The Hamilton-Jacobi analysis by Peter Bergmann and Arthur Komar of classical general relativity

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Reference

See *Observables and Hamilton-Jacobi approaches to general relativity. I. The Earlier History*, arXiv:2106.11894

Overview

1. Bergmann's initial Hamilton-Jacobi analysis of general relativity
2. Komar's isolation of solution equivalence classes
3. Contrast with geometrodynamics

1. BERGMANN INITIAL HAMILTON-JACOBI ANALYSIS OF GENERAL RELATIVITY

Bergmann and Hamilton Jacobi approaches to general relativity

It is not widely recognized that it was Peter Bergmann who pointed out to Peres prior to the publication of his groundbreaking paper in [?] that his S should be interpreted "as the Hamilton-Jacobi functional for the gravitational field."

$$\mathcal{H}_\mu \left(g_{ab}, \frac{\delta S}{\delta g_{cd}} \right) = 0$$

The $\mathcal{H}_\mu \left(g_{ab}, p^{cd} \right)$ are the secondary constraints in general relativity.

Temporal independence and form invariance

Bergmann proved in [?] that in a theory in which the Hamiltonian is constrained to vanish S could not depend explicitly on the time. The argument applied equally well to spatial dependence, as noted first in [?]. Thus $S = S[g_{ab}(\vec{x})]$.

Bergmann also showed that the Hamilton-Jacobi equations were form invariant under canonical transformations generated by diffeomorphism invariants.

The fact that the numerical value of S is altered under the action of $\mathcal{H}_0$ presented a puzzle. Could this be inconsistent with the accepted notion of 'frozen time'?

2. KOMAR'S ISOLATION OF SOLUTION EQUIVALENCE CLASSES

Komar's fixation of momenta

Komar observed in [?] that although there were only four Hamilton-Jacobi equations the principal function S delivered $6 \times \infty$ expressions for the momenta,

$$p^{ab}(\vec{x}) = \frac{\delta S}{\delta g_{ab}(\vec{x})}$$

and therefore the $p^{ab}(\vec{x})$ are not uniquely determined. Two additional constraints needed to be imposed, with $A = 1, 2$,

$$\alpha^0_A \left[g_{ab}(\vec{x}), \frac{\delta S}{\delta g_{cd}(\vec{x})} \right] - \alpha_A(\vec{x}) = 0$$

Proof of diffeomorphism invariance

From the fact that

$$\frac{\delta \alpha_A^0}{\delta g_{ab}} + \frac{\delta \alpha_A^0}{\delta p^{cd}} \frac{\delta^2 S}{\delta g_{cd} \delta g_{ab}} = 0,$$

and similarly for the $\mathcal{H}_\mu$ it follows that

$$\{H_\mu, \alpha_A^0\} = \frac{\delta^2 S}{\delta g_{ab} \delta g_{cd}} \left(-\frac{\delta H_\mu}{\delta p^{ab}} \frac{\delta \alpha_A^0}{\delta p^{cd}} + \frac{\delta H_\mu}{\delta p^{cd}} \frac{\delta \alpha_A^0}{\delta p^{ab}} \right) = 0.$$

In other words, the α_A^0 must be diffeomorphism invariants (and they must also commute with each other.)

Equivalence classes under diffeomorphisms

The constant values of $\alpha^0_A\left[g_{ab}(\vec{x}), p^{cd}(\vec{x})\right]$ identify equivalence classes under the action of the spacetime diffeomorphism group.

In [**?**] he showed that there existed invariant functionals β^A_0 that were canonically conjugate to the α^0_A.

However, as formulated at this stage by Komar, one cannot yet obtain solutions of Einstein's equations by setting $\beta^A(\vec{x}) = \frac{\delta S}{\delta\alpha_A(\vec{x})}$. One still requires a temporal coordinate - like the 'intrinsic' q^0 that appears in the free particle action.

The intrinsic coordinate mystery

Compare the non-vanishing action increment for the free relativistic particle

$$dS_p = p_\mu dq^\mu$$

with constraint $p^2 + m^2 = 0$ to that for general relativity,

$$dS_{gr} = \int d^3x \, p_{ab} dg^{ab},$$

with constraints $\mathcal{H}_\mu = 0$.

The intrinsic coordinate mystery

Choose the 'intrinsic time' $t = q^0$ as the evolution paramter and also solve for p_0 resulting in

$$dS_p = -\left(\vec{p}^2 + m^2\right)^{1/2} dt + p_a dq^a$$

This yields the complete Hamilton principal function

$$S_p(q^a, t; \alpha^b) = -\left(\vec{\alpha}^2 + m^2\right)^{1/2} t + \alpha_a q^a$$

The analogue of the gravitational α_0^A in this case is p^a. The analogue of the canonical conjugate β_B^0 would be the reparameterization constant $q^a - p^a q^0/p^0$. The general solution is obtained from

$$\beta^a = \frac{\partial S_p}{\partial \alpha^a}$$

Komar's intrinsic curvature coordinates

[?] explicitly recognized this type of emergence of intrinsic time evolution.

[?] proposed that intrinsic curvature-based coordinates could be constructed using Weyl curvature scalars. [?] proved that these scalars depended only on g_{ab} and p^{cd}

Why did Bergmann and Komar not proceed with the use of intrinsic coordinates in their Hamilton-Jacobi treatment? A quote from [?] is revealing: "Although intrinsic coordinates lead, in principle, to a complete set of observables in general relativity, their defects, of which the most glaring is their deviation from Lorentz coordinates, render this procedure illusory. It appears preferable to retain coordinates that are approximately, or asymptotically Lorentzian and hence not to destroy one's intuition."

Hamilton-Jacobi approach with intrinsic coordinates

As shown in *Restoration of four-dimensional diffeomorphism covariance in canonical general relativity: An intrinsic Hamilton-Jacobi approach*, arXiv:1508.01277v6, it is in principle possible to carry out a canonical change of variable in the non-vanishing increment dS_{GR} to intrinsic spacetime coordinates $x^\mu = X^\mu(g_{ab}, p^{cd})$, analogous to the parameter choice $t = q^0$ in the free relativistic particle model. This is corrected version of [**?**]. Current work with Kurt Sundermeyer and Jürgen Renn is in preparation.

Hamilton-Jacobi approach with intrinsic coordinates

Make a canonical change of variables such that

$$dS_{gr} = \int d^3x \, p^{ab} dg_{ab}$$

$$= \int d^3x \left(\pi_\mu dX^\mu + p^A dg_A + \frac{\delta G}{\delta g_{ab}} dg_{ab} + \frac{\delta G}{\delta g_A} dg_A + \frac{\delta G}{\delta X^\mu} dX^\mu \right)$$

Find $G[g_{ab}, X_A, g_B]$ such that $p_{ab} = \frac{\delta G}{\delta g_{ab}}$. Then

$$dS'_{gr} := d(S_{gr} - G) = \int d^3x \left(\pi_\mu dX^\mu + p^A dg_A \right).$$

Hamilton-Jacobi approach with intrinsic coordinates

Next choose the X^μ as intrinsic coordinates, i.e., set $x^\mu = X^\mu$. Eliminate the canonical conjugates to X^μ, π_ν, by solving the constraints. Then we have the Hamilton-Jacobi equation

$$\pi_0 \left[g_A, \frac{\delta S'_{gr}}{\delta g_B}, x^\mu \right] + \frac{\partial S'_{gr}}{\partial x^0} = 0.$$

From the complete solutions $S'_{gr}\left[g_A, x^\mu; \alpha_B\right]$ one can obtain the full set of physically distinct solutions of Einstein's equations from

$$\beta^A = \frac{\delta S'_{gr}}{\delta \alpha_A}.$$

3. CONTRAST WITH GEOMETRODYNAMICS

Contrast with geometrodynamics

The contrast of this program with Wheeler's geometrodynamics cannot be overstated. The multifingered time approach assumed that the full four-dimensional diffeomorphism symmetry had been lost. States should be labeled by the $2 \times \infty^3$ diffeomorphism invariants $\alpha_A(\vec{x})$, and not by three-geometries.

References I