Nature of Singularities in Vector-Tensor Theories of Gravity

V. H. Satheeshkumar
Departamento de Física, UNIRIO
Rio de Janeiro, Brazil

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Existence of new singularities in Einstein-Aether theory

R. Chan,[a] M.F.A. da Silva[b] and V.H. Satheeshkumar[c]

[a]Coordenação de Astronomia e Astrofísica, Observatório Nacional (ON), Rio de Janeiro, RJ 20921-400, Brazil
[b]Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro (UERJ), Rio de Janeiro, RJ 20550-900, Brazil
[c]Departamento de Física, Universidade Federal do Estado do Rio de Janeiro (UNIRIO), Rio de Janeiro, RJ 22290-240, Brazil

E-mail: chan@on.br, mfasnic@gmail.com, vhsatheeshkumar@gmail.com

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Spherically symmetric analytic solutions and naked singularities in Einstein–Aether theory

R. Chan[1,a], **M. F. A. da Silva**[2,b], **V. H. Satheeshkumar**[3,c]

[1] Coordenação de Astronomia e Astrofísica, Observatório Nacional (ON), Rio de Janeiro, RJ 20921-400, Brazil
[2] Departamento de Física Teórica, Universidade do Estado do Rio de Janeiro (UERJ), Rio de Janeiro, RJ 20550-900, Brazil
[3] Departamento de Física, Universidade Federal do Estado do Rio de Janeiro (UNIRIO), Rio de Janeiro, RJ 22290-240, Brazil

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Abstract We analyze all the possible spherically symmetric exterior vacuum solutions allowed by the Einstein–Aether theory with static aether. We show that there are three classes of solutions corresponding to different values of a combination of the free parameters, $c_{14} = c_1 + c_4$, which are: lytical solutions for the metric functions, up to the inversion of a transcendental function, assuming an aether vector proportional to the timelike Killing field. These solutions do not depend on the parameters c_2 and c_3 of the EA theory but only depend on the combination $c_1 + c_4$, which for simplicity we

Summary!

For the Friedmann-Lematre-Robertson-Walker metric, there exist three singular solutions in the Einstein-Aether theory which are not singular in the General Relativity, all of them for $k = -1$ with $\Lambda > 0$, $\Lambda = 0$ and $\Lambda < 0$. This result is cross-verified by showing the focusing of timelike geodesics using the Raychaudhuri equation.

For the most general spherically symmetric static metric, there exist analytical solutions for $c_{14} = 16/9$, $48/25$ and -16 which have no horizons, neither Killing horizon nor universal horizon, thus we have naked singularities. But the corresponding cases in General Relativity do have horizons.

Introduction

Introduction

The Vector-Tensor theories of gravity contain a dynamical, timelike, four-vector field besides the metric. In some models, the four-vector is unconstrained, while in others it is constrained to have unit norm.

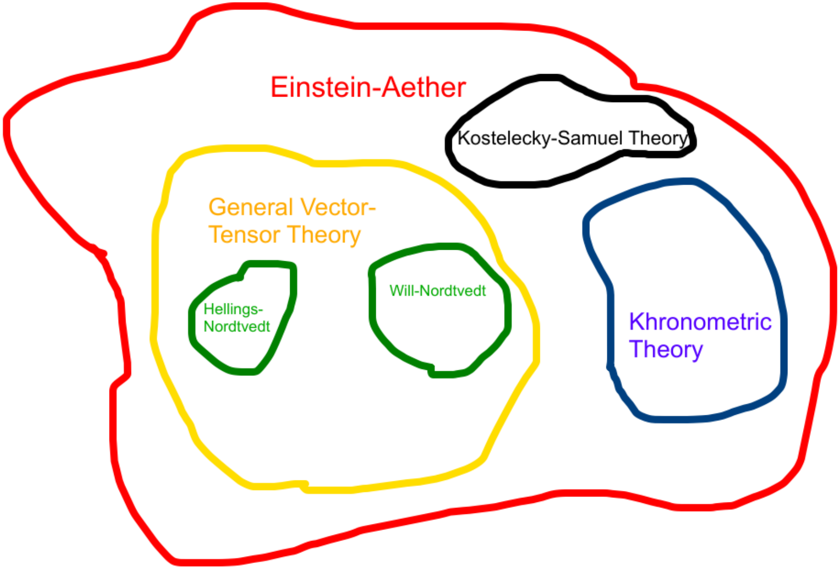
Constrained Theories

Einstein-Aether theory [2001]
Khronometric theory [2010] $\leftarrow$ Hořava-Lifshitz Gravity [2009]
Kostelecky-Samuel theory [1989] $\rightarrow$ Bumblebee Gravity [2004]

Unconstrained Theories

General vector-tensor theory [1981]
Hellings-Nordtvedt theory [1973]
Will-Nordtvedt theory [1972]

Introduction



Introduction

Einstein-aether theory is a generally covariant model in which local Lorentz invariance is broken by a dynamical unit timelike vector field dubbed the aether.

Einstein-aether theory has the most general diffeomorphism-invariant action involving a spacetime metric and a vector field with the field equations being the second-order differential equations in terms of not only the metric but also the aether field.

Introduction

The general action of the EA theory is given by,

$$S = \int \sqrt{-g}\,(L_{\rm Einstein} + L_{\rm aether} + L_{\rm matter})d^4x, \tag{1}$$

where,

$$L_{\rm Einstein} = \frac{1}{16\pi G}\left(R - 2\Lambda\right), \tag{2}$$

$$L_{\rm aether} = \frac{1}{16\pi G}[-K^{ab}{}_{mn}\nabla_a u^m \nabla_b u^n + \lambda(g_{ab}u^a u^b + 1)], \tag{3}$$

with

$$K^{ab}{}_{mn} = c_1 g^{ab} g_{mn} + c_2 \delta^a_m \delta^b_n + c_3 \delta^a_n \delta^b_m - c_4 u^a u^b g_{mn}, \tag{4}$$

and the c_i being dimensionless coupling constants, and λ a Lagrange multiplier enforcing the unit timelike constraint on the aether. The last term, $L_{\rm matter}$ is the matter Lagrangian.

In the weak-field, slow-motion limit EA theory reduces to Newtonian gravity with a value of Newton's constant G_{N} related to the parameter G in the action,

$$G = G_N \left(1 - \frac{c_1 + c_4}{2}\right). \tag{5}$$

The coupling constant G of EA theory is equal to the usual Newtonian gravitational constant G_N for $c_1 = -c_4$ and not necessarily $c_1 = c_4 = 0$.

The Newtonian limit is recovered only for $c_1 + c_4 < 2$.

If $c_1 + c_4 > 2$ gravity is repulsive, while for $c_1 + c_4 = 2$ the coupling constant G is zero, which means that the action blows up.

Initial Singularities

The most general isotropic and homogeneous universe is described by a FLRW metric,

$$ds^2 = -dt^2 + B(t)^2 \left[\frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2 \right], \tag{6}$$

where, $B(t)$ is the scale factor and k is a Gaussian curvature the space at a given time. According to observations by WMAP and Planck experiments, this metric is a good description of our universe as it is spatially homogeneous and isotropic when averaged over large scales. This leaves us with a choice of

$$u^a = (1, 0, 0, 0). \tag{7}$$

The standard definitions of the Hubble parameter $H(t)$, the deceleration parameter $q(t)$ and the redshift are given by, respectively,

$$H(t) = \frac{\dot{B(t)}}{B(t)}, \tag{8}$$

$$q(t) = -\frac{\ddot{B(t)}B(t)}{\dot{B(t)}^2}, \tag{9}$$

where the symbol dot denotes the differentiation with respect to the time coordinate. The Friedmann-Lemaître equations are given by,

$$\left(1+\frac{\beta}{2}\right)\left(\frac{\dot{B(t)}}{B(t)}\right)^2 = \frac{\Lambda}{3} - \frac{k}{B(t)^2}, \tag{10}$$

$$\left(1+\frac{\beta}{2}\right)\frac{\ddot{B(t)}}{B(t)} = \frac{\Lambda}{3}. \tag{11}$$

Initial Singularities

A singularity always implies focusing of geodesics, although focusing alone cannot imply a singularity as pointed out by Landau. Having already established the presence of singularities, we now use the focusing of timelike geodesics to reinforce our results.

$$\frac{d\theta}{d\tau} = -\frac{\theta^2}{3} - \sigma_{\mu\nu}\sigma^{\mu\nu} + w_{\mu\nu}w^{\mu\nu} - R_{\mu\nu}k^{\mu}k^{\nu}, \tag{12}$$

where $\theta, \sigma^{\mu\nu}, w^{\mu\nu}$ are respectively the expansion, shear and twist of the congruence of geodesics, and τ is the proper time along a geodesic with a tangent vector field, $k^{\mu} = dx^{\mu}/d\tau$. The Raychaudhuri equation has geometrical meaning and has no connection *a priori* to the gravitational theory which only enters through the term $-R_{\mu\nu}\xi^{\mu}\xi^{\nu}$.

For the FLRW metric, ($\tau = t$), assuming the vector $k^\mu = \delta_t^\mu$ as the four-velocity, both shear $\sigma^{\mu\nu}$ and twist $w^{\mu\nu}$ are zero, while the expansion and curvature terms for timelike geodesics are given by,

$$\theta = \frac{3}{2}\frac{\dot{B}(t)}{B(t)}, \tag{13}$$

$$-R_{\mu\nu}k^\mu k^\nu = -R_{tt}, \tag{14}$$

where R_{tt} is the component tt of the Ricci tensor. Thus, the expansion rate of congruence of timelike geodesics is given by

$$\frac{d\theta}{dt} = -\frac{3}{4}\left[\frac{\dot{B}(t)}{B(t)}\right]^2 + 3\frac{\ddot{B}(t)}{B(t)} = -3H^2\left(q + \frac{1}{4}\right). \tag{15}$$

$$\boxed{\Lambda > 0, k = -1}$$

The Friedmann-Lemaître equations yield the following two solutions for $\Lambda > 0, k = -1$.

$$B_1(t) = \frac{1}{2\sqrt{\Lambda}}\left[-\frac{3\,e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}}(t_0-t)}}{\epsilon\sqrt{\Lambda+3}+\sqrt{\Lambda}} + \frac{\epsilon\sqrt{\Lambda+3}+\sqrt{\Lambda}}{e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}}(t_0-t)}}\right], \tag{16}$$

$$B_2(t) = \frac{1}{2\sqrt{\Lambda}}\left[-\frac{3\,e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}}(t-t_0)}}{\epsilon\sqrt{\Lambda+3}+\sqrt{\Lambda}} + \frac{\epsilon\sqrt{\Lambda+3}+\sqrt{\Lambda}}{e^{\sqrt{\frac{2\Lambda}{3(\beta+2)}}(t-t_0)}}\right]. \tag{17}$$

The solution only exists for $\beta + 2 > 0$.

Initial Singularities

The Kretschmann scalar for the metric is singular at

$$t_{sing}(B_1) \qquad = t_0 - \sqrt{\frac{3(\beta+2)}{8\Lambda}} \ln\left[\frac{2}{3}\Lambda + \frac{2}{3}\epsilon\sqrt{\Lambda(\Lambda+3)} + 1\right], \tag{18}$$

$$t_{sing}(B_2) \qquad = t_0 + \sqrt{\frac{3(\beta+2)}{8\Lambda}} \ln\left[\frac{2}{3}\Lambda + \frac{2}{3}\epsilon\sqrt{\Lambda(\Lambda+3)} + 1\right]. \tag{19}$$

The metric is not singular for $\beta = 0$ since in this time the curvature invariant being $\frac{8\Lambda^2}{3}$.
But, the metric is singular for $\beta + 2 > 0$ with $\beta \neq 0$. This means it is a new singularity.

Initial Singularities

The Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau}(B_1) = \frac{3\Lambda}{2(\beta+2)} \frac{1}{\left[-3(\beta+2) + e^{\frac{2}{3}\sqrt{\frac{6\Lambda}{\beta+2}}(-t+C1)}\right]^2} \times \left[e^{\frac{4}{3}\sqrt{\frac{6\Lambda}{\beta+2}}(-t+C1)} - 10(\beta+2)e^{\frac{2}{3}\sqrt{\frac{6\Lambda}{\beta+2}}(-t+C1)} + 9(\beta+2)^2\right], \quad (20)$$

$$\frac{d\theta}{d\tau}(B_2) = \frac{3\Lambda}{2(\beta+2)} \frac{1}{\left[-(3\beta+2) + e^{-\frac{2}{3}\sqrt{\frac{6\Lambda}{\beta+2}}(-t+C2)}\right]^2} \times \left[e^{-\frac{4}{3}\sqrt{\frac{6\Lambda}{\beta+2}}(-t+C2)} - 10(\beta+2)e^{-\frac{2}{3}\sqrt{\frac{6\Lambda}{\beta+2}}(-t+C2)} + 9(\beta+2)^2\right]. \quad (21)$$

The expansion rate is negative only for a certain interval of time, assuring the convergence of the congruences of timelike geodesics.

$\boxed{\Lambda = 0, k = -1}$

The Friedmann-Lemaître equations yield the following two solutions for $\Lambda = 0, k = -1$,

$$B_1(t) = 1 + \sqrt{\frac{2}{2+\beta}}(t_0 - t) \tag{22}$$

$$B_2(t) = 1 + \sqrt{\frac{2}{2+\beta}}(t - t_0) \tag{23}$$

The solution only exists for $\beta + 2 > 0$.

Initial Singularities

The Kretschmann scalar for the metric is singular at

$$t_{sing}(B_1) = t_0 + \sqrt{\frac{2+\beta}{2}}, \tag{24}$$

$$t_{sing}(B_2) = t_0 - \sqrt{\frac{2+\beta}{2}}. \tag{25}$$

For $\beta = 0$ which corresponds to GR, the solution exists and is never singular with the curvature invariant being null. But, the metric is singular for $\beta + 2 > 0$ such that $\beta \neq 0$. This means it is a new singularity.

The Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau}(B_1) = -\frac{3}{\left[-2t + C1\sqrt{2(\beta+2)}\right]^2}, \tag{26}$$

$$\frac{d\theta}{d\tau}(B_2) = -\frac{3}{\left[-2t + C2\sqrt{2(\beta+2)}\right]^2}. \tag{27}$$

$\boxed{\Lambda < 0, k = -1}$

The Friedmann-Lemaître equations yield the following two solutions for $\Lambda < 0, k = -1$,

$$B_1(t) = \sqrt{\frac{3}{|\Lambda|}} \sin\left(\sqrt{\frac{2|\Lambda|}{3(\beta+2)}}(t - t_0) + \sin^{-1}\sqrt{\frac{|\Lambda|}{3}}\right) \tag{28}$$

$$B_2(t) = \sqrt{\frac{3}{|\Lambda|}} \sin\left(\sqrt{\frac{2|\Lambda|}{3(\beta+2)}}(t_0 - t) + \sin^{-1}\sqrt{\frac{|\Lambda|}{3}}\right) \tag{29}$$

The solution only exists for $\beta + 2 > 0$.

Initial Singularities

The Kretschmann scalar for the metric is singular at

$$t_{sing}(B_1) = t_0 - \sqrt{\frac{3(\beta+2)}{2|\Lambda|}} \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \quad (30)$$

$$t_{sing}(B_2) = t_0 + \sqrt{\frac{3(\beta+2)}{2|\Lambda|}} \sin^{-1} \sqrt{\frac{|\Lambda|}{3}} \quad (31)$$

For $\beta = 0$ the solution exists and is never singular with the curvature invariant being $\frac{8|\Lambda|^2}{3}$. But, the metric is singular for $\beta + 2 > 0$ with $\beta \neq 0$. This means it is a new singularity.

Initial Singularities

The Raychaudhuri equation is given by

$$\frac{d\theta}{d\tau}(B_1) = -\frac{|\Lambda|}{2(\beta+2)} \frac{3\cos^2\left[\sqrt{\frac{2|\Lambda|}{3(\beta+2)}}(-t+C1)\right]-4}{\cos^2\left[\sqrt{\frac{2|\Lambda|}{3(\beta+2)}}(-t+C1)\right]-1}, \tag{32}$$

$$\frac{d\theta}{d\tau}(B_2) = -\frac{|\Lambda|}{2(\beta+2)} \frac{3\cos^2\left[\sqrt{\frac{2|\Lambda|}{3(\beta+2)}}(-t+C2)\right]-4}{\cos^2\left[\sqrt{\frac{2|\Lambda|}{3(\beta+2)}}(-t+C2)\right]-1}. \tag{33}$$

The negative expansion rate is assuring the convergence of the congruences of timelike geodesics.

Naked Singularities

We start with the most general spherically symmetric static metric

$$ds^2 = -e^{2A(r)}dt^2 + e^{2B(r)}dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\phi^2. \tag{34}$$

The aether field is assumed to be unitary and timelike, chosen as

$$u^a = (e^{-A(r)}, 0, 0, 0). \tag{35}$$

This choice is not the most general and is restricted to the scenario where aether is static. The aether must tip in a black hole solution as it cannot be timelike be aligned with the null Killing vector on the horizon. As that is not the case with our choice, our solutions are valid only outside the Killing horizon. This is good enough for solar system tests and even for astrophysical solutions to describe the exterior spacetime to a source.

The timelike Killing vector of the metric is giving by

$$\chi^a = (-1, 0, 0, 0). \tag{36}$$

The Killing and the universal horizon are obtained finding the largest root of

$$\chi^a \chi_a = 0, \tag{37}$$

and

$$\chi^a u_a = 0, \tag{38}$$

respectively, where χ^a is the timelike Killing vector. In our case,

$$\chi^a \chi_a = -e^{2A(r)}, \tag{39}$$

$$\chi^a u_a = e^{A(r)}. \tag{40}$$

Solution for $c_{14} = 16/9$

From the Kretschmann scalar, we can get the singularities which are at

$$r_{sing1} = 0,$$
$$r_{sing2} = 8\alpha. \tag{41}$$

Since the radial coordinate is always positive, the second singularity does not exist, since $\alpha \leq 0$. So, the singularity at r_{sing1} is physical and

$$\lim_{r \to r_{sing1}^-} K = +\infty, \tag{42}$$

$$\lim_{r \to r_{sing1}^+} K = -\infty, \tag{43}$$

which is independent of α.

Naked Singularities

$$\chi^a \chi_a = -r^{-\frac{3}{2}} \left(r - 4\alpha + \sqrt{(r - 8\alpha) r} \right)^{\frac{3}{2}} = 0, \tag{44}$$

$$\chi^a u_a = r^{-\frac{3}{4}} \left(r - 4\alpha + \sqrt{(r - 8\alpha) r} \right)^{\frac{3}{4}} = 0. \tag{45}$$

We can see easily again these equations do not have any root with $\alpha < 0$, hence, there exists no horizon, neither Killing nor universal horizon.

Naked Singularities

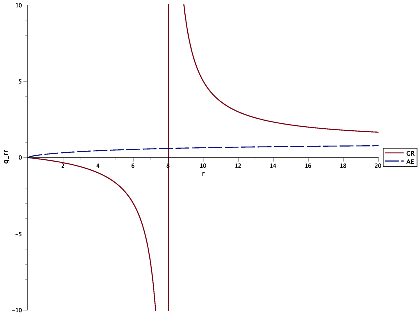


FIGURE: Plot of the metric component g_{rr}, for EA parameters $c_{14} = 16/9$, $\alpha = -1$ and an arbitrary Schwarzschild mass $M = 4$. The continuous red line represents the GR Schwarzschild metric. The dashed blue line represents the EA solution. Calculating the limit $r \to +\infty$ we obtain $g_{rr} = e^{2B} \to 1$, as we can see in this figure. Calculating the limit $r \to r_{sing1} = 0$ we obtain $g_{rr} = e^{2B} \to 0$, as we can see in this figure.

Solution for $c_{14} = 48/25$

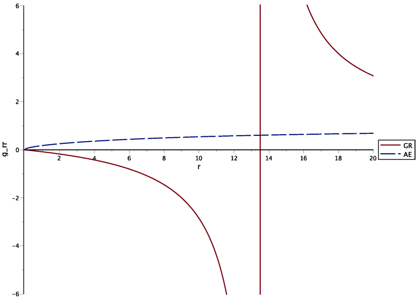


Figure: Plot of the metric component g_{rr}, for EA parameters $c_{14} = 48/25$, $|\gamma| = 1$ and an arbitrary Schwarzschild mass $M = 27/4$. The continuous red line represents the GR Schwarzschild metric. The dashed blue line represents the EA solution. Calculating the limit $r \to +\infty$ we obtain $g_{rr} = e^{2B} \to 1$, as we can see in this figure. Calculating the limit $r \to r_{...} = 0$ we obtain $g_{rr} = e^{2B} \to 0$, as we can see in this figure.

Naked Singularities

Solution for $c_{14} = -16$

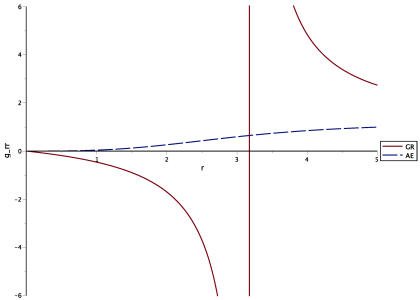


Figure: Plot of the metric component g_{rr}, for EA parameters $c_{14} = -16$, $|\kappa| = 1$ and an arbitrary Schwarzschild mass $M = 32^{\frac{1}{3}}/2$. The continuous red line represents the GR Schwarzschild metric. The dashed blue line represents the EA solution. Calculating the limit $r \to +\infty$ we obtain $g_{rr} = e^{2B} \to 1$, as we can see in this figure. Calculating the limit $r \to r$[illegible] $= 0$ we obtain $g_{rr} = e^{2B} \to 0$, as we can see in this figure.

Conclusions

Conclusions

New singularities (both cosmological and black hole) appear in Einstein-aether theory that are otherwise absent in General Relativity.

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