Onset of synchronization in coupled Mixmaster oscillators

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1 Motivation

2 General sync dynamics

3 Evolution of spatial points

4 Basic question: Sync in cosmology?

5 Inhomogeneous Mixmaster sync dynamics

6 The reset

7 The story so far

8 The entropy problem

9 The horizon problem
We want to understand:

- The generic nature of the cosmological singularity (BKL conjecture)
- Inhomogeneous cosmological evolution (generic Einstein flows)
- Entropy problem
- Horizon problem

We introduce consideration of a new phenomenon in cosmology:

- Synchronization, or ‘Sync’.

This is here developed in the context of $G_0$ cosmologies:

**Inhomogeneous Mixmaster dynamics**
For $x \in \mathbb{R}^n$, consider the dynamical system, (the ‘transmitter’):

$$\dot{x} = f(x). \quad (1)$$

**SYNC SETUP METHOD**

- **Step 1:** Split the transmitter as follows: $x = (u, v)$, with

  $$u = (x^1, \ldots, x^m), \quad v = (x^{m+1}, \ldots, x^n),$$

  $$\dot{u} = f_1(u, v) \quad (2)$$
  $$\dot{v} = f_2(u, v) \quad \text{subsystem-b} \quad (3)$$
Step 2 Create the ‘receiver’, a new subsystem identical to subsystem-B: new variables $u', v'$, with,

$$u' = u \quad (4)$$
$$\dot{v}' = f_2(u, v') \quad (5)$$

Step 3: Define the transmitter-receiver system:

$$\dot{v} = f_2(u, v) \quad (6)$$
$$\dot{v}' = f_2(u, v') \quad (7)$$

Note: In general, we may consider the effects of some coupling between the transmitter and receiver.
Step 4: Define the sync function:

$$\Omega = \nu - \nu'$$

(8)

and the sync dynamical equations,

$$\dot{\Omega} = g(\Omega)$$

(9)

governing the sync dynamics corresponding to the transmitter-receiver system.

Step 5: The receiver will sync with the transmitter provided:

$$\Omega \to 0, \text{ as } \tau \to \infty.$$

(10)

In the case of sync, both systems, the transmitter and the receiver, will evolve in perfect unison.
Basic fact:
In inhomogeneous spacetime, distinct Mixmaster spatial points evolve as separate homogeneous Mixmaster universes.

In terms of dimensionless variables,

\[ N = (N_1, N_2, N_3) : \text{spatial curvature, Bianchi type} \]
\[ \Sigma = (\Sigma_+, \Sigma_-) : \text{anisotropy of the Hubble flow}, \]

the dimensionless state vector fields for spatial points \( A, B \) are,

\[ X_A(t) = (N_1, N_2, N_3, \Sigma_+, \Sigma_-), \]
\[ X_B(t) = (M_1, M_2, M_3, \Pi_+, \Pi_-). \]
Mixmaster point $A$ evolves according to,

\begin{align}
N'_1 &= (q - 4\Sigma_+) N_1, \\
N'_2 &= (q + 2\Sigma_+ + 2\sqrt{3}\Sigma_-) N_2, \\
N'_3 &= (q + 2\Sigma_+ - 2\sqrt{3}\Sigma_-) N_3, \\
\Sigma'_+ &= -(2 - q)\Sigma_+ - 3S_+, \\
\Sigma'_- &= -(2 - q)\Sigma_- - 3S_-, \\
\end{align}

with the constraint,

\begin{align}
\Sigma^2_+ + \Sigma^2_- + \frac{3}{2} \left( N^2_1 + N^2_2 + N^2_3 - 2(N_1 N_2 + N_2 N_3 + N_3 N_1) \right) &= 1,
\end{align}
where

\[
q = 2(\Sigma_+^2 + \Sigma_-^2), \tag{17}
\]

\[
S_+ = \frac{1}{2} \left( (N_2 - N_3)^2 - N_1(2N_1 - N_2 - N_3) \right), \tag{18}
\]

\[
S_- = \frac{\sqrt{3}}{2} (N_3 - N_2)(N_1 - N_2 - N_3). \tag{19}
\]

Mixmaster point \( B \) has identical eqns to the \( N \)-eqns for \( A \) but for the \( X_B(t) = (M_1, M_2, M_3, \Pi_+, \Pi_-) \) variables, while the shear eqns are,

\[
\Pi'_+ = -(2 - p)\Pi_+ - 3Q_+, \tag{20}
\]

\[
\Pi'_- = -(2 - p)\Pi_- - 3Q_- \tag{21}
\]

with
\[ p = 2(\Pi^2_+ + \Pi^2_-) \]

- the \( Q \)'s like the \( S \)'s in the Eqns. (18), (19) but with the \( M \)'s in the corresponding places of \( N \)'s,

- and the constraint identical to (16), but with the \((M, \Pi)\)'s in the places of the \((N, \Sigma)\)'s.
Basic question: Sync in cosmology?

Question:
Is it possible for different Mixmaster spatial points to synchronize towards the initial singularity? If yes, what are the implications for cosmology?

Answer:
Indeed! Most interesting!
Think of $A, B$ as a ‘transmitter-receiver pair’

- $A$ in the past null cone of $B$
- $A$ sends a signal containing the $N$-variables received and adopted by $B$
- Set $B$ variables $M$ equal to $N$-variables of $A$.

The $B$-eqns become,

$$M_i = N_i, \quad i = 1, 2, 3 \quad (22)$$

$$\Pi'_+ = -(2 - p)\Pi_+ - 3S_+, \quad (23)$$

$$\Pi'_- = -(2 - p)\Pi_- - 3S_- \quad (24)$$

leading to a generic transmitter-receiver cosmological system.
Definition

The cosmological sync function $\Omega = (\Omega_+, \Omega_-)$, is $\Omega = \Sigma - \Pi$, that is:

$$\Omega_+ = \Sigma_+ - \Pi_+, \quad \Omega_- = \Sigma_- - \Pi_-.$$ \hspace{1cm} (25)

Synchronization of $A, B$ if,

$$\Omega \to (0, 0), \quad \text{as} \quad \tau \to -\infty.$$ \hspace{1cm} (26)

Otherwise, $A, B$ evolve autonomously.

Kinds of sync:

- Complete sync (identical systems)
- Phase sync (slightly coupled and slightly distinct systems)
Generalized sync (large couplings and different systems)
All three kinds are in principle possible in inhomogeneous cosmology!

**Theorem**

*The two coupled Mixmaster points A, B synchronize when we have:*

- Zero coupling, equal shear-magnitudes \( p = q \)
- Linear coupling (via Liapunov exponents)
- More general shear-magnitudes relations \( p = f(q) \)
- Arbitrary \( f(\Omega) \) coupling
- Kuramoto coupling (from mean-field response of spatial points)
Corollary

When sync happens, $B$ reconstructs remaining signal of $A$ from only the partial information received.

**Sketch of proof for zero-coupling case with $p = q$**

The transmitter-receiver system is given by the Eqns. (14), (15), (23), (24), namely,

\[
\begin{align*}
\Sigma'_+ &= -(2-q)\Sigma_+ - 3S_+ , \quad \text{(27)} \\
\Sigma'_- &= -(2-q)\Sigma_- - 3S_- , \quad \text{(28)} \\
\Pi'_+ &= -(2-p)\Pi_+ - 3S_+ , \quad \text{(29)} \\
\Pi'_- &= -(2-p)\Pi_- - 3S_- . \quad \text{(30)}
\end{align*}
\]
The sync eqns in the zero-coupling case and assuming $q = p$, read:

\[
\begin{align*}
\Omega_+' &= -(2 - q)\Omega_+ \quad (31) \\
\Omega_-' &= -(2 - q)\Omega_- \quad (32)
\end{align*}
\]

There is a Liapunov function,

\[
V(\Omega_+, \Omega_-) = \frac{1}{2} (\Omega_+^2 + \Omega_-^2), \quad (33)
\]

with $V(\Omega_+, \Omega_-) > 0$, and

\[
\frac{dV}{d\tau} = -(2 - q)(\Omega_+^2 + \Omega_-^2) < 0. \quad (34)
\]
The state $\Omega = 0$ is globally asymptotically stable. Also exponentially fast decay for $V$,

$$V \leq V_0 e^{-4\tau},$$  \hspace{1cm} (35)

All trajectories $\Omega(\tau)$ flow downhill toward the state $\Omega = 0$ exponentially fast, and synchronization is stable asymptotically.
Theorem

The two synchronized Mixmaster points $A, B$ in inhomogeneous spacetime proceed oscillating in perfect unison towards the singularity, in the sense that,

- The Kasner exponents of $A, B$ satisfy,

$$p^B_\alpha \to p^A_\alpha, \quad \alpha = 1, 2, 3, \quad \text{as} \quad \Omega \to 0,$$

(36)

- The two $u$-parameters associated with the Kasner exponents will also converge,

$$u^B \to u^A$$

(37)

for $u^A, B \in (1, \infty)$,

- the boundary $u$-values of each sector, namely the values $u = 1$, and $u = \infty$ will also converge (Taub points will go to Taub points, midpoints to midpoints on the two Kasner circles)
The story so far

On approach to the generic past singularity:

- Every spacetime point is constantly sending and receiving signals from other causally connected spacetime points
- Points shift and adjust their states of oscillation, constantly resetting them
- **Inhomogeneous Mixmaster points are resettable oscillators**
- **Sync organizes them in perfect harmony**
- No resetting possible in homogeneous Mixmaster
The universe losses information every time two spatial points become synced.
The universe evolves towards the initial singularity in a series of simpler, more synced, states.

Consider the entropy $S_1$ at time $t_1$, and the entropy $S_0$ at an earlier time $t_0$. Since the state at time $t_0$ is simpler, because of sync, than that at time $t_1$, it follows that

$$S_0 < S_1.$$  \hfill (38)

As a result of sync, entropy reduces as we proceed to earlier times (more synced states)!

Initial state is the simplest state possible, a state of minimum entropy.

Inhomogeneous, synced universe has the most special initial state!
The horizon problem represents the biggest example of sync ever!

Figure: The horizon problem
For more on sync

https://www.ted.com/talks/steven_strogatz_the_science_of_sync

Thanks!