Dynamical system analysis of Bianchi-I spacetimes in *f(R)* gravity

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First part

Short review on dark energy and modified gravity theories including *f(R)* gravity

Current cosmic acceleration

- **Current expansion of the Universe is accelerating ("Dark Energy Problem")**

Type Ia Supernova (SN)

Nobel Prize in Physics 2011

Dr. Saul Perlmutter

Dr. Brian P. Schmidt

Dr. Adam G. Riess

From [the URL of *Nobelprize.org*].

Two main approaches

(1) **General relativistic (GR) approach**

→ **"Dark Energy"**

(with its negative pressure)

(2) **To extend gravity theories**

Reviews: E.g.,

[Copeland, Sami and Tsujikawa, Int. J. Mod. Phys. D **15**, 1753 (2006)]

[Nojiri and Odintsov, Phys. Rept. **505**, 59 (2011); Int. J. Geom. Meth. Mod. Phys. **4**, 115 (2007)]

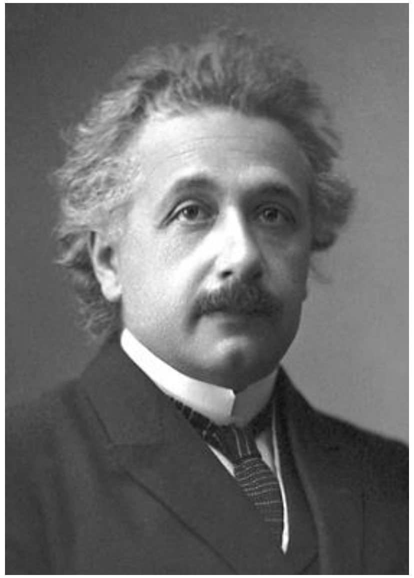
[Capozziello and Faraoni, *Beyond Einstein Gravity* (Springer, 2010)]

[Clifton, Ferreira, Padilla and Skordis, Phys. Rept. **513**, 1 (2012)]

[Joyce, Jain, Khoury and Trodden, Phys. Rept. **568**, 1 (2015)]

[Cai, Capozziello, De Laurentis and Saridakis, Rept. Prog. Phys. **79** (2016), 106901]

[KB, Capozziello, Nojiri and Odintsov, Astrophys. Space Sci. **342**, 155 (2012)]

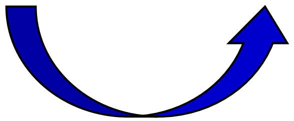


Dr. Albert Einstein

From [the URL of *Nobelprize.org*].

Beyond

GR



General relativistic approach

(i) **Cosmological constant**

Canonical field

(ii) **Scalar field** : ▪**X matter**, **Quintessence** ←

[Chiba, Sugiyama and Nakamura, Mon. Not. Roy. Astron. Soc. **289**, L5 (1997)]
[Caldwell, Dave and Steinhardt, Phys. Rev. Lett. **80**, 1582 (1998)]

Cf. Pioneering work: [Fujii, Phys. Rev. D **26**, 2580 (1982)]

▪**Phantom** ← Wrong sign kinetic term

[Caldwell, Phys. Lett. B **545**, 23 (2002)]

▪**K-essence** ← Non canonical kinetic term

[Chiba, Okabe and Yamaguchi, Phys. Rev. D **62**, 023511 (2000)]

[Armendariz-Picon, Mukhanov and Steinhardt, Phys. Rev. Lett. **85**, 4438 (2000)]

▪**Tachyon** ← String theories

[Padmanabhan, Phys. Rev. D **66**, 021301 (2002)]

General relativistic approach (2)

(iii) Cosmic fluids

- **Chaplygin gas** ⟵ $P = -A/\rho$

  $A > 0$: Constant

  [Kamenshchik, Moschella and Pasquier, Phys. Lett. B **511**, 265 (2001)]

- **Viscous fluid**

  [Brevik, Obukhov and Timoshkin, Astrophys. Space Sci. **355**, 399 (2015)]

Extension of gravitational theories

Cf. Application to inflation:

[Starobinsky, Phys. Lett. B **91**, 99 (1980)]

- ***f*(*R*) gravity**

$f(R)$: Arbitrary function of the Ricci scalar *R*

[Capozziello, Carloni and Troisi, Recent Res. Dev. Astron. Astrophys. **1**, 625 (2003)]

[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D **70**, 043528 (2004)]

[Nojiri and Odintsov, Phys. Rev. D **68**, 123512 (2003)]

- **Scalar-tensor theories** ← $f_1(\phi)R$

$f_i(\phi)$ $(i=1,2)$: Arbitrary function of a scalar field ϕ

[Boisseau, Esposito-Farese, Polarski and Starobinsky, Phys. Rev. Lett. **85**, 2236 (2000)]

[Gannouji, Polarski, Ranquet and Starobinsky, JCAP **0609**, 016 (2006)]

cf. **Brans-Dicke theories** [Brans and Dicke, Phys. Rev. **124**, 925 (1961)]

Extension of gravitational theories (2)

- **Ghost condensates**

[Arkani-Hamed, Cheng, Luty and Mukohyama, JHEP **0405**, 074 (2004)]

- **Higher-order curvature term**

  ↳ Gauss-Bonnet term with a coupling to a scalar field: $f_2(\phi)\mathcal{G}$

$$\mathcal{G} \equiv R^2 - 4R_{\mu\nu}R^{\mu\nu} + R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$$

[Nojiri, Odintsov and Sasaki, Phys. Rev. D **71**, 123509 (2005)]

$R_{\mu\nu}$: Ricci curvature tensor

$R_{\mu\nu\rho\sigma}$: Riemann tensor

- $f(\mathcal{G})$ **gravity** $\leftarrow \dfrac{R}{2\kappa^2} + f(\mathcal{G})$

$$\kappa^2 \equiv 8\pi G$$

G : Gravitational constant

[Nojiri and Odintsov, Phys. Lett. B **631**, 1 (2005)]

Extension of gravitational theories (3)

- **DGP (Dvali-Gabadadze-Porrati) braneworld scenario**

[Dvali, Gabadadze and Porrati, Phys. Lett B **485**, 208 (2000)]

[Deffayet, Dvali and Gabadadze, Phys. Rev. D **65**, 044023 (2002)]

- ***f*(*T*) gravity** Extended teleparallel Lagrangian density described by the torsion scalar T

[Bengochea and Ferraro, Phys. Rev. D **79**, 124019 (2009)]

[Linder, Phys. Rev. D **81**, 127301 (2010) [Erratum-ibid. D **82**, 109902 (2010)]]

→ **Teleparallelism**

: One could use the Weitzenböck connection, which has no curvature but torsion, rather than the curvature defined by the Levi-Civita connection.

[Hayashi and Shirafuji, Phys. Rev. D **19**, 3524 (1979) [Addendum-ibid. D **24**, 3312 (1982)]]

Extension of gravitational theories (4)

- **Galileon gravity** [Nicolis, Rattazzi and Trincherini, Phys. Rev. D **79**, 064036 (2009)]

  Review: [Tsujikawa, Lect. Notes Phys. **800**, 99 (2010)]

$\Box\phi(\partial^\mu\phi\partial_\mu\phi) \longleftarrow$ Longitudinal graviton (i.e. a branebending mode)

The equations of motion are invariant under the Galilean shift: $\partial_\mu\phi \to \partial_\mu\phi + b_\mu$

⇨ One can keep the equations of motion up to the second-order.

$\Box$: Covariant d'Alembertian

This property is welcome to avoid the appearance of an extra degree of freedom associated with ghosts.

Extension of gravitational theories (5)

- **Horndeski theory** ← Generalization of Galileon gravity

[Horndeski, Int. J. Theor. Phys. **10**, 363 (1974)]

[Kobayashi, Yamaguchi and Yokoyama, Prog. Theor. Phys. **126**, 511 (2011)]

- **Non-local gravity** ← Quantum effects

[Deser and Woodard, Phys. Rev. Lett. **99**, 111301 (2007)]

- **Horava-Lifshitz gravity**

[Horava, Phys. Rev. D **79**, 084008 (2009)]

Extension of gravitational theories (6)

- **Massive gravity**

  [van Dam and Veltman, Nucl.Phys. **B22**, 397 (1970)]

  [Zakharov, JETP Lett. **12**, 312, (1970)]

  [de Rham and Gabadadze, Phys.Rev. D **82**, 044020 (2010)]

  [de Rham, Gabadadze and Tolley, Phys. Rev. Lett. **106**, 231101 (2011)]

- **Bi-gravity**

  [Hassan and Rosen, Phys. Rev. Lett. **108**, 041101 (2012)]

  [Hassan and Rosen, JHEP **1202**, 126 (2012)]

$f(R)$ gravity

Action $S = \int d^4x \sqrt{-g}\, \frac{f(R)}{2\kappa^2}$

$\kappa^2 = 8\pi G$

G : Gravitational constant

Cf. $f(R) = R$: General Relativity

[Capozziello, Carloni and Troisi, Recent Res. Dev. Astron. Astrophys. **1**, 625 (2003)]

[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D **70**, 043528 (2004)]

[Nojiri and Odintsov, Phys. Rev. D **68**, 123512 (2003)]

Gravitational field equation

$f'(R) = df(R)/dR$

$$f'(R)R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}f(R) + g_{\mu\nu}\Box f'(R) - \nabla_\mu\nabla_\nu f'(R) = 0$$

$\Box \equiv g^{\mu\nu}\nabla_\mu\nabla_\nu$: Covariantd'Alembertian

∇_μ : Covariant derivative

$f(R)$ gravity (2)

- In the flat FLRW background, gravitational field equations read

$\rho_{\rm eff}, \ p_{\rm eff}$: Effective energy density and pressure from the term $f(R) - R$

$$H^2 = \frac{\kappa^2}{3}\rho_{\rm eff}\,, \quad \dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\rm eff} + p_{\rm eff}\right)$$

$H \equiv \frac{\dot{a}}{a}$: Hubble parameter

$$\rho_{\rm eff} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(-f(R) + Rf'(R)\right) - 3H\dot{R}f''(R)\right]$$

$$p_{\rm eff} = \frac{1}{\kappa^2 f'(R)}\left[\frac{1}{2}\left(f(R) - Rf'(R)\right) + \left(2H\dot{R} + \ddot{R}\right)f''(R) + \dot{R}^2 f'''(R)\right]$$

Effective equation of state (EoS) parameter: →

$$w_{\rm eff} = \frac{p_{\rm eff}}{\rho_{\rm eff}} = \frac{\left(f(R) - Rf'(R)\right)/2 + \left(2H\dot{R} + \ddot{R}\right)f''(R) + \dot{R}^2 f'''(R)}{\left(-f(R) + Rf'(R)\right)/2 - 3H\dot{R}f''(R)}$$

Cosmic acceleration in *f(R)* gravity

$$f(R) = R - \frac{\mu^{2(n+1)}}{R^n}$$

[Carroll, Duvvuri, Trodden and Turner, Phys. Rev. D 70, 043528 (2004)]

μ : Mass scale n : Constant

Second term becomes important as *R* decreases.

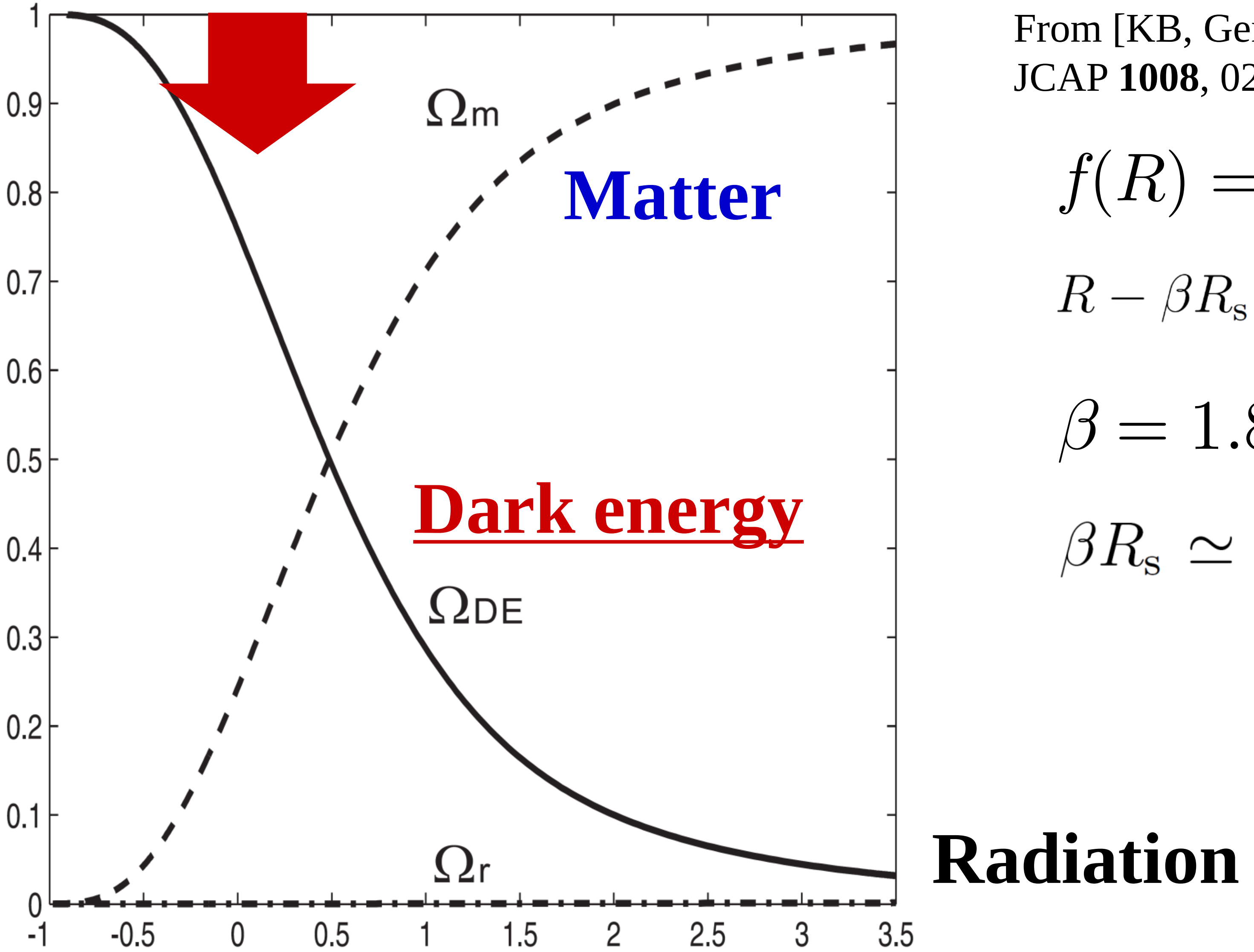
$$\Rightarrow \quad a \propto t^q, \quad q = \frac{(2n+1)(n+1)}{n+2}$$

If $q > 1$, accelerated expansion ($\ddot{a} > 0$) can be realized.

$$w_{\rm eff} = -1 + \frac{2(n+2)}{3(2n+1)(n+1)}$$

$\longrightarrow$ For $n = 1$, $q = 2$ and $w_{\rm eff} = -2/3$

Evolution of the energy fractions



z : Redshift

From [KB, Geng and Lee, JCAP **1008**, 021 (2010)].

$$f(R) =$$

$$R - \beta R_{\mathrm{s}} \left(1 - e^{-R/R_{\mathrm{s}}}\right)$$

$$\beta = 1.8$$

$$\beta R_{\mathrm{s}} \simeq 18 H_0^2 \Omega_{\mathrm{m}}^{(0)}$$

Evolution of the energy fractions (2)

$$z \equiv \frac{1}{a} - 1$$: Redshift

$$\Omega_i \equiv \frac{\rho_i}{\rho_{\mathrm{c}}}$$: Energy fraction

Radiation: r

Matter: m

Dark energy: DE

ρ_{c} : Critical density

Conditions for the viability of *f*(*R*) gravity

(1) $f'(R) > 0 \longleftarrow$ Positivity of the effective gravitational coupling

$$f'(R) \equiv df(R)/dR \qquad G_{\mathrm{eff}} = G/f'(R) > 0$$

(2) $f''(R) > 0 \longleftarrow$ Stability condition: $M^2 \approx 1/(3f''(R)) > 0$

$$f''(R) \equiv d^2 f(R)/dR^2$$

[Dolgov and Kawasaki, Phys. Lett. B 573, 1 (2003)]

M : Mass of a new scalar degree of freedom ("scalaron") in the weak-field regime.

(3) $f(R) \to R - 2\Lambda$ **for** $R \gg R_0$ $\leftarrow$ **Existence of a matter-dominated stage**

R_0 : Current curvature

Λ : Cosmological constant

Conditions for the viability of *f*(*R*) gravity (2)

(4) $0 < m \equiv Rf''(R)/f'(R) < 1$ $\longleftarrow$ **Stability of the late-time de Sitter point**

[Amendola, Gannouji, Polarski and Tsujikawa, Phys. Rev. D 75, 083504 (2007)]

[Amendola and Tsujikawa, Phys. Lett. B 660, 125 (2008)]

Cf. For general relativity, $m = 0.$

[Faraoni and Nadeau, Phys. Rev. D 75, 023501 (2007)]

(5) Constraints from the violation of the equivalence principle (Solar-system constraints)

[Chiba, Phys. Lett. B 575, 1 (2003)]

[Chiba, Smith and Erickcek, Phys. Rev. D 75, 124014 (2007)]

$M = M(R)$ $\longleftarrow$ Scale-dependence

: "Chameleon mechanism"

Cf. [Khoury and Weltman, Phys. Rev. D 69, 044026 (2004)]

Models of *f*(*R*) gravity (examples)

(i) Hu-Sawicki model [Hu and Sawicki, Phys. Rev. D 76, 064004 (2007)]

Cf. [Nojiri and Odintsov, Phys. Lett. B 657, 238 (2007); Phys. Rev. D 77, 026007 (2008)]

$$f_{\mathrm{HS}} = R - \frac{c_1 R_{\mathrm{HS}} \left(R/R_{\mathrm{HS}}\right)^p}{c_2 \left(R/R_{\mathrm{HS}}\right)^p + 1}$$

$c_1,\ c_2,\ p(>0),\ R_{\mathrm{HS}}(>0)$: Constant parameters

(ii) Starobinsky's model [Starobinsky, JETP Lett. 86, 157 (2007)]

$$f_{\mathrm{S}} = R + \lambda R_{\mathrm{S}} \left[\left(1 + \frac{R^2}{R_{\mathrm{S}}^2} \right)^{-n} - 1 \right]$$

$\lambda(>0),\ n(>0),\ R_{\mathrm{S}}$: Constant parameters

Models of *f*(*R*) gravity (examples) (2)

(iii) Tsujikawa's model

[Tsujikawa, Phys. Rev. D 77, 023507 (2008)]

$$f_{\rm T} = R - \mu R_{\rm T} \tanh\left(\frac{R}{R_{\rm T}}\right)$$

$\mu (> 0),\ R_{\rm T} (> 0)$: Constant parameters

(iv) Exponential gravity model

[Cognola, Elizalde, Nojiri, Odintsov, Sebastiani and Zerbini, Phys. Rev. D 77, 046009 (2008)]

[Linder, Phys. Rev. D 80, 123528 (2009)]

$$f_{\rm E} = R - \beta R_{\rm E} \left(1 - e^{-R/R_{\rm E}}\right)$$

$\beta,\ R_{\rm E}$: Constant parameters

Second part

Dynamical system analysis of Bianchi-I spacetimes in *f(R)* gravity

I. Purposes of this study

- We present a dynamical system analysis in terms of new expansion-normalized variables for homogeneous and anisotropic Bianchi-I spacetimes in $f(R)$ gravity under the presence of anisotropic matter.

※ Homogeneous and isotropic spacetime : [Leach, Carloni and Dunsby, Class. Quant. Grav. **23**, 4915 (2006)]
[Goheer, Leach and Dunsby, Class. Quant. Grav. **24**, 5689 (2007)]

※ Kasner type vacuum solution : [Barrow and Clifton, Class. Quant. Grav. **23**, L1 (2006)]
[Clifton and Barrow, Class. Quant. Grav. **23**, 2951 (2006)]

- We demonstrate that the Einstein's equations are reduced to an autonomous 5-dimensional system of ordinary differential equations for new variables.

- We make the autonomous system analysis in vacuum for the power-law form of $f(R)$ and show that the dynamics can be solved exactly.

II. Homogeneous and anisotropic Bianchi-I spacetimes in *f*(*R*) gravity in the presence of anisotropic matter

- Action

$$S = \frac{1}{2\kappa}\int d^4x\sqrt{-g}\,f(R) + S_M$$

$\kappa = 8\pi G$ G : Gravitational constant

$c = \hbar = 1$

S_M : Matter contributions to the total action

- Homogeneous and anisotropic Bianchi-I metric

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1}^{3} e^{2\beta_i(t)}(dx^i)^2$$

$a(t)$: Average scale factor

β_i $(i = 1, 2, 3)$: Quantities which characterize the anisotropies

$$\beta_1 + \beta_2 + \beta_3 = 0$$

$\longrightarrow$ We introduce the following relations.

$$\beta_\pm = \beta_1 \pm \beta_2 \Rightarrow \sigma^2 = \dot{\beta}_1^2 + \dot{\beta}_2^2 + \dot{\beta}_3^2 = \frac{3}{2}\dot{\beta}_+^2 + \frac{1}{2}\dot{\beta}_-^2$$

: Total amount of anisotropy in the metric

※ $\sigma = 0$: Homogeneous and isotropic case

＊The dot denotes the time derivative.

$$H = \frac{\dot{a}}{a}$$: Average Hubble parameter

- Scalar curvature: $R = 6\dot{H} + 12H^2 + \sigma^2$

- Energy-momentum tensor of an anisotropic barotropic fluid

$$T_\mu^\nu = \mathrm{diag}\left(-\rho, p_1, p_2, p_3\right) = \mathrm{diag}\left(-\rho, \omega_1\rho, \omega_2\rho, \omega_3\rho\right)$$

- Anisotropic equation of state

$$p_i = (\omega + \mu_i)\rho$$

ω : Average barotropic parameter (constant)

$$\omega_i = \omega + \mu_i \,,$$

$$\mu_1 + \mu_2 + \mu_3 = 0$$

$$\longrightarrow \mu_\pm = \mu_1 \pm \mu_2$$

- Set of field equations (Four equations for $H(t)$, $\rho(t)$, $\beta_\pm(t)$)

(i) $$3H^2 = \frac{\kappa}{f'}\left(\rho + \frac{Rf' - f}{2\kappa} - \frac{3Hf''\dot{R}}{\kappa}\right) + \frac{\sigma^2}{2}$$

$$* \; f' = \frac{df(R)}{dR}$$

(ii) $$2\dot{H} + 3H^2 = -\frac{\kappa}{f'}\left(\omega\rho + \frac{\dot{R}^2 f''' + \left(2H\dot{R} + \ddot{R}\right) f''}{\kappa} - \frac{Rf' - f}{2\kappa}\right) - \frac{\sigma^2}{2}$$

(iii) $\ddot{\beta}_\pm + \left(3H + \frac{\dot{R}f''}{f'}\right)\dot{\beta}_\pm = \frac{\kappa\rho}{F}\mu_\pm$

※ For perfect fluid: $\mu_+ = \mu_- = 0$

$\longrightarrow \dot{\sigma} + \left(3H + \frac{\dot{R}f''}{f'}\right)\sigma = 0$

(iv) $\dot{\rho} + \left(3H(1+\omega) + \boldsymbol{\delta}\cdot\dot{\boldsymbol{\beta}}\right)\rho = 0$

$$\boldsymbol{\delta}\cdot\dot{\boldsymbol{\beta}} = \mu_1\dot{\beta}_1 + \mu_2\dot{\beta}_2 + \mu_3\dot{\beta}_3 = \frac{3}{2}\mu_+\dot{\beta}_+ + \frac{1}{2}\mu_-\dot{\beta}_-$$

- We introduce the logarithmic time.

$$N = \epsilon \ln a$$

※ For the expansion universe: $\epsilon = +1$

※ Value of a at $t = 0$: $a_0 = 1$

※ $\dot{N} = \epsilon H$

- We use the following dimensionless expansion-normalized dynamical variables.

$$u_1 = \frac{\dot{R}f''}{f'H},\ u_2 = \frac{R}{6H^2},\ u_3 = \frac{f}{6f'H^2},\ u_4^+ = \frac{\dot{\beta}_+^2}{4H^2},\ u_4^- = \frac{\dot{\beta}_-^2}{12H^2},\ u_5 = \frac{\kappa\rho}{3f'H^2}$$

(i) $\longrightarrow$ $g = 1 + u_1 - u_2 + u_3 - u_4^+ - u_4^- - u_5 = 0$: Energy constraint

The 5-dimensional system of autonomous first order differential equations, which are equivalent to (ii)-(iv):

$$\epsilon\frac{du_1}{dN} = 1 + u_2 - 3u_3 - u_4 - 3\omega u_5 \; - u_1\left(u_1 + u_2 - u_4\right)$$

$$\epsilon\frac{du_2}{dN} = u_1 u_2 \gamma\left(\frac{u_2}{u_3}\right) - 2u_2\left(u_2 - u_4 - 2\right)$$

$$\epsilon\frac{du_3}{dN} = u_1 u_2 \gamma\left(\frac{u_2}{u_3}\right) - u_3\left(u_1 + 2u_2 - 2u_4 - 4\right)$$

$$\epsilon\frac{du_4^+}{dN} = -2u_4^+\left(1 + u_1 + u_2 - u_4\right) + 3\mu_+\sqrt{u_4^+}u_5$$

$$\epsilon\frac{du_4^-}{dN} = -2u_4^-\left(1 + u_1 + u_2 - u_4\right) + \mu_-\sqrt{3u_4^-}u_5$$

$$\epsilon\frac{du_5}{dN} = -u_5\left(3\omega - 1 + u_1 + 2u_2 - 2u_4 + 3\mu_+\sqrt{u_4^+} + \mu_-\sqrt{3u_4^-}\right)$$

- $u_4 = u_4^+ + u_4^- = \dfrac{\sigma^2}{6H^2}$
- $\gamma(R) = \dfrac{f'}{Rf''}$
- $\dfrac{u_2}{u_3} = \dfrac{Rf'}{f}$

※ $\epsilon\dfrac{dg}{dN} = -(u_1 + 2u_2 - 2u_4^+ - 2u_4^- - 1)g$

- Forms of $f(R)$ and the corresponding expressions of $\gamma = \frac{f'}{Rf''}$

(1) $f(R) = R^{1+\delta}$ $(\delta \neq 0)$

$f(R) = \alpha R^{1+\delta} + \Lambda$ (α, Λ: Constants) ⇒ $\gamma = \delta^{-1}$

(2) $f(R) = \alpha \ln R + \Lambda$ ⇒ $\gamma = \delta^{-1}, \ \delta \to -1$

(3) $f(R) = e^{\alpha R}$ (**Exponential gravity**) ⇒ $\gamma = \frac{u_3}{u_2}$

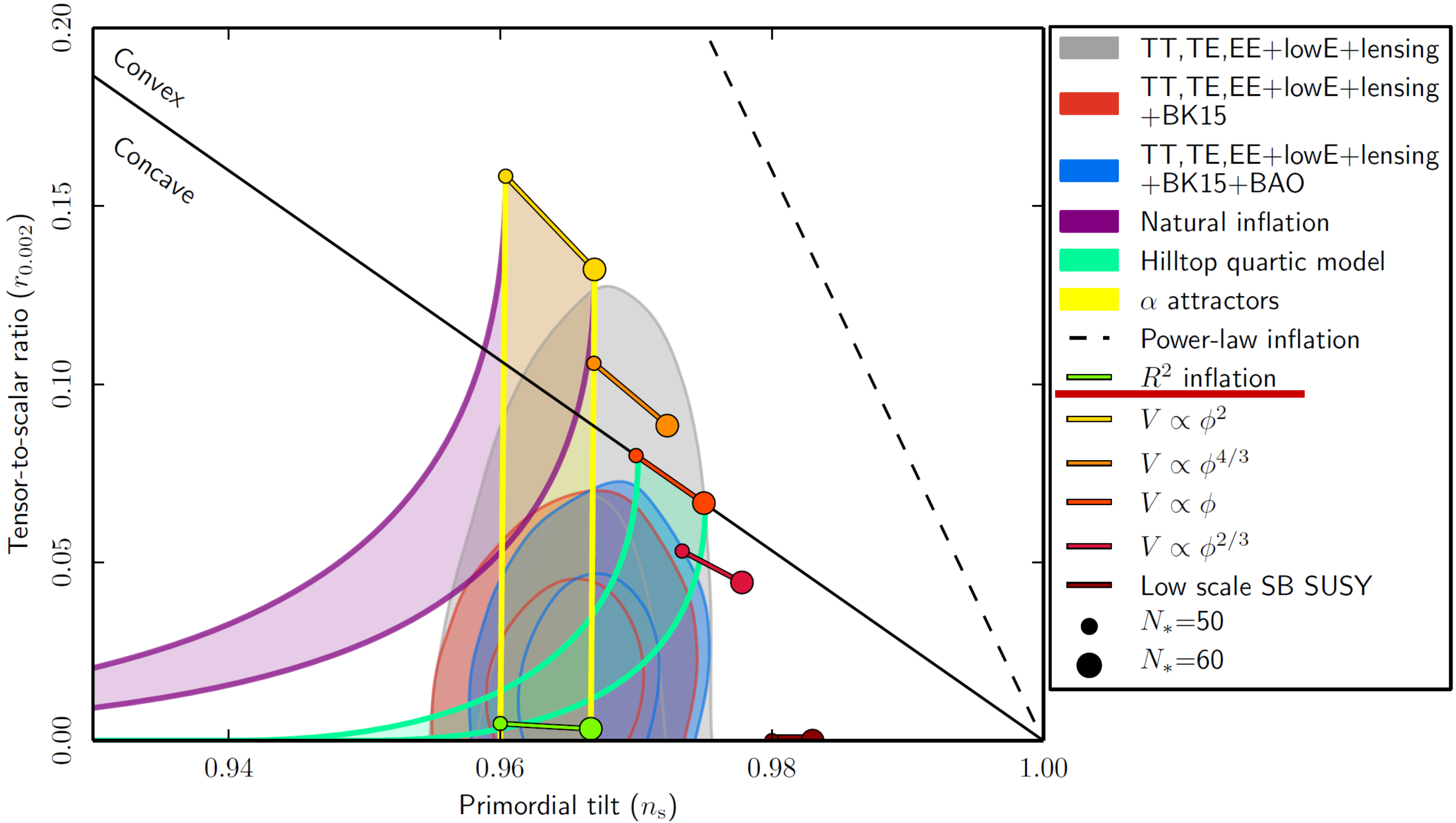
(4) $f(R) = R + \frac{\alpha}{R}$ ⇒ $\gamma = \frac{u_2}{u_3 - u_2}$

(5) $f(R) = R^a + \alpha R^b$ ⇒ $\gamma = \frac{u_2}{(b+a-1)u_2 - abu_3}$

(a, b : Constants, $a \neq b$)

※ $a = 1, \ b = 2$: **Starobinsky inflation** [Starobinsky, Phys. Lett. **91B**, 99 (1980)]

(This is consistent with the Planck results. (Cf. Next slide))



[Y. Akrami *et al*. [Planck Collaboration], Astron. Astrophys. **641**, A10 (2020)]

III. Applications to cosmology

A. $f(R) = R^{1+\delta}$ $(\delta \neq 0, -1)$, **case of vacuum (** $u_5 = 0$ **)**

$$u_3 = u_2 - u_1 + u_4 - 1\,, \qquad \frac{u_2}{u_3} = \frac{Rf'}{f} = 1 + \delta$$

$$\delta u_2 = (1+\delta)(u_1 - u_4 + 1)$$

⇨ Two-dimensional phase space spanned by the variables u_1 and u_4 :

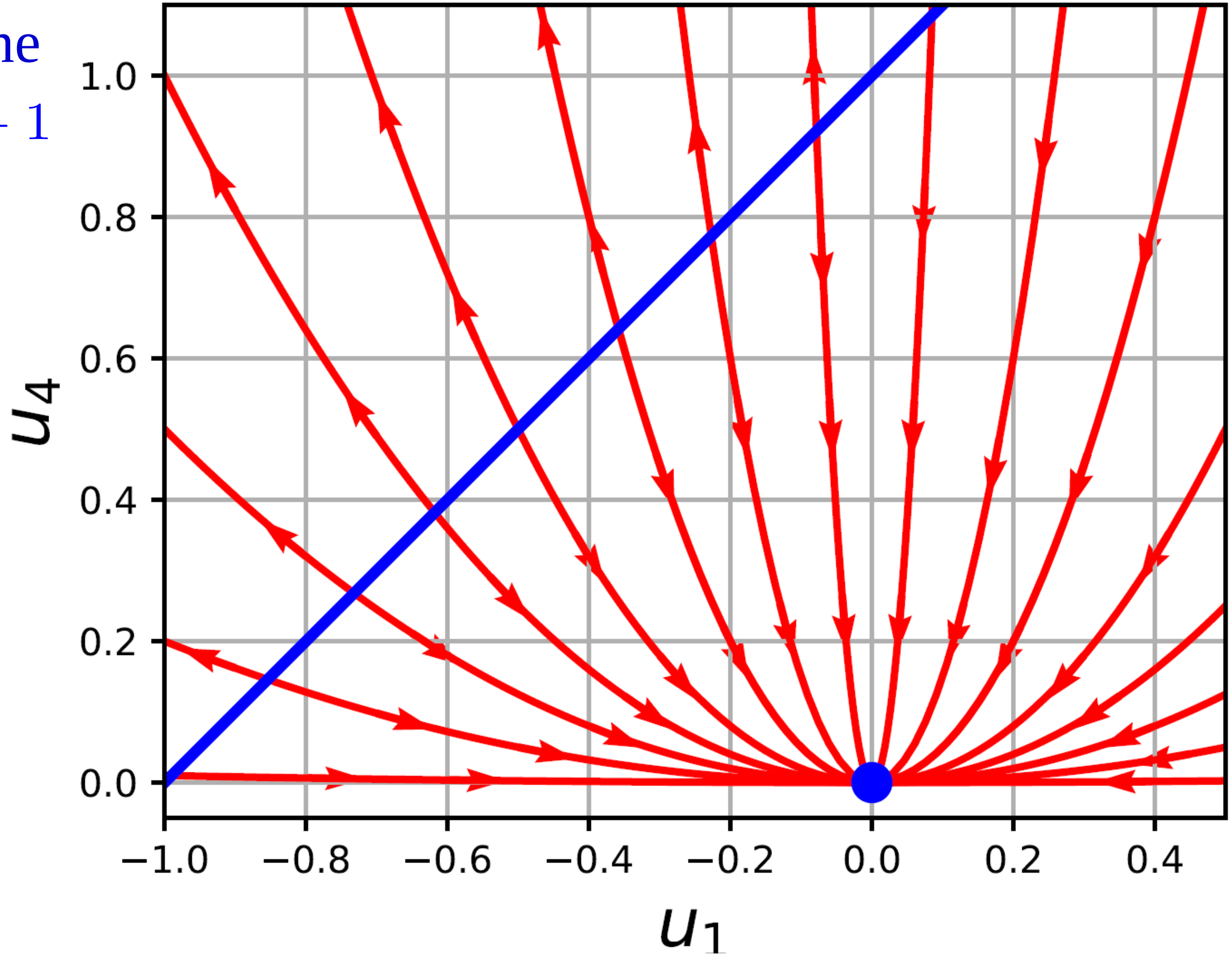
$$\frac{du_1}{dN} = \phi_1(u_1, u_4) = -\delta^{-1}(1+2\delta)(u_1 - u_1^*)(u_1 - u_4 + 1)$$

$$\frac{du_4}{dN} = \phi_4(u_1, u_4) = -2\delta^{-1}(1+2\delta)u_4\,(u_1 - u_4 + 1)$$

$$u_1^* = \frac{2(\delta - 1)}{1 + 2\delta}$$

Invariant line

$u_1 - u_4 = -1$



$(0, 0)$: Fixed point

de Sitter solution

FIG. 1 Phase space (u_1, u_4) for $\delta = 1$

⇨ The vacuum solutions in the case of $f(R) = R^{1+\delta}\ (\delta \neq -1)$, for $u_5 = 0$, are exactly soluble.

※ This is consistent with the past studies.

[Barrow and A.C. Ottewill, J. Phys. **A16**, 2757 (1983)]
[Maeda, Phys. Rev. D **37**, 858 (1988)]
[Barrow and Hervik, Phys. Rev. D **74**, 124017(2006)]

< Case of $\delta = 1$ >

⇨ The stable fixed point corresponds to the de Sitter solution.
(Solution of Starobinsky inflation)

$$a(t) = e^{Ht} \longrightarrow u_1 = u_4 = 0$$

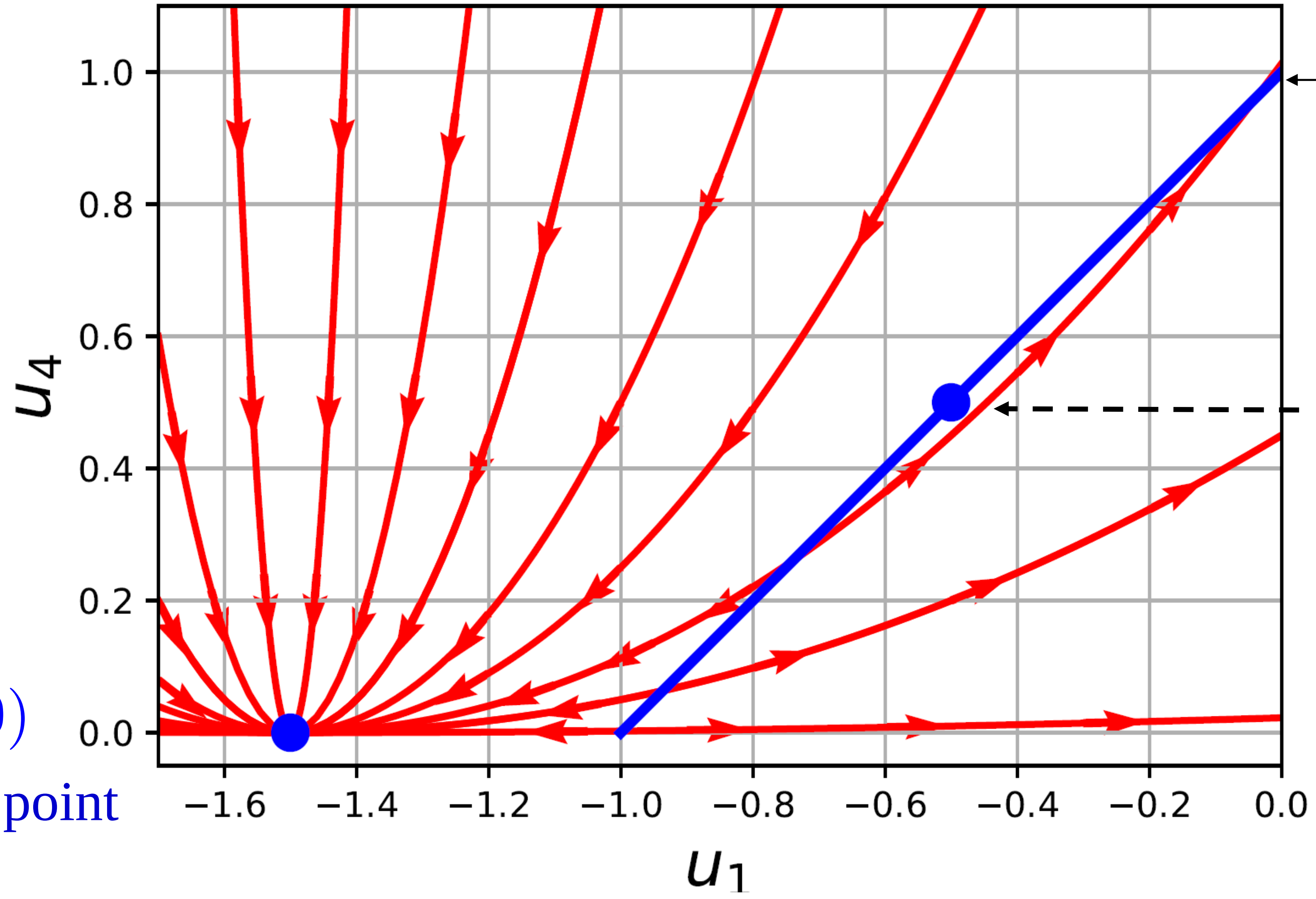
H : Arbitrary constant

- $R = 12H^2 \longrightarrow u_2 = 2, \quad u_3 = \dfrac{2f}{Rf'}$

⇨ Energy constraint $\longrightarrow Rf'(R) = 2f(R)$ ⇨ Unique solution : $f(R) = \alpha R^2$

⇨ The case of R^2 is unique among all vacuum *f*(*R*) theories with respect to the existence of a de Sitter solution with arbitrary *H*.

Invariant line
$u_1 - u_4 = -1$



$(-\frac{3}{2}, 0)$
: Fixed point

An attractive and a repulsive segment located, respectively, above and below the depicted point $(-\frac{1}{2}, \frac{1}{2})$.

FIG. 2 Phase space (u_1, u_4) for $\delta = \frac{1}{10}$

$<$ Case of $\delta \neq 1$ $>$

$$\dot{H} = \Delta H^2, \quad \Delta = \frac{\delta - 1}{\delta(1 + 2\delta)}$$

$$\Rightarrow \quad H(t) = \frac{H_0}{1 - \Delta H_0 (t - t_0)}, \quad H(t_0) = H_0$$

- $\Delta > 0$ ($-\frac{1}{2} < \delta < 0$ or $\delta > 1$) $\longrightarrow$ **Big rip singularity**

  （future finite time singularity）

- $\Delta < 0$ ($\delta < -\frac{1}{2}$ or $0 < \delta < 1$) $\longrightarrow$ $H \sim t^{-1}$: Power-law expansion

B. $f(R) = R^{1+\delta}\ (\delta \neq 0, -1)$, case in the presence of anisotropic matter

Autonomous system

$$\frac{du_1}{dN} = 1 + (\delta - 2)u_3 - u_4 - 3\omega u_5 - u_1\left(u_1 + (1+\delta)u_3 - u_4\right)$$

$$\frac{du_3}{dN} = u_3\left(\delta^{-1}u_1 - 2(1+\delta)u_3 + 2u_4 + 4\right)$$

$$\frac{du_4^+}{dN} = -2u_4^+\left(1 + u_1 + (1+\delta)u_3 - u_4\right) + 3\mu_+\sqrt{u_4^+}\,u_5$$

$$\frac{du_4^-}{dN} = -2u_4^-\left(1 + u_1 + (1+\delta)u_3 - u_4\right) + \mu_-\sqrt{3u_4^-}\,u_5$$

- $u_4 = u_4^+ + u_4^-$
- $u_5 = 1 + u_1 - \delta u_3 - u_4$: Energy constraint

- Four isolated fixed points (the solutions with $u_4^+ = u_4^- = 0$), namely the following values for the pair (u_1, u_3):

$$(-1,0)\,, \quad (1-3\omega,0)\,, \quad \left(\frac{2(\delta-1)}{1+2\delta}, \frac{4\delta-1}{\delta(1+2\delta)}\right), \quad \left(-\frac{3\delta(\omega+1)}{1+\delta}, \frac{4\delta+1-3\omega}{2(1+\delta)^2}\right)$$

In the case with anisotropic fluids, all isotropic fixed points are unstable (there is no asymptotically stable isotropic solutions in the presence of anisotropic matter).

※ For the case of exponential gravity ($f(R) = e^{\alpha R}$ $(\alpha \neq 0)$), we obtain the same consequences.

IV. Conclusions

- **We have analyzed the cosmological solutions for homogeneous and anisotropic Bianchi-I spacetimes in *f(R)* gravity under the existence of anisotropic matter.**
- **It has been demonstrated that the Einstein's equations are reduced to an autonomous 5-dimensional system of ordinary differential equations for new variables.**
- **By making the autonomous system analysis of the vacuum solutions for the power-law forms of *f*(*R*), we have shown that the dynamics can be solved exactly, and that only for the case of R^2, there exists a stable de Sitter solution (the solution of the Starobinsky inflation).**

Backup slides

Gravitational field equation

$$G_{\mu\nu} = \kappa^2 T_{\mu\nu}$$

Gravity **Matter**

$G_{\mu\nu}$: Einstein tensor

$T_{\mu\nu}$: Energy-momentum tensor

$$\kappa^2 \equiv 8\pi / {M_{\rm Pl}}^2$$

$M_{\rm Pl}$: Planck mass

(1) **General relativistic approach** $\longrightarrow$ **Dark Energy**

(2) **Extension of gravity theories**

Condition for accelerated expansion

Flat Friedmann-Lemaître-Robertson-Walker (FLRW) space-time

$$ds^2 = -dt^2 + a^2(t)\sum_{i=1,2,3}\left(dx^i\right)^2$$

$a(>0)$: Scale factor

Equation of $a(t)$ for a single perfect fluid

$$\frac{a}{a} = -\frac{\kappa^2}{6}(1+3w)\rho$$

ρ : Energy density

P : Pressure

∗ The dot denotes the time derivative.

$a > 0$: Accelerated expansion

$\Rightarrow$

$$w \equiv \frac{P}{\rho} < -\frac{1}{3}$$

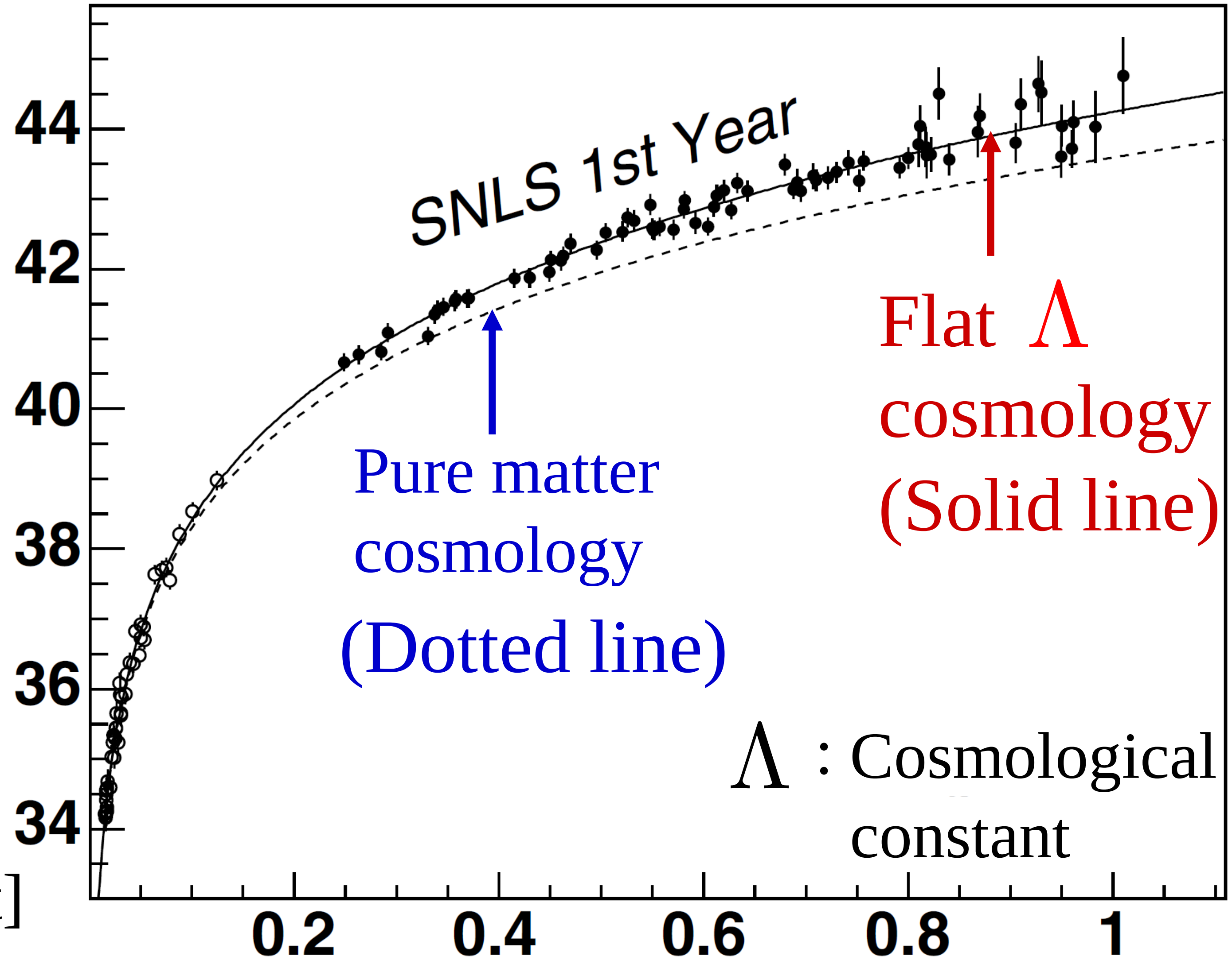
Cf. $w = -1$

: Cosmological constant

w : Equation of state (EoS) parameter

Distance estimator

SNLS data

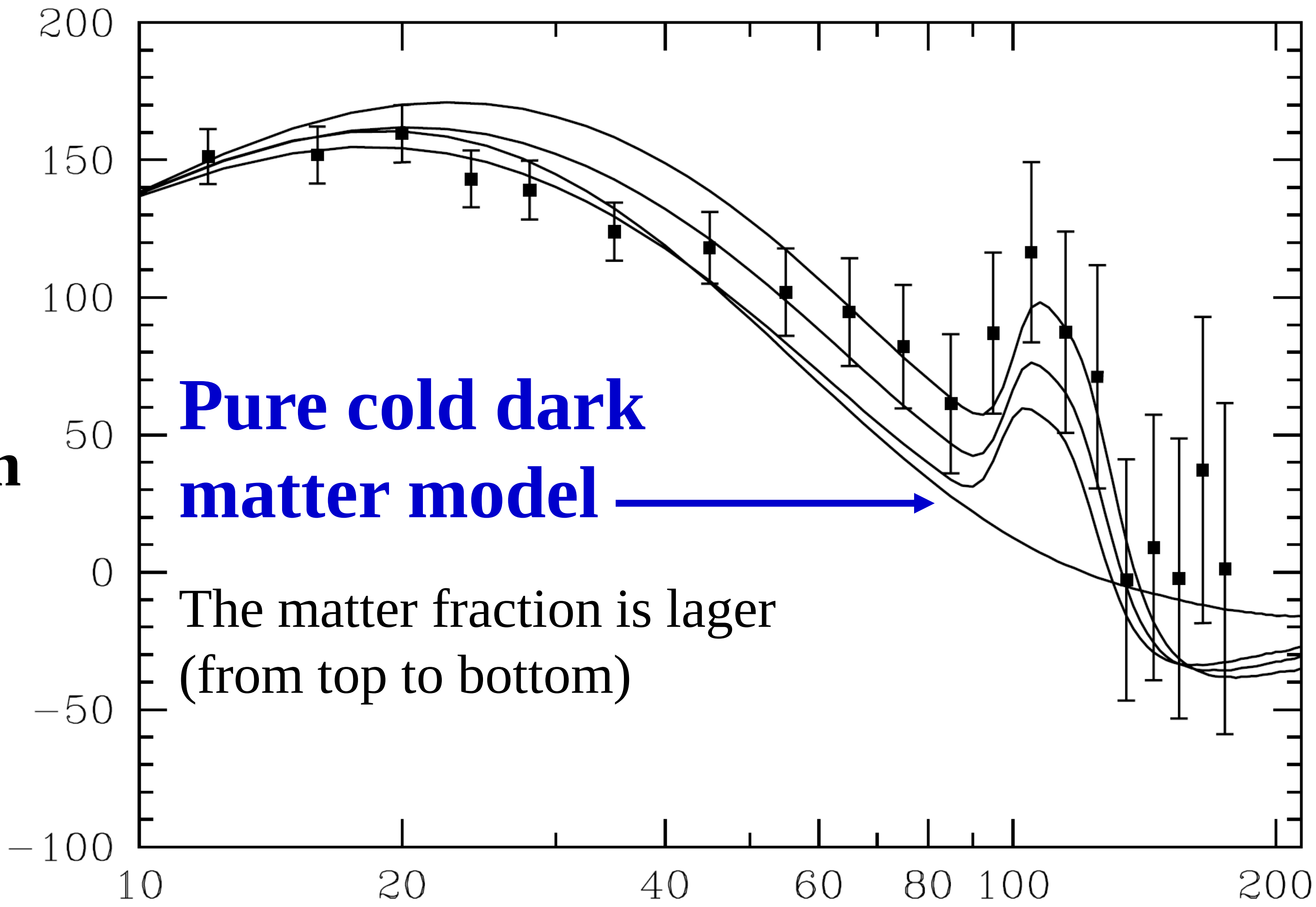


[Present] [Past]

Redshift

From [Astier *et al*. [The SNLS Collaboration], Astron. Astrophys. **447**, 31 (2006)].

Baryon acoustic oscillation (BAO)

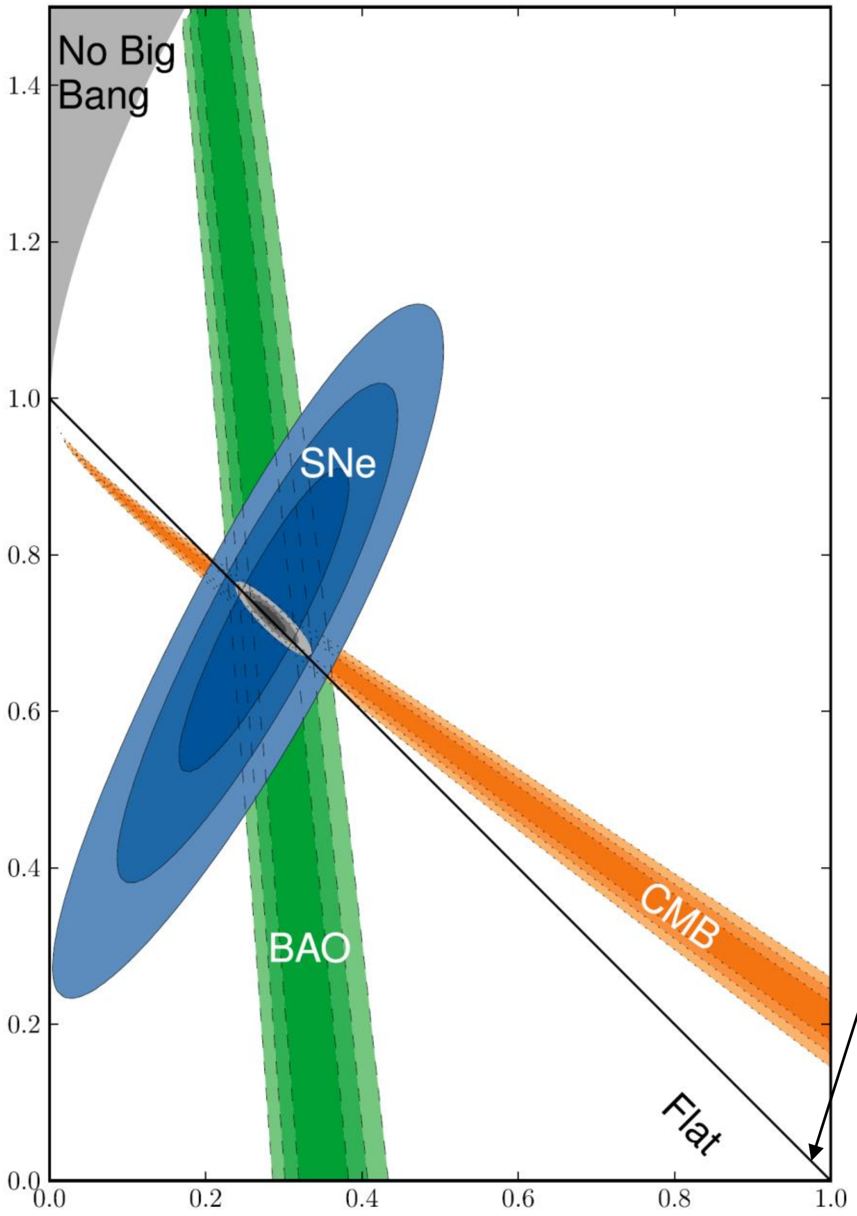


From [Eisenstein *et al*. [SDSS Collaboration], Astrophys. J. **633**, 560 (2005)].

SNe, BAO, and CMB

CMB: Cosmic Microwave Background Radiation

Fraction of dark enegy



From [Suzuki *et a*l., Astrophys. J. **746**, 85 (2012)].

Flat universe

Fraction of matter

Current three cosmic compositions

Dark Energy : 68.7%

Dark Matter : 26.4%

Other Matter (Baryon) : 4.9%

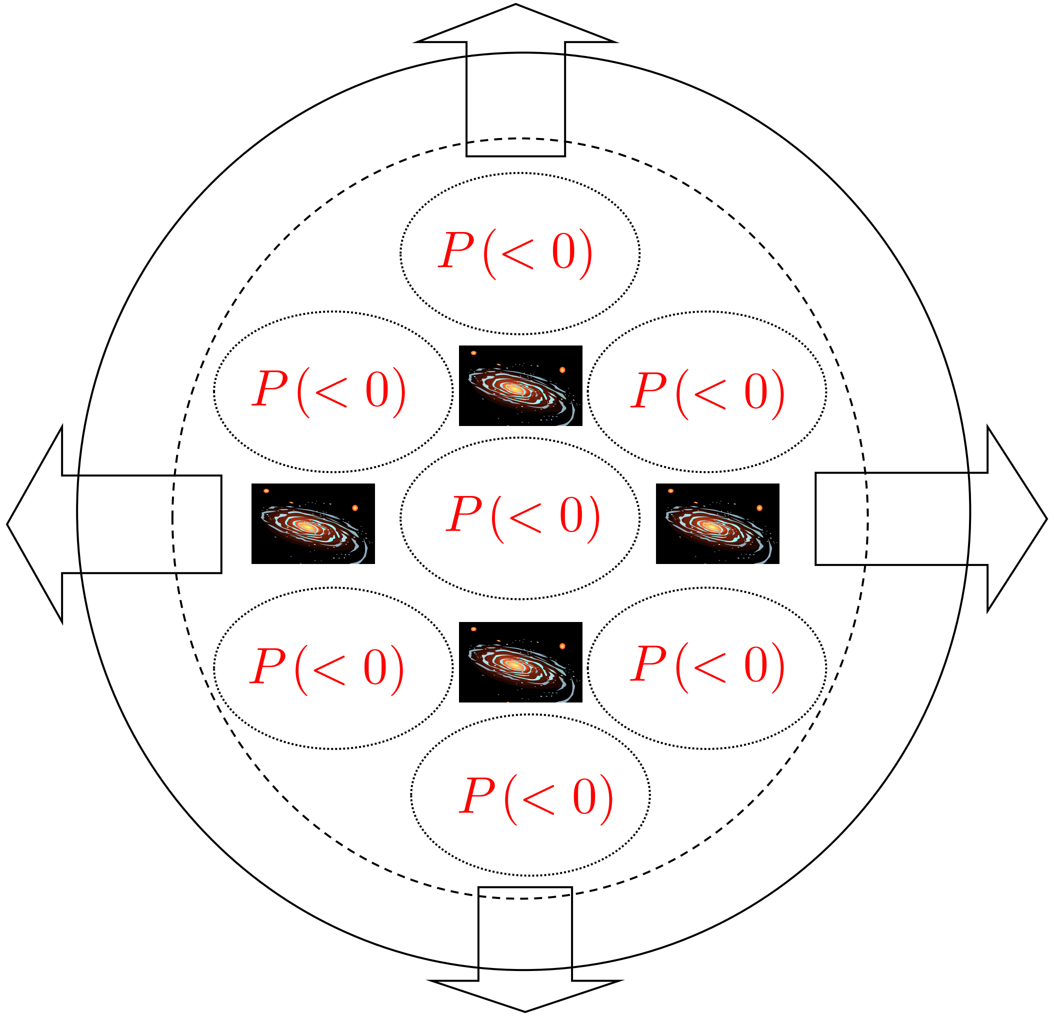
[N. Aghanim et al. [Planck Collaboration], Astron. Astrophys. 641, A6 (2020)]

Current cosmic acceleration

Universal Gravitation $<$ **|Negative Pressure|**

$P(<0)$

: Pressure



Planck data for the current w

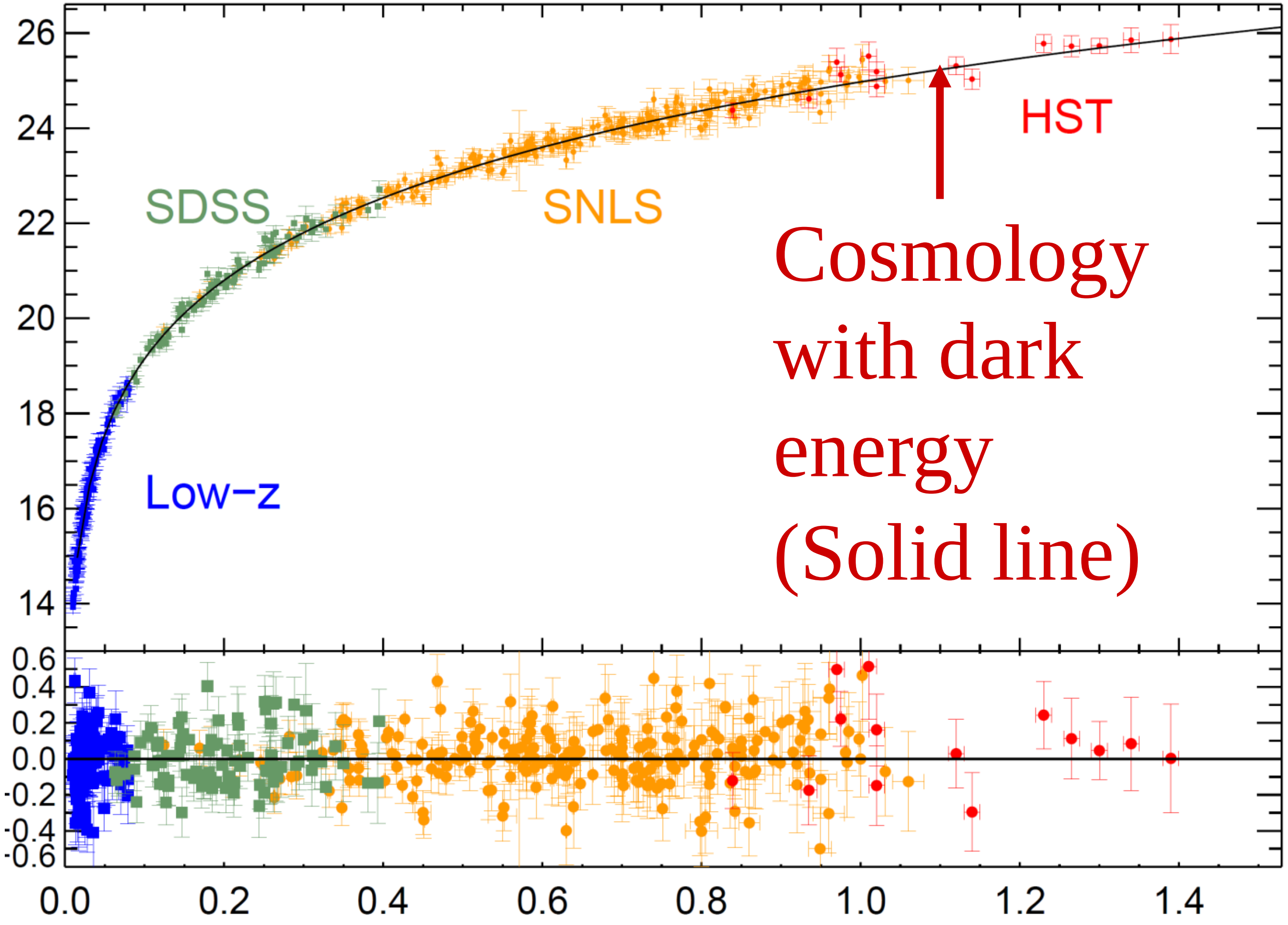
$$w = -1.019^{+0.075}_{-0.080} \quad \textit{Planck}\ \mathrm{TT, TE, EE+lowP+lensing+ext}\,.$$

$(95\%\ \mathrm{CL})$ [Ade *et al*. [Planck Collaboration], arXiv:1502.01589 [astro-ph.CO]]

⇨ **The current expansion of the universe is accelerating.**

SNLS data (2)

Distance estimator



[Present] [Past]

Redshift

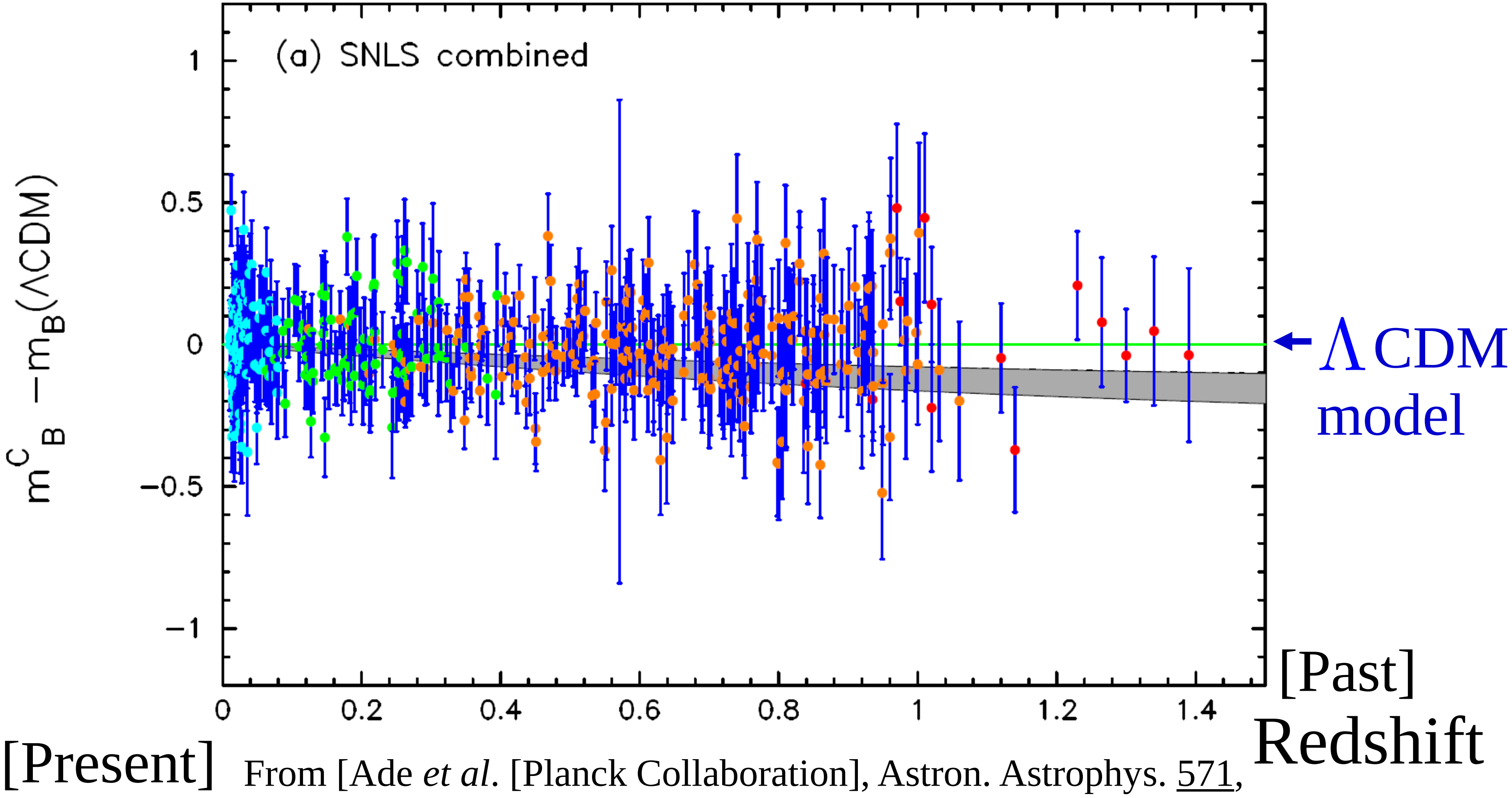
[Conley *et al*. [SNLS Collaboration], Astrophys. J. Suppl. <u>192</u>, 1 (2011)].

Cf. [Suzuki *et al*., Astrophys. J. <u>746</u>, 85 (2012)]

[arXiv:1303.5076 [astro-ph.CO]]

Planck 2013 results of SNLS

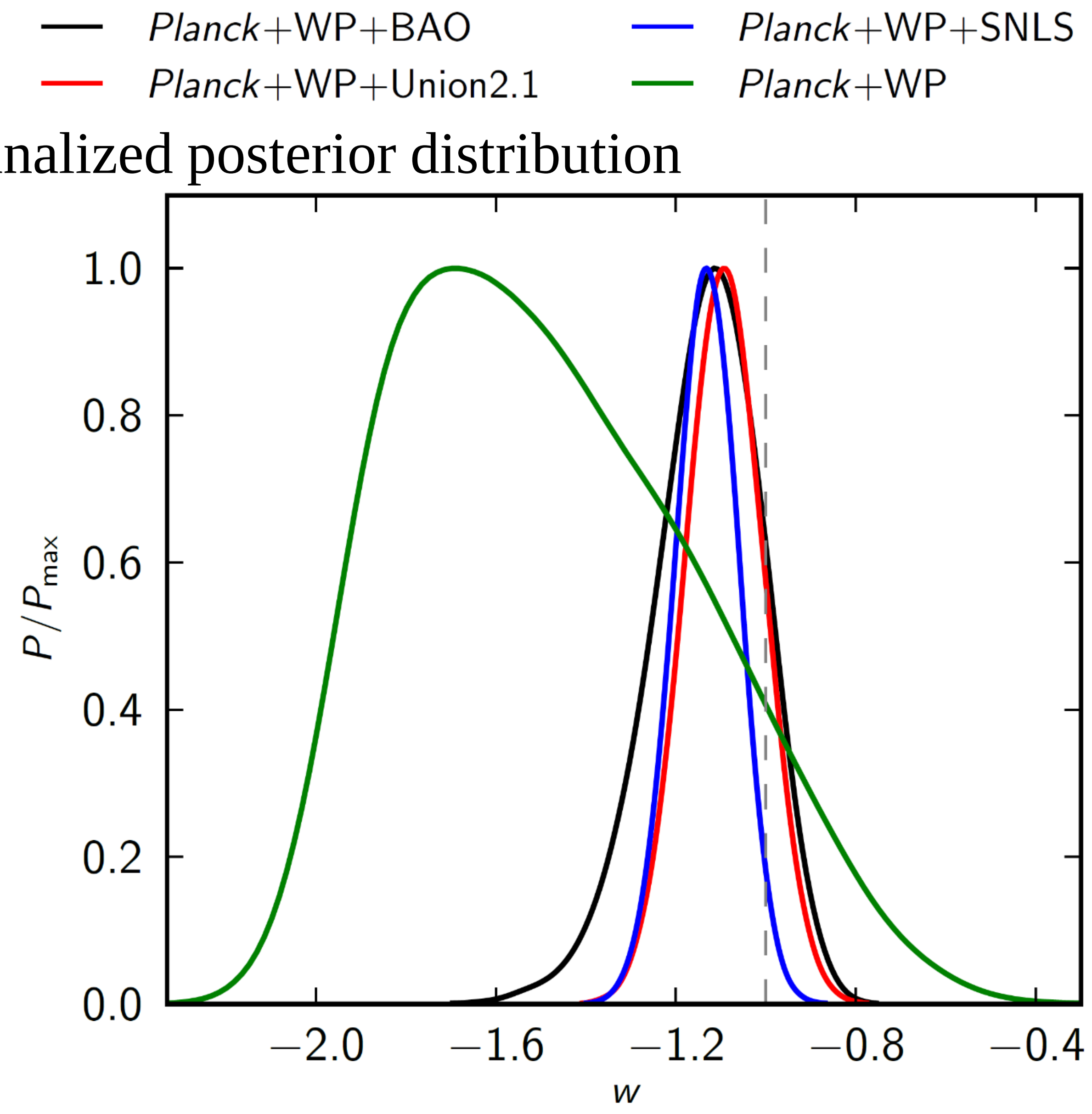
Magnitude residuals of the ΛCDM model that best fits the SNLS combined sample



From [Ade *et al*. [Planck Collaboration], Astron. Astrophys. 571, A16 (2014) [arXiv:1303.5076 [astro-ph.CO]]].

Planck data for the current w

Marginalized posterior distribution



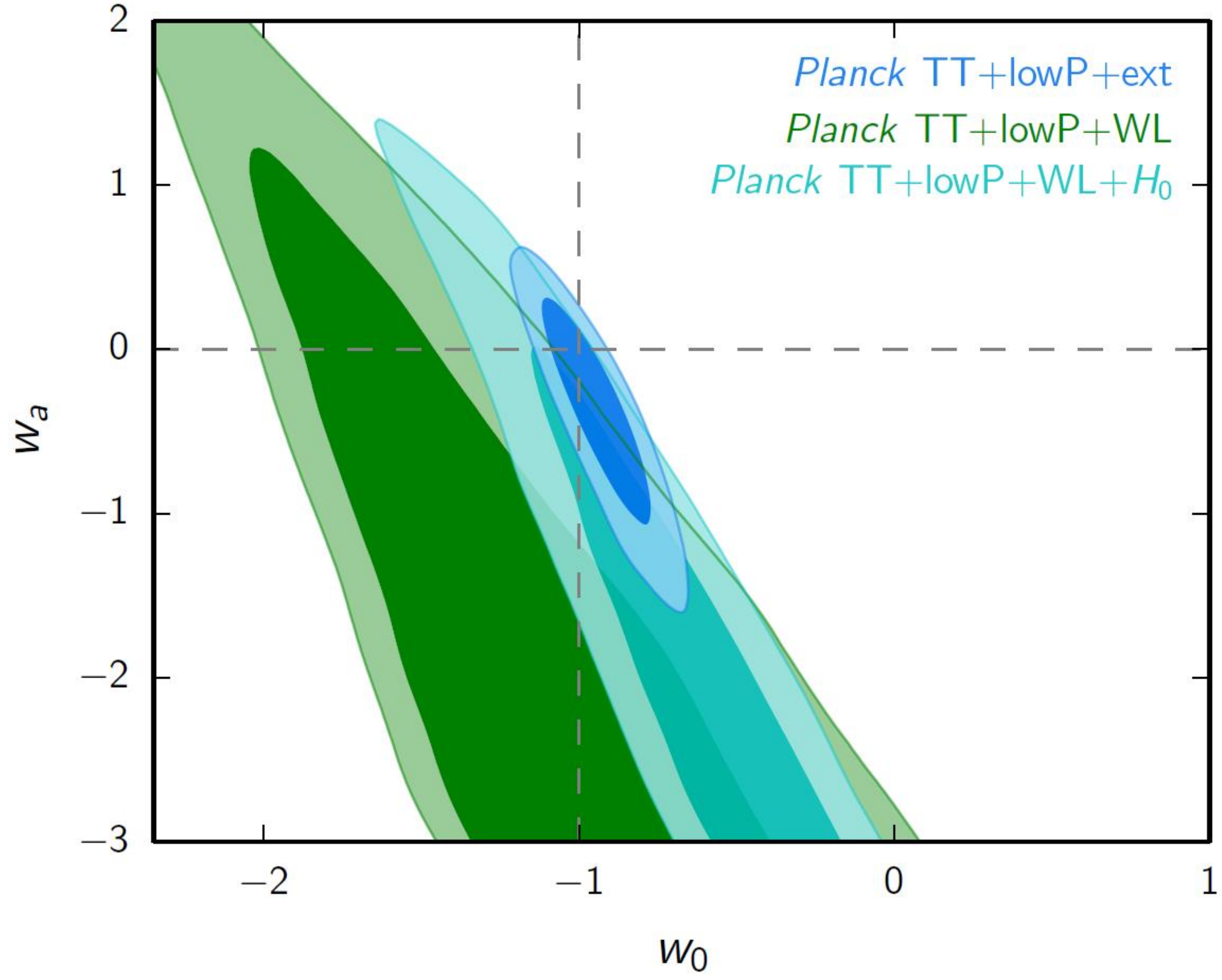
From [Ade *et al.* [Planck Collaboration], Astron. Astrophys. <u>571</u>, A16 (2014) [arXiv:1303.5076 [astro-ph.CO]]].

$$w = \text{constant}$$

WP: WMAP

Planck data for the time-dependent w

From [Ade *et al*. [Planck Collaboration], arXiv:1502.01589 [astro-ph.CO]].



$$w = w_0 + (1 - a)w_a$$

Cosmological constant

- Observational upper bound:

  Energy density of the current universe

$$\rho_{\mathrm{cr}0} = 4.2 \times 10^{-47} \mathrm{GeV}^4$$

- Planck density predicted by quantum field theory

$$< \rho_{\mathrm{v}} > = \int_0^{k_c} \frac{d^3k}{(2\pi)^3} \frac{1}{2} \sqrt{k^2 + m^2} \cong \frac{k_c^4}{16\pi^2} = \frac{M_{\mathrm{Pl}}^4}{16\pi^2}$$

$k_c = M_{\mathrm{Pl}}$: Cut-off scale

⇨ $$\frac{\rho_{\mathrm{cr}0}}{< \rho_{\mathrm{v}} >} \simeq 3.0 \times 10^{-121}$$

[Weinberg, Rev. Mod. Phys., 61, 1 (1989)]

: Unnaturally small

Cosmology in Teleparallelism

- **Teleparallel Dark Energy**

[Geng, Lee, Saridakis and Wu, Phys. Lett. B 704, 384 (2011)]

[Geng, Lee and Saridakis, JCAP 1201, 002 (2012)]

[Gu, Lee and Geng, Phys. Lett. B 718, 722 (2013)]

[Li, Wu and Geng, Phys. Rev. D 89, 044040 (2014)]

[Geng, Gu and Lee, Phys. Rev. D 88, 024030 (2013)]

[Li, Lee and Geng, Eur. Phys. J. C 73, 2315 (2013)]

- **Density Perturbations**

[Wu and Geng, Phys. Rev. D 86, 104058 (2012)]

[Wu and Geng, JHEP 1211, 142 (2012)]

[Geng and Wu, JCAP 1304, 033 (2013)]

- **Higher dimensional theories**

[Geng, Lai, Luo and Tseng, Phys. Lett. B 737, 248 (2014)]

[Geng, Luo and Tseng, Class. Quant. Grav. 31, 185004 (2014)]

Canonical scalar field

$$S_\phi = \int d^4x \sqrt{-g} \left[-\frac{1}{2} g^{\mu\nu} \partial_\mu \phi \partial_\nu \phi - V(\phi) \right]$$

$g = \det(g_{\mu\nu})$

ϕ : Scalar field

$V(\phi)$: Potential of ϕ

- For a homogeneous scalar field $\phi = \phi(t)$

$$\longrightarrow \rho_\phi = \frac{1}{2}\dot{\phi}^2 + V(\phi), \quad P_\phi = \frac{1}{2}\dot{\phi}^2 - V(\phi)$$

$$\Longrightarrow \boxed{w_\phi = \frac{P_\phi}{\rho_\phi} = \frac{\dot{\phi}^2 - 2V(\phi)}{\dot{\phi}^2 + 2V(\phi)}}$$

If $\dot{\phi}^2 \ll V(\phi)$, $\underline{w_\phi \approx -1.}$

$\longrightarrow$ Accelerated expansion can be realized.