Onset of synchronization in coupled Mixmaster oscillators

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Motivation

We want to understand:

- The generic nature of the cosmological singularity (BKL conjecture)
- Inhomogeneous cosmological evolution (generic Einstein flows)
- Entropy problem
- Horizon problem

We introduce consideration of a new phenomenon in cosmology:

- Synchronization, or 'Sync'.

This is here developed in the context of G_0 *cosmologies*:

Inhomogeneous Mixmaster dynamics

General sync dynamics I

For $x \in \mathbb{R}^n$, consider the dynamical system, (the 'transmitter'):

$$\dot{x} = f(x). \tag{1}$$

SYNC SETUP METHOD

- STEP 1: Split the transmitter as follows: $x = (u, v)$, with $u = (x^1, \ldots, x^m), v = (x^{m+1}, \ldots, x^n)$,

$$\dot{u} = f_1(u, v) \tag{2}$$

$$\dot{v} = f_2(u, v) \quad \text{subsystem-b} \tag{3}$$

General sync dynamics II

- STEP 2 Create the 'receiver', a new subsystem identical to subsystem-B: new variables u', v', with,

$$u' = u \tag{4}$$
$$\dot{v}' = f_2(u, v') \tag{5}$$

- STEP 3: Define the transmitter-receiver system:

$$\dot{v} = f_2(u, v) \tag{6}$$
$$\dot{v}' = f_2(u, v') \tag{7}$$

Note: In general, we may consider the effects of some coupling between the transmitter and receiver.

General sync dynamics III

- STEP 4: Define the sync function:

$$\Omega = v - v', \tag{8}$$

and the sync dynamical equations,

$$\dot{\Omega} = g(\Omega) \tag{9}$$

governing the sync dynamics corresponding to the transmitter-receiver system.

- STEP 5: The receiver will sync with the transmitter provided:

$$\Omega \to 0, \ as \ \tau \to \infty. \tag{10}$$

In the case of sync, both systems, the transmitter and the receiver, will evolve in perfect unison.

Evolution of spatial points I

Basic fact:

In inhomogeneous spacetime, distinct Mixmaster spatial points evolve as separate homogeneous Mixmaster universes.

In terms of dimensionless variables,

$$\begin{aligned} N &= (N_1, N_2, N_3): \quad \text{spatial curvature, Bianchi type} \\ \Sigma &= (\Sigma_+, \Sigma_-): \quad \text{anisotropy of the Hubble flow,} \end{aligned}$$

the dimensionless state vector fields for spatial points A, B are,

$$\begin{aligned} X_A(t) &= (N_1, N_2, N_3, \Sigma_+, \Sigma_-), \\ X_B(t) &= (M_1, M_2, M_3, \Pi_+, \Pi_-). \end{aligned}$$

Evolution of spatial points II

Mixmaster point A evolves according to,

$$
\begin{aligned}
N_1' &= (q - 4\Sigma_+)N_1, &(11)\\
N_2' &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-)N_2, &(12)\\
N_3' &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-)N_3, &(13)\\
\Sigma_+' &= -(2-q)\Sigma_+ - 3S_+, &(14)\\
\Sigma_-' &= -(2-q)\Sigma_- - 3S_-, &(15)
\end{aligned}
$$

with the constraint,

$$\Sigma_+^2 + \Sigma_-^2 + \frac{3}{2}\left(N_1^2 + N_2^2 + N_3^2 - 2(N_1N_2 + N_2N_3 + N_3N_1)\right) = 1, \quad (16)$$

Evolution of spatial points III

where

$$
\begin{aligned}
q &= 2(\Sigma_+^2 + \Sigma_-^2), && (17)\\
S_+ &= \frac{1}{2}\left((N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3)\right), && (18)\\
S_- &= \frac{\sqrt{3}}{2}(N_3 - N_2)(N_1 - N_2 - N_3). && (19)
\end{aligned}
$$

Mixmaster point B has identical eqns to the N-eqns for A but for the $X_B(t) = (M_1, M_2, M_3, \Pi_+, \Pi_-)$ variables, while the shear eqns are,

$$
\begin{aligned}
\Pi'_+ &= -(2-p)\Pi_+ - 3Q_+, && (20)\\
\Pi'_- &= -(2-p)\Pi_- - 3Q_-, && (21)
\end{aligned}
$$

with

Evolution of spatial points IV

- $p = 2(\Pi_+^2 + \Pi_-^2)$
- the Q's like the S's in the Eqns. (18), (19) but with the M's in the corresponding places of N's,
- and the constraint identical to (16), but with the (M, Π)'s in the places of the (N, Σ)'s.

Basic question: Sync in cosmology?

Question:

Is it possible for different Mixmaster spatial points to synchronize towards the initial singularity? If yes, what are the implications for cosmology?

Answer:

Indeed! Most interesting!

- Think of A, B as a 'transmitter-receiver pair'
- A in the past null cone of B
- A sends a signal containing the N-variables received and adopted by B
- Set B variables M equal to N-variables of A.

The B-eqns become,

$$M_i = N_i, \quad i = 1, 2, 3 \tag{22}$$
$$\Pi'_+ = -(2-p)\Pi_+ - 3S_+, \tag{23}$$
$$\Pi'_- = -(2-p)\Pi_- - 3S_-. \tag{24}$$

leading to a generic transmitter-receiver cosmological system.

Inhomogeneous Mixmaster sync dynamics II

Definition

The cosmological sync function $\Omega = (\Omega_+, \Omega_-)$, is $\Omega = \Sigma - \Pi$, that is:

$$\Omega_+ = \Sigma_+ - \Pi_+, \quad \Omega_- = \Sigma_- - \Pi_-. \tag{25}$$

Synchronization of A, B if,

$$\Omega \to (0,0), \quad \text{as} \quad \tau \to -\infty. \tag{26}$$

Otherwise, A, B evolve autonomously.

Kinds of sync:

- Complete sync (identical systems)
- Phase sync (slightly coupled and slightly distinct systems)

Inhomogeneous Mixmaster sync dynamics III

- Generalized sync (large couplings and different systems)

All three kinds are in principle possible in inhomogeneous cosmology!

Theorem

The two coupled Mixmaster points A, B synchronize when we have:

- *Zero coupling, equal shear-magnitudes ($p = q$)*
- *Linear coupling (via Liapunov exponents)*
- *More general shear-magnitudes relations ($p = f(q)$)*
- *Arbitrary $f(\Omega)$ coupling*
- *Kuramoto coupling (from mean-field response of spatial points)*

Inhomogeneous Mixmaster sync dynamics IV

Corollary

When sync happens, B reconstructs remaining signal of A from only the partial information received.

Sketch of proof for zero-coupling case with $p = q$
The transmitter-receiver system is given by the Eqns. (14), (15),(23), (24), namely,

$$\begin{aligned} \Sigma'_+ &= -(2-q)\Sigma_+ - 3S_+, & (27)\\ \Sigma'_- &= -(2-q)\Sigma_- - 3S_-, & (28)\\ \Pi'_+ &= -(2-p)\Pi_+ - 3S_+, & (29)\\ \Pi'_- &= -(2-p)\Pi_- - 3S_-. & (30) \end{aligned}$$

Inhomogeneous Mixmaster sync dynamics V

The sync eqns in the zero-coupling case and assuming $q = p$, read:

$$\Omega'_+ = -(2-q)\Omega_+ \tag{31}$$
$$\Omega'_- = -(2-q)\Omega_-. \tag{32}$$

There is a Liapunov function,

$$V(\Omega_+, \Omega_-) = \frac{1}{2}\left(\Omega_+^2 + \Omega_-^2\right), \tag{33}$$

with $V(\Omega_+, \Omega_-) > 0$, and

$$\frac{dV}{d\tau} = -(2-q)(\Omega_+^2 + \Omega_-^2) < 0. \tag{34}$$

Inhomogeneous Mixmaster sync dynamics VI

The state $\Omega = 0$ is globally asymptotically stable. Also exponentially fast decay for V,

$$V \leq V_0 e^{-4\tau}, \tag{35}$$

All trajectories $\Omega(\tau)$ flow downhill toward the state $\Omega = 0$ exponentially fast, and synchronization is stable asymptotically.

The reset

Theorem

The two synchronized Mixmaster points A, B in inhomogeneous spacetime proceed oscillating in perfect unison towards the singularity, in the sense that,

- *The Kasner exponents of A, B satisfy,*

$$p_\alpha^B \to p_\alpha^A, \quad \alpha = 1, 2, 3, \quad as \;\; \Omega \to 0, \tag{36}$$

- *The two u-parameters associated with the Kasner exponents will also converge,*

$$u^B \to u^A \tag{37}$$

  for $u^{A,B} \in (1, \infty)$,

- *the boundary u-values of each sector, namely the values $u = 1$, and $u = \infty$ will also converge (Taub points will go to Taub points, midpoints to midpoints on the two Kasner circles)*

The story so far

On approach to the generic past singularity:

- Every spacetime point is constantly sending and receiving signals from other causally connected spacetime points
- Points shift and adjust their states of oscillation, constantly resetting them
- Inhomogeneous Mixmaster points are resettable oscillators
- Sync organizes them in perfect harmony
- No resetting possible in homogeneous Mixmaster

The entropy problem I

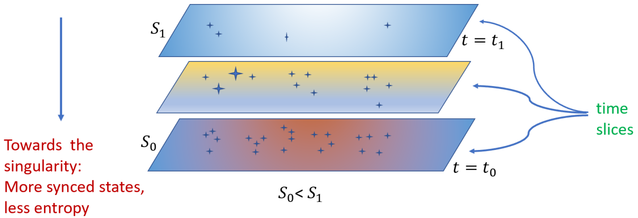

Figure: The cosmological entropy problem

- The universe losses information every time two spatial points become synced.

The entropy problem II

- The universe evolves towards the initial singularity in a series of simpler, more synced, states
- Consider the entropy S_1 at time t_1, and the entropy S_0 at an earlier time t_0. Since the state at time t_0 is simpler, because of sync, than that at time t_1, it follows that

$$S_0 < S_1. \tag{38}$$

- As a result of sync, entropy reduces as we proceed to earlier times (more synced states)!
- Initial state is the simplest state possible, a state of minimum entropy.
- Inhomogeneous, synced universe has the most special initial state!

The horizon problem

The horizon problem represents the biggest example of sync ever!

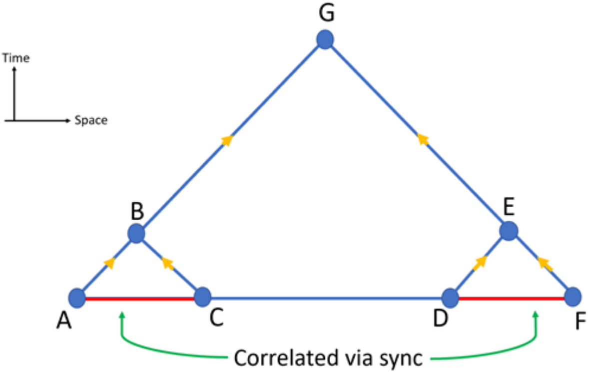

Figure: The horizon problem

For more on sync

https://www.ted.com/talks/steven_strogatz_the_science_of_sync

Thanks!