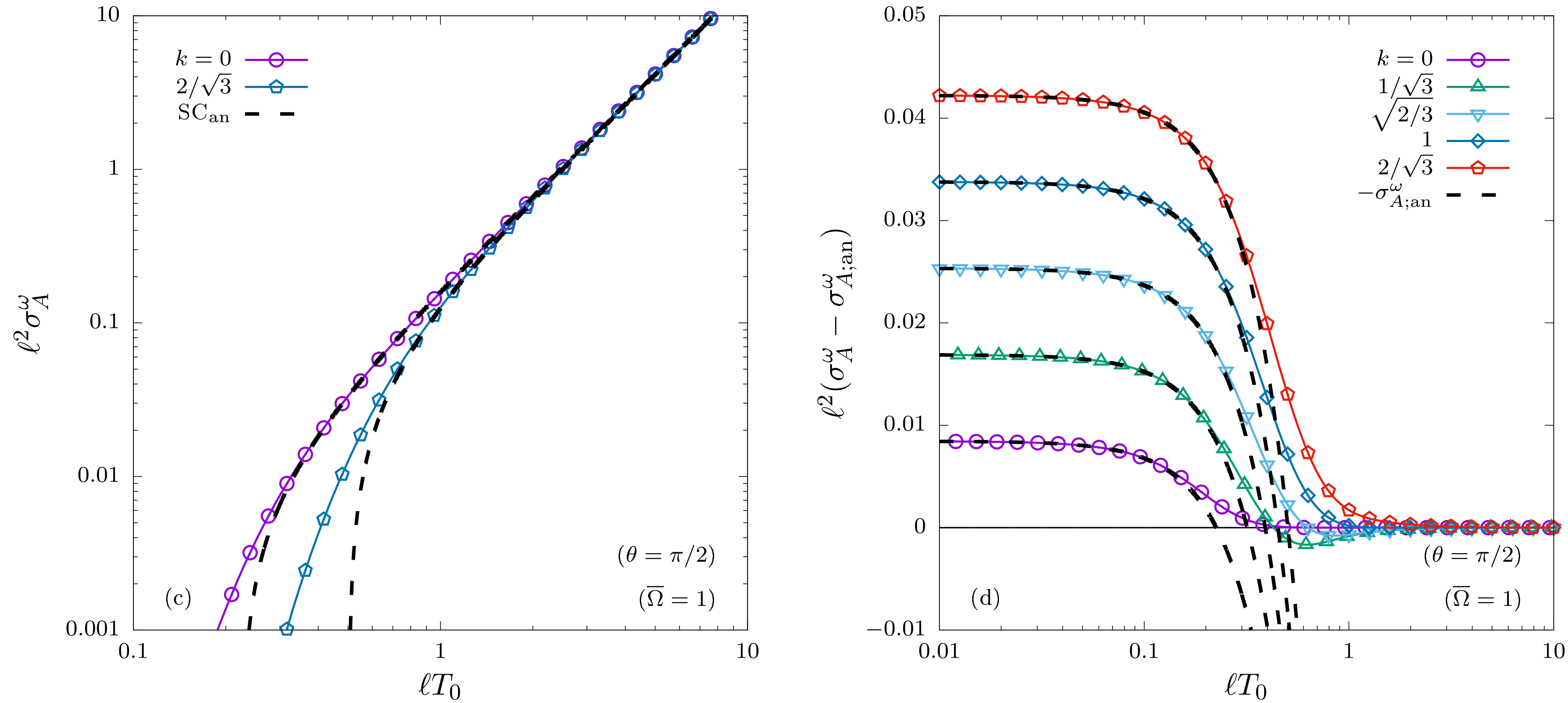
$$\lim_{\overline{\Omega}\to 1} \sigma_A^\omega \left(\theta = \frac{\pi}{2}\right) = \frac{\Gamma_k}{2\pi^2 \ell^2} \sum_{j=1}^\infty \frac{(-1)^{j+1}}{(\sinh \frac{j\beta_0}{2\ell})^{2+2k}} \times {}_2F_1\left(k, 2+k; 1+2k; -\operatorname{cosech}^2 \frac{j\beta_0}{2\ell}\right). \tag{20}$$

Equatorial plane (EP) profiles



- $T = T_0 \Gamma \cos \bar{r}$ decreases with $\bar{r}$ slower at larger $\overline{\Omega}$.
- Deviations from asymptotic formula appear at low T_0, high $k = \ell M$ and/or high $\bar{r}$.
- σ^ω_A in EP independent of $\bar{r}$ when $\overline{\Omega} = 1$ (critical rotation).

Critical rotation $\overline{\Omega} = 1$: temperature dependence



- Asympyotic behaviour reached later for higher k.
- All terms including $O(T^0)$ confirmed by comparison with numerical results.

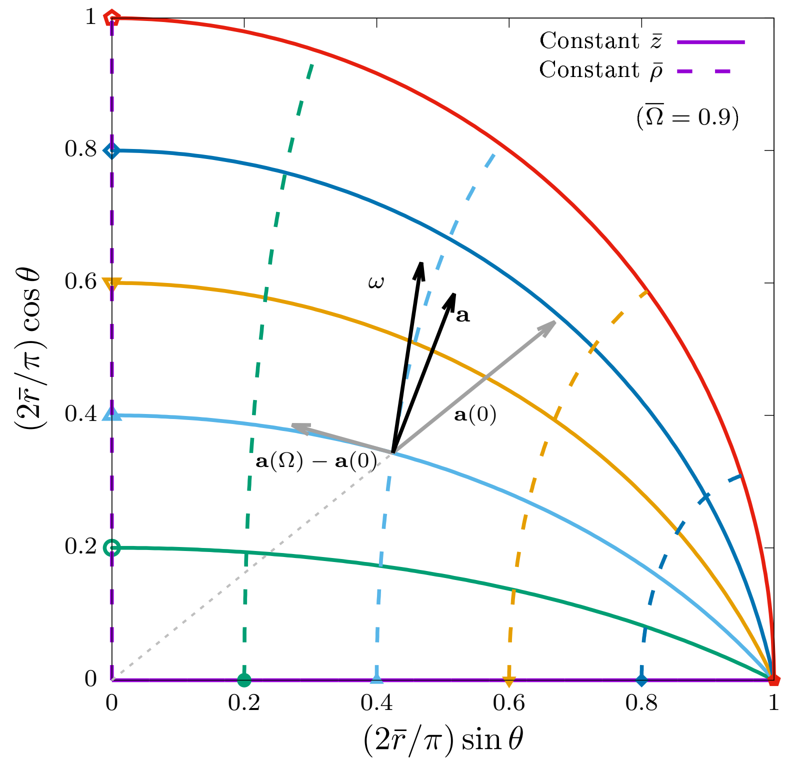
Axial flux

- J_A^μ obeys the conservation equation

$$\nabla_\mu J_A^\mu = \frac{1}{\sqrt{-g}} \frac{\partial (J_A^{\overline{z}} \sqrt{-g})}{\partial \overline{z}} = -2MPC,$$

where $PC = -i\overline{\psi}\gamma^5\psi$ and

$$\overline{z} = \tan \overline{r} \cos\theta, \qquad \overline{\rho} = \sin\overline{r}\sin\theta.$$



- CVE $\Rightarrow$ non-vanishing axial flux F_A:

$$F_A = \ell^{-1} \int_0^1 d\overline{\rho} \int_0^{2\pi} d\varphi \sqrt{-g} J_A^{\overline{z}}(\overline{z})$$

$$= \frac{\Gamma_k}{2\pi\ell(1+k)} \sum_{j=1}^\infty \frac{(-1)^{j+1} \sinh \frac{\Omega j \beta_0}{2} (\sinh \frac{j\beta_0}{2\ell})^{-1-2k}}{(\sinh^2 \frac{j\beta_0}{2\ell} - \sinh^2 \frac{\Omega j \beta_0}{2})(1+\overline{z}^2)^k}$$

$$\times {}_2F_1\left(k, 1+k; 1+2k; -\frac{(1+\overline{z}^2)^{-1}}{\sinh^2 \frac{j\beta_0}{2\ell}}\right). \quad (21)$$

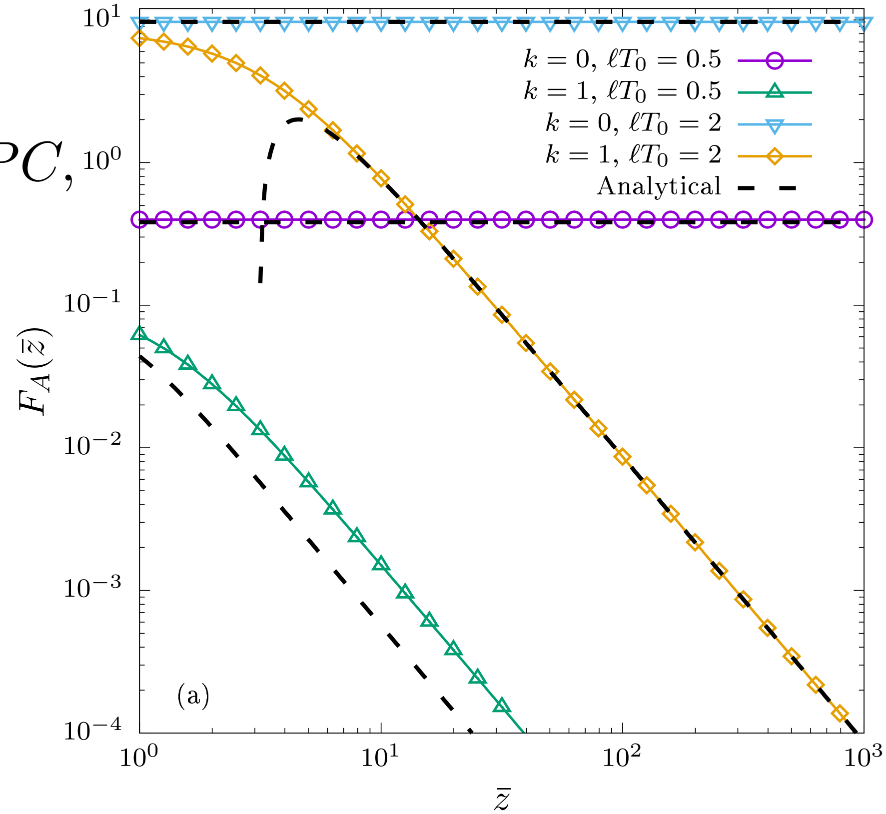
Transparency to chiral particles

- The conservation equation entails

$$F_A(\bar{z}) - F_A(0) = -\frac{2M}{\ell}\int_V d^3x\sqrt{-g}\,PC,$$

showing that $F_A(\pm\infty) = F_A(0)$ when $k = 0 \Rightarrow$ adS is transparent to chiral particles.

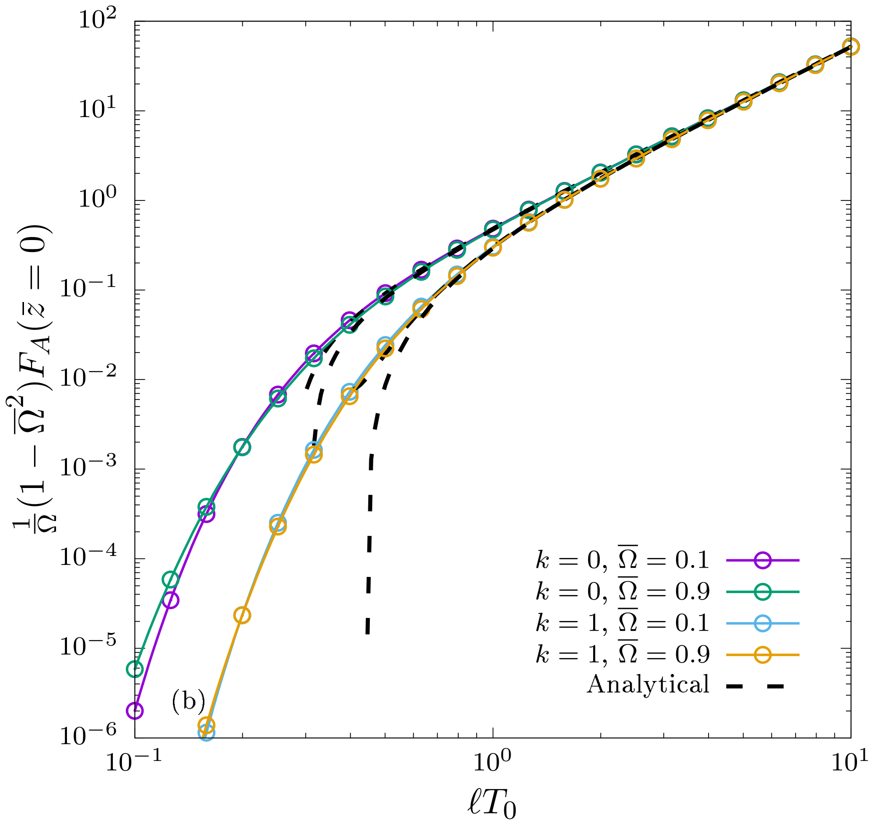
- At finite mass, $F_A(\bar{z})$ can be expanded for large $\bar{z}$ as

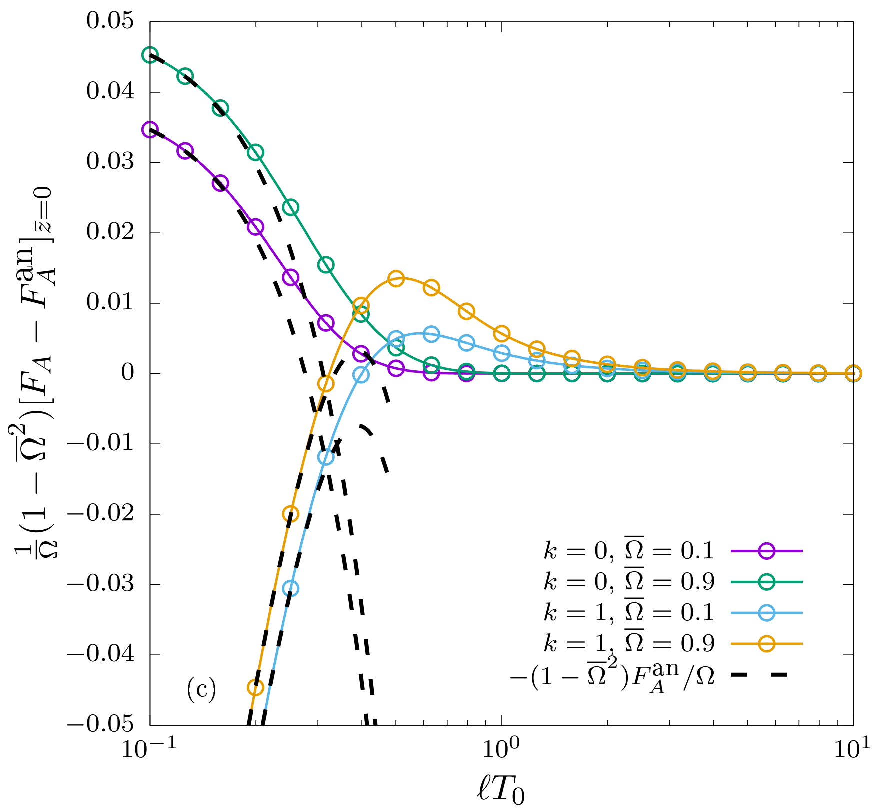


$$F_A(\bar{z}) = \frac{\Omega}{1-\overline{\Omega}^2}\frac{\Gamma(1+k)}{2\sqrt{\pi}\,\Gamma(\frac{1}{2}+k)} \times \left(\frac{\ell T_0}{\sqrt{1+\bar{z}^2}}\right)^{2k}\left\{\left[\zeta(2+2k)\left(1-\frac{1}{2^{1+2k}}\right)(2\ell T_0)^2 - \frac{\zeta(2k)}{6}\left(1-\frac{1}{2^{2k-1}}\right)(3+2k+\overline{\Omega}^2) + O(T_0^{-2})\right] + O(z^{-2})\right\} \quad (22)$$

- Since $F_A(\bar{z}) \simeq z^{-2k}$, a finite mass always ensures that $F(\pm\infty) = 0 \Rightarrow$ adS is opaque to non-chiral particles.

High temperature asymptotics





$$F_A(\overline{z}) = \frac{\Omega}{1-\overline{\Omega}^2}\left\{\frac{\pi\ell^2T_0^2}{6} - \frac{3+\overline{\Omega}^2}{24\pi} - \frac{k^2(1+\overline{z}^2)}{2\pi}\ln\frac{\pi\ell T_0}{\sqrt{1+\overline{z}^2}} - \frac{k(1+\overline{z}^2)}{4\pi}[1+k-2k\mathcal{C}-2k\psi(1+k)] + O(T_0^{-1})\right\}. \quad (23)$$

All terms are confirmed by comparison with numerical results.

Conclusion

- New KMS relation allows rotating thermal propagator $S^F_{\beta_0,\Omega}$ to be written in terms of the vacuum propagator $S^F_{\rm vac}$.
- Exact expression for $S^F_{\rm vac}$ allows vortical effects to be studied on adS.
- Axial vortical effect confirmed and curvature correction revealed.
- For critical rotation ($\ell\Omega = 1$), σ^ω_A becomes constant in the equatorial plane.
- Axial flux F_A of massless particles originates from southern hemisphere and escapes through northern hemisphere.
- For massive particles, $F_A \sim \overline{z}^{-2k}$ and no flux can penetrate the boundary of adS.
- Possible extensions: supercritical rotation, finite chemical potential, ...