Gravitational tidal forces bring Newton's equivalence principle to life in quantum mechanics

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FH, P. Sadeghi, N. Fleury and A. Leblanc, arXiv:2102.09534

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Outline

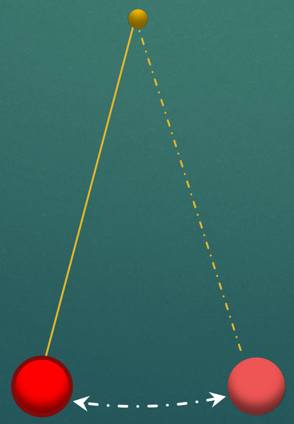
1. The equivalence principle & quantum mechanics
2. Two famous experiments with cold neutrons
3. Three ways of exploiting gravitational tidal forces
4. A remarkable mass-independence when $m_g = m_i$
5. What does general relativity say about this?
6. Experimental challenges
7. Conclusion

1 The equivalence principle & quantum mechanics

The equivalence principle in classical mechanics might be traced back to John Philoponus (John of Alexandria 490~570 AD), who first observed that two balls of different masses fall at the same rate.

Only much later did Simon Stevin (~1586 using the Delft Church tower) and Galileo Galilei (~1610 using inclined planes) redo the experiments.

A more specific experiment to test the distinction between gravitational mass m_g and inertial mass m_i goes back to Isaac Newton (~1680) who used pendulums.

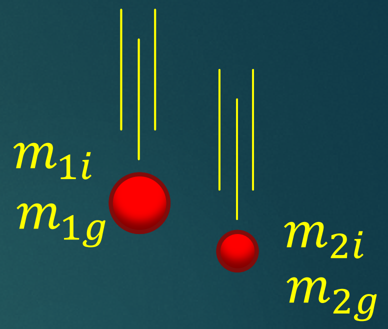


1 The equivalence principle & quantum mechanics

According to Newton's 2nd Law: $F = m_i a$

According to universal gravity: $F = m_g g$

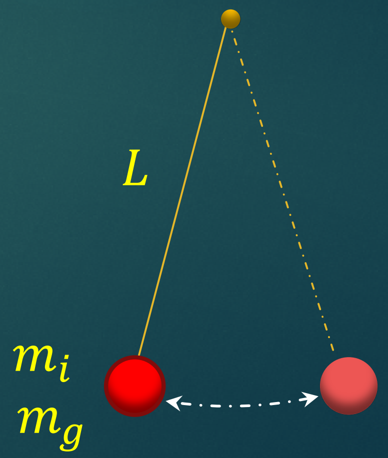
$$a = \frac{m_g}{m_i} g$$



According to Newton's 2nd Law: $M = m_i L \ddot{\theta}$

According to universal gravity: $M = m_g g \theta$

$$\ddot{\theta} + \frac{m_g}{m_i}\frac{g}{L}\theta = 0$$



Classical experiments confirm the universality of free fall at the 10^{-14} precision level (P. Touboul *et al.*, PRL (2017))

Quantum experiments confirm the universality of free fall at:

H. Albers *et al.*, EPJD (2020) (10^{-7} precision level)

G. Rosi *et al.*, NC (2017) (10^{-9} precision level)

P. Asenbaum *et al.*, PRL (2020) (10^{-12} precision level)

1 The equivalence principle & quantum mechanics

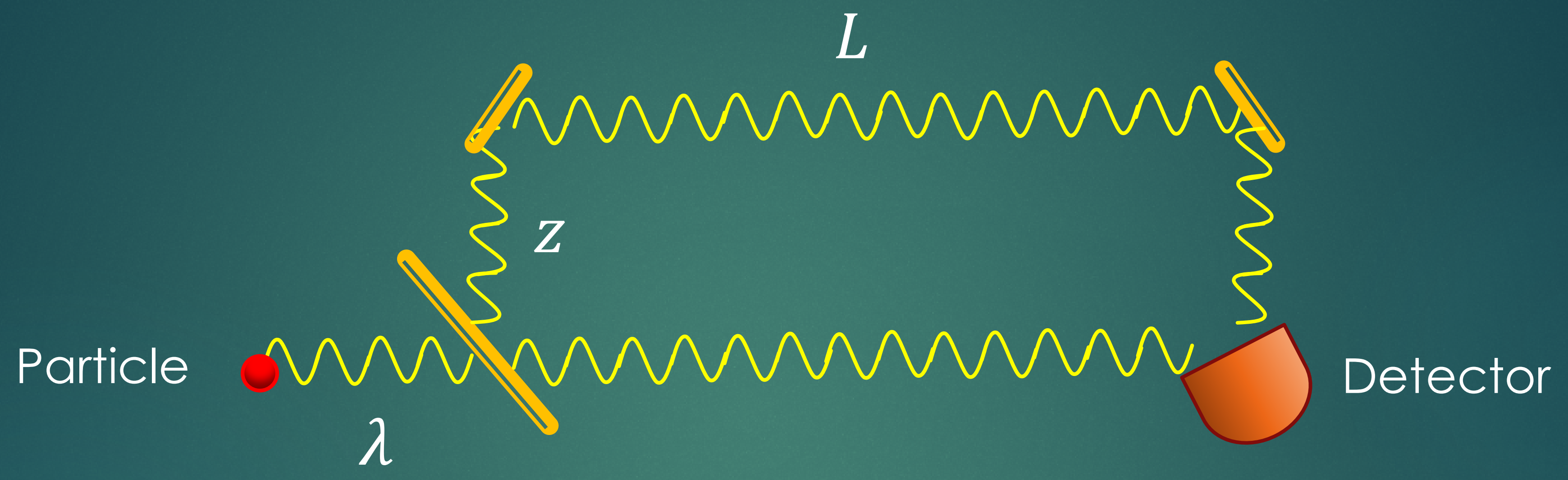
Quantum mechanics does usually not display an isolated ratio: $\frac{m_g}{m_i}$

Because

the Schrödinger equation with a gravitational potential reads:

$$-\frac{\hbar^2}{2m_i}\nabla^2\psi + m_g V(r)\,\psi = i\hbar\partial_t\psi$$

2 Two famous experiments with cold neutrons

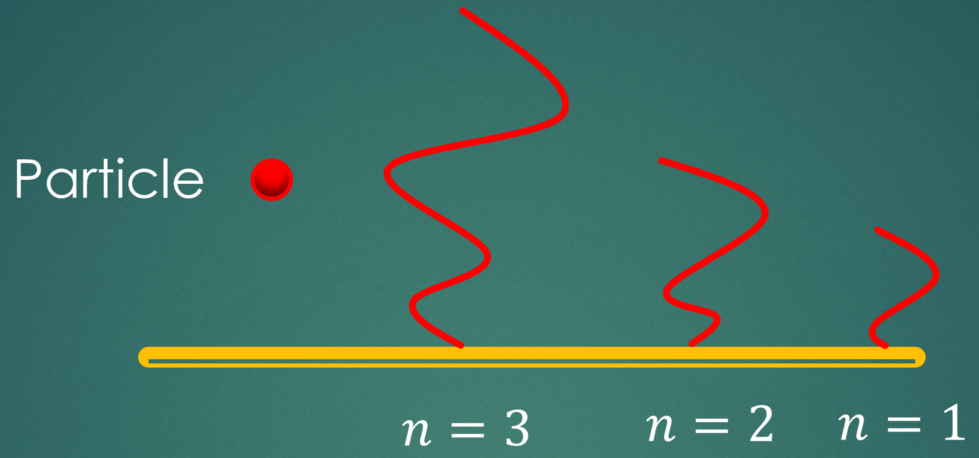


$$\Delta\phi = -m_i m_g \frac{gz\lambda L}{2\pi\hbar^2}$$

COW experiment

R. Colella, A.W. Overhauser and S.A. Werner, PRL (1975)

2 Two famous experiments with cold neutrons



$$E_n \approx -\left[\frac{m_g^2}{m_i}\frac{9\pi^2\hbar^2 g^2}{8}\left(n-\frac{1}{4}\right)^2\right]^{\frac{1}{3}}$$

Q-bounce experiment

R.V.V. Nesvizhevsky *et al.*, PRD (2003)

2 Two famous experiments with cold neutrons

It seems then that, because of the form of the Schrödinger equation

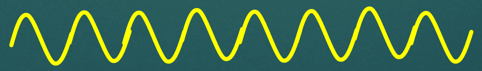
$$-\frac{\hbar^2}{2m_i}\nabla^2\psi + m_g V(r)\psi = i\hbar\partial_t\psi,$$

the dynamics of a quantum particle cannot involve the ratio $\frac{m_g}{m_i}$

Both results, however, were limited to the linear approximation of the gravitational potential:

$$m_g V(r) = -m_g\frac{GM}{r} = -m_g\frac{GM}{R_E + z} \cong m_g\frac{GM}{R_E^2}z$$

… and did not fully exploit the spatial extension of the wavefunction:

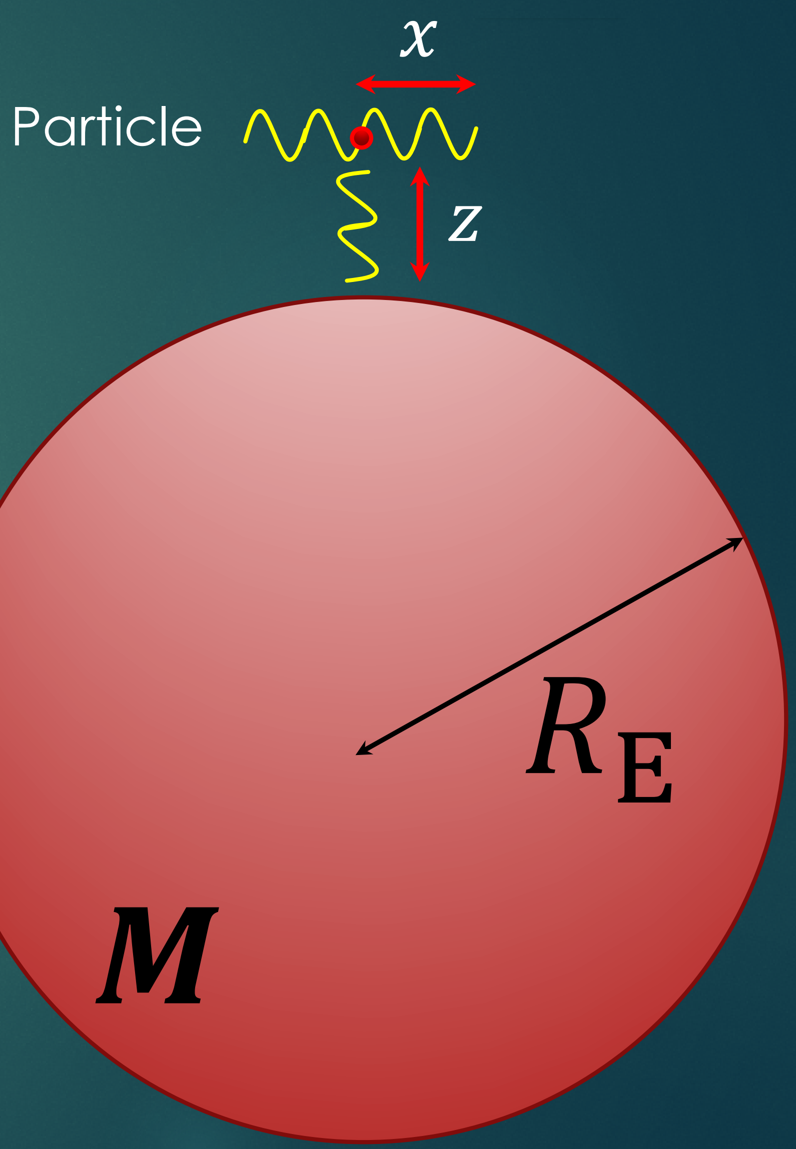


3 Three ways of exploiting gravitational tidal forces

I The gravitational potential in the Schrödinger equation in the non-inertial frame on the ground is:

$$m_g V(r) = \tfrac{1}{2} m_g \frac{GM}{R_E^3}\left(x^2 + y^2 - 2\zeta^2\right)$$

$$\zeta = \tfrac{1}{2} R_E - z$$



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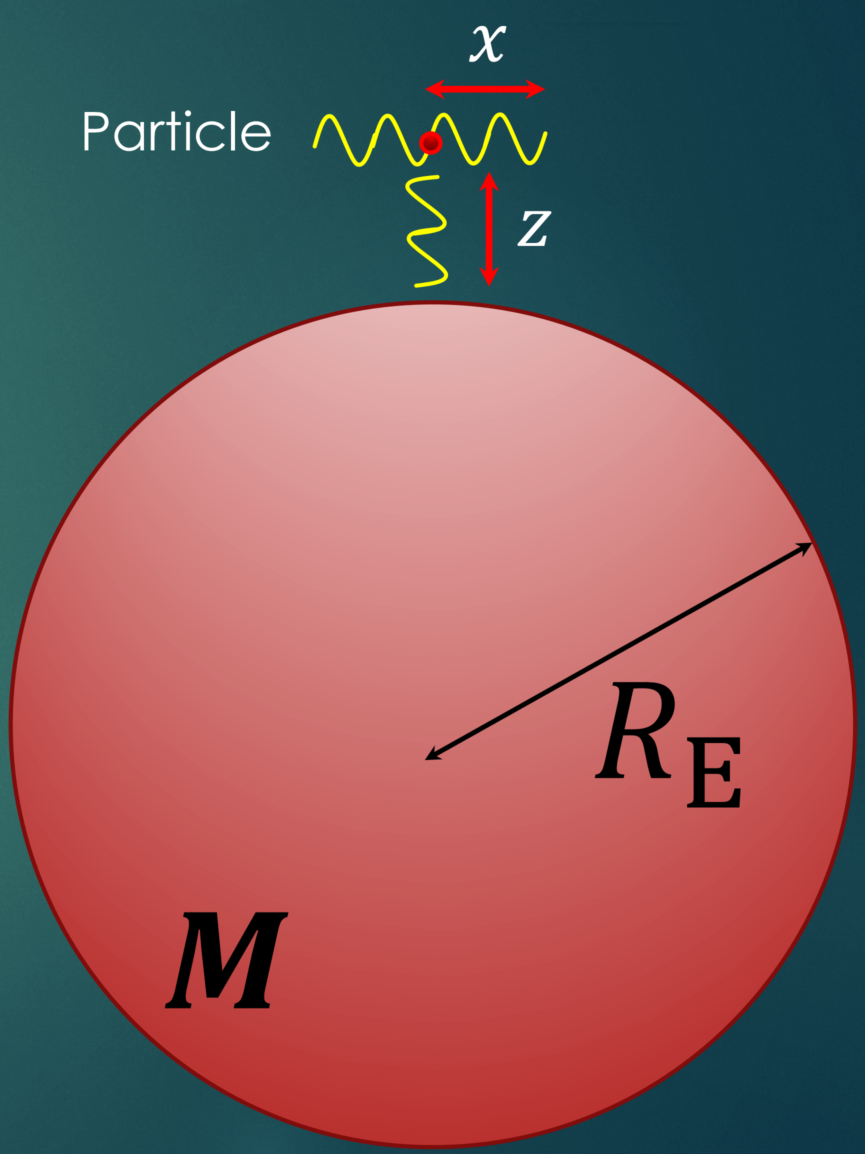
$$m_g V(r) = \tfrac{1}{2} m_g \frac{GM}{R_E^3}\left(x^2 + y^2 - 2\zeta^2\right)$$

Simple harmonic oscillator potentials along x & y.

Quantized energy along the x- and y-axes:

$$E_{n_x, n_y} = \hbar\omega \sqrt{\frac{m_g}{m_i}}\left(n_x + n_y + 1\right)$$



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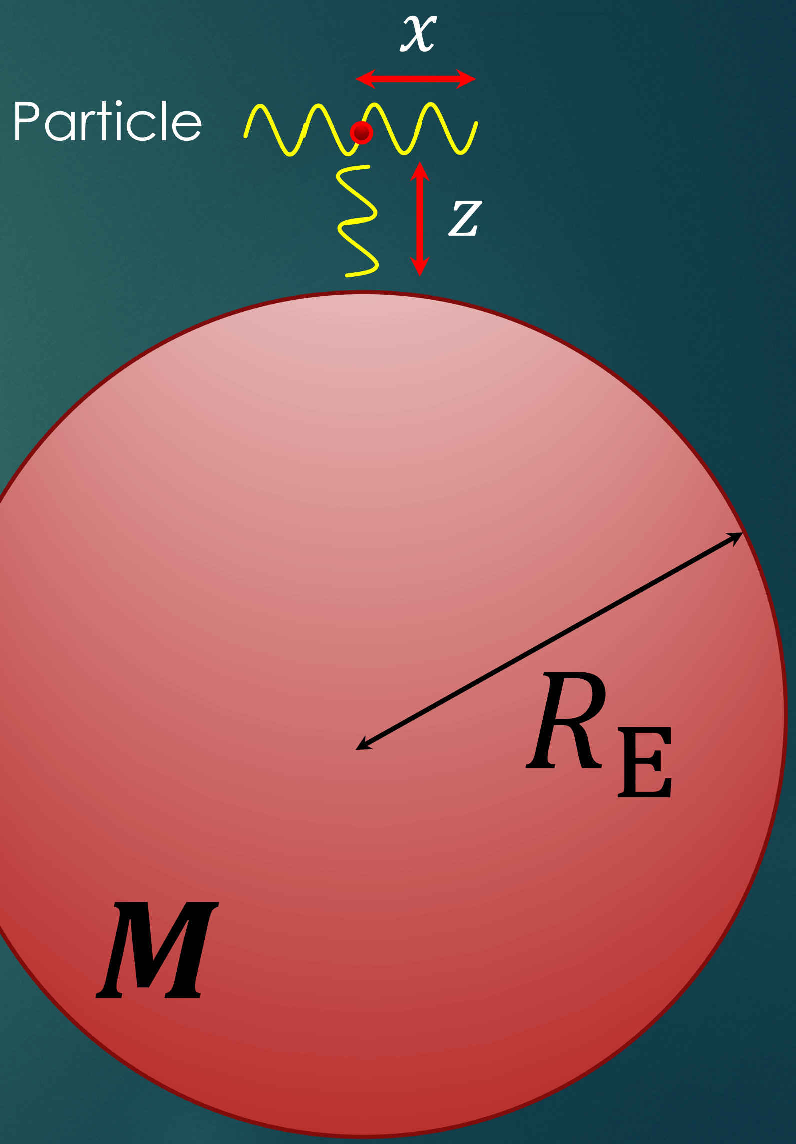
$$m_g V(r) = \tfrac{1}{2} m_g \frac{GM}{R_E^3}\left(x^2 + y^2 - 2\zeta^2\right)$$

Inverted harmonic oscillator potential along z.

$$\psi(\zeta) \sim e^{\pi\eta}\,\Gamma\!\left(\tfrac{1}{2} - i\eta\right) \mathcal{D}_{i\eta-\frac{1}{2}}\left(e^{\frac{3\pi i}{4}} 2^{\frac{3}{4}} \zeta \left[\frac{m_i m_g g}{\hbar^2 R_E}\right]^{\frac{1}{4}}\right)$$

$$\eta = \frac{E}{\hbar}\sqrt{\frac{m_i}{m_g}\frac{R_E}{2g}}$$



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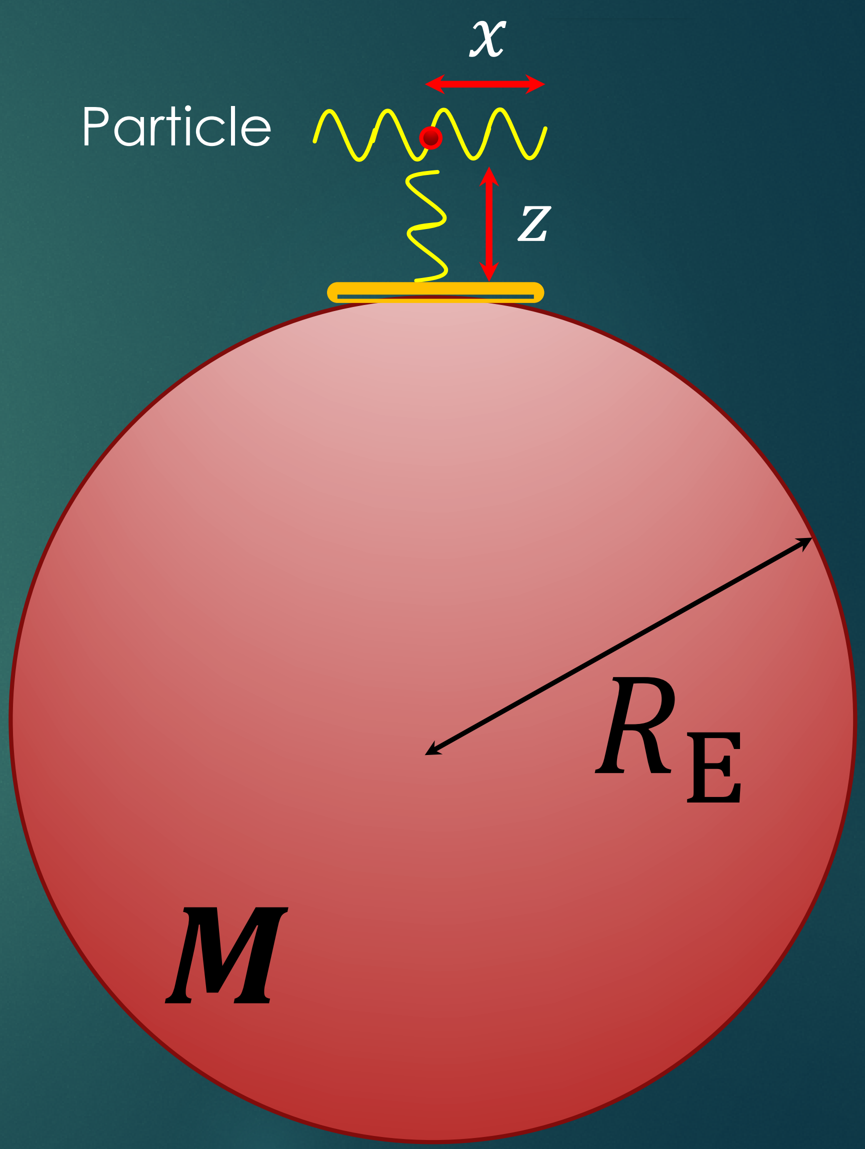
$$m_g V(r) = \tfrac{1}{2} m_g \frac{GM}{R_E^3}\left(x^2 + y^2 - 2\zeta^2\right)$$

Inverted harmonic oscillator potential along z.

Quantized energy along the z axis:

$$E_{n_\zeta} \approx -\left[\frac{m_g^2}{m_i}\frac{9\pi^2\hbar^2 g^2}{8}\left(n-\frac{1}{4}\right)^2\right]^{\frac{1}{3}}$$



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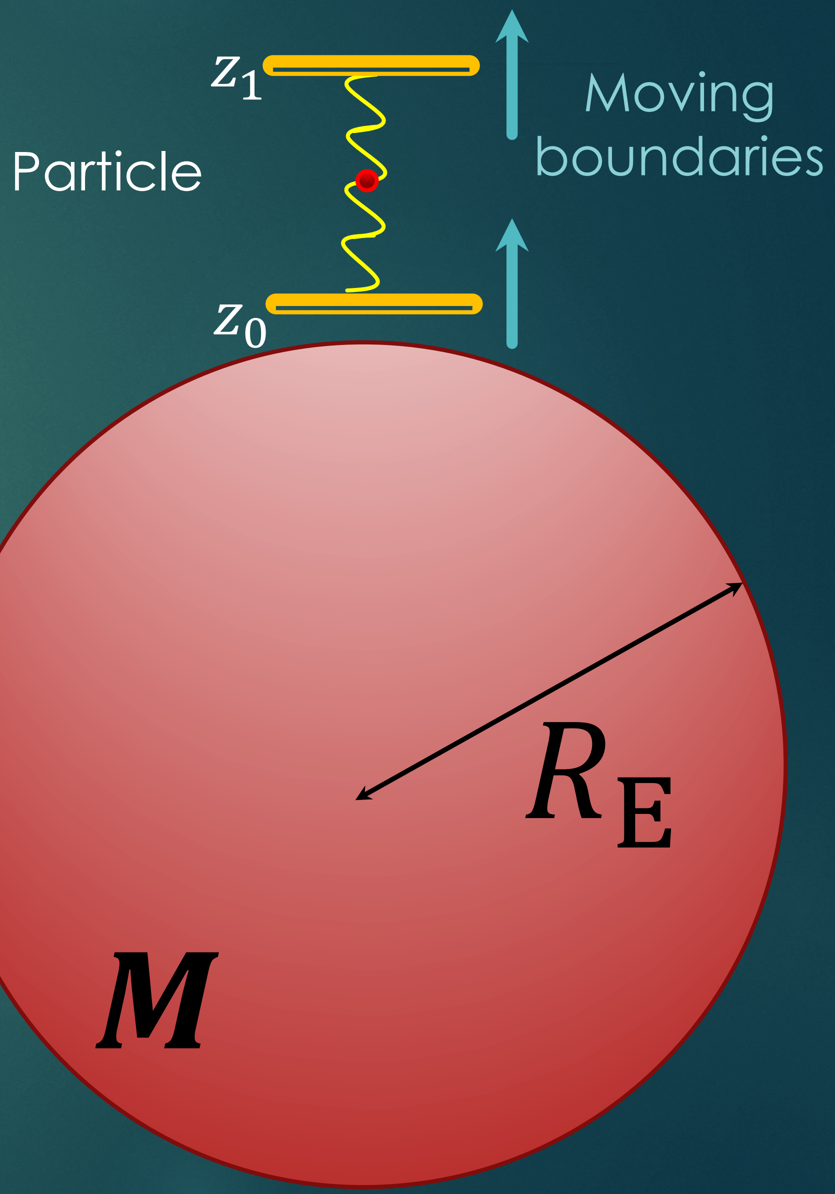
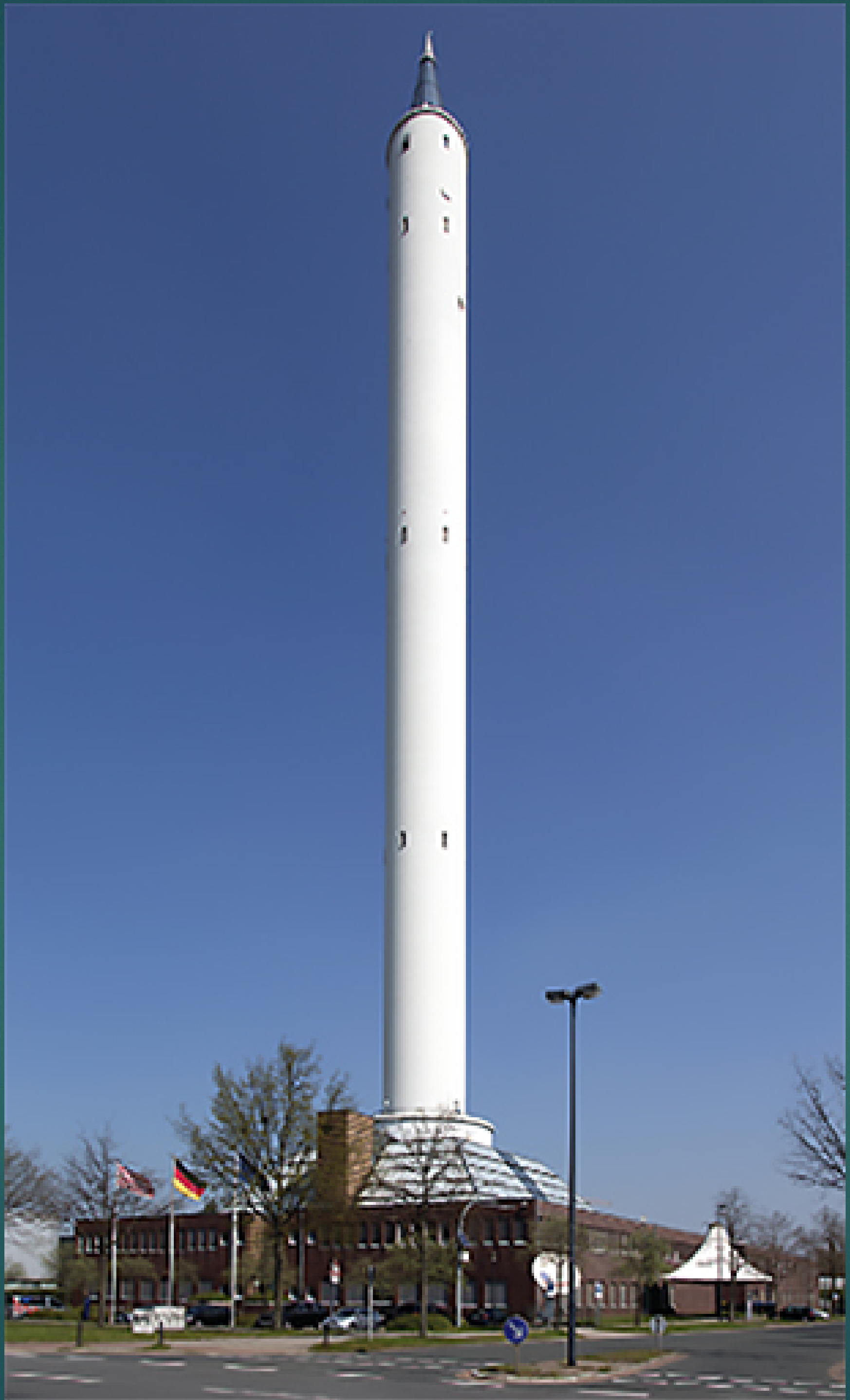
II The gravitational potential in the Schrödinger equation in the non-inertial frame on the ground is:

$$m_g V(r) = \frac{1}{2} m_g \frac{GM}{R_E^3}\left(x^2 + y^2 - 2\zeta^2\right)$$

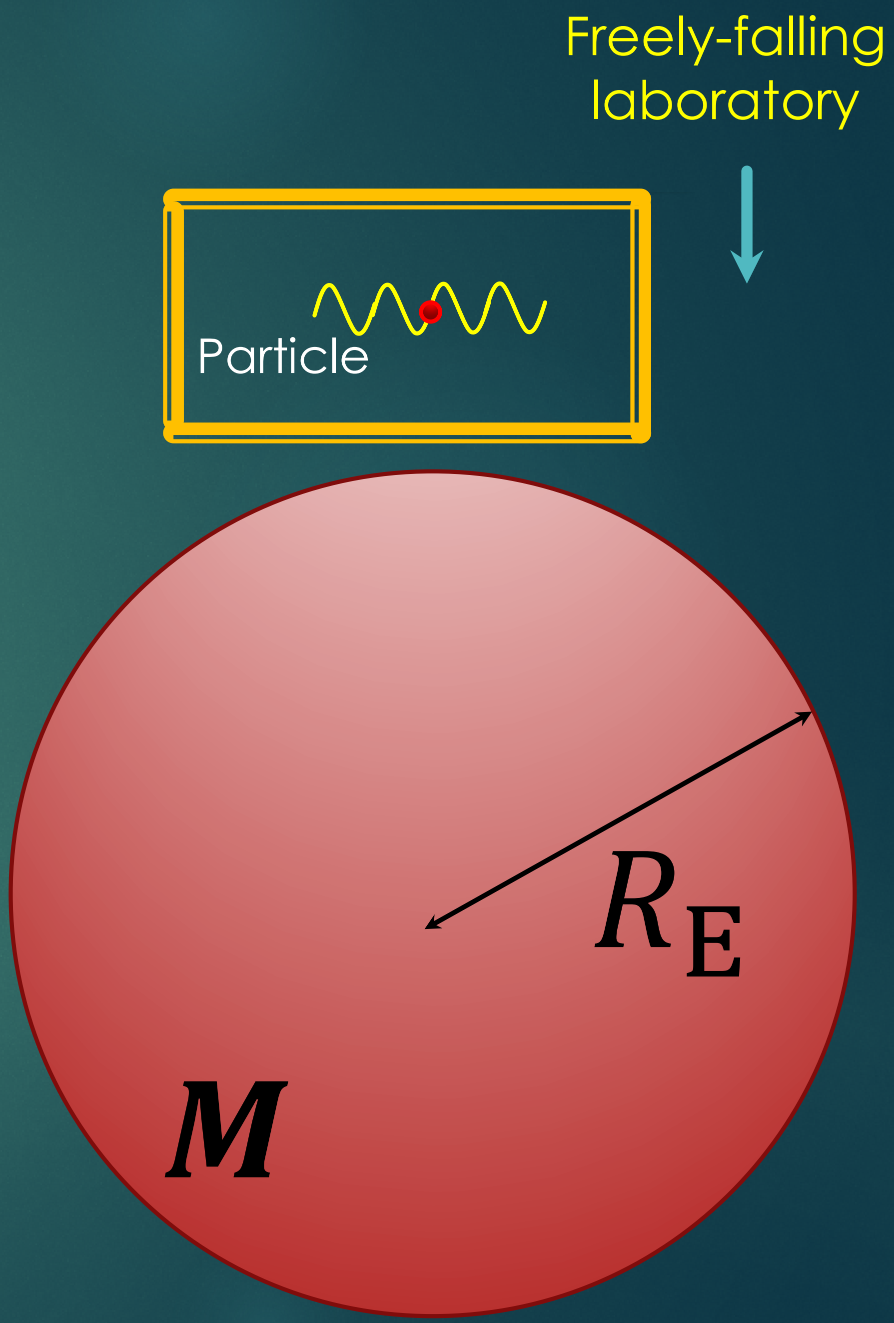
Inverted harmonic oscillator potential along z.

$$E_{n_\zeta} = \frac{n^2(s-1)^2\pi^2\hbar^2}{8 m_i z_0} e^{-\sqrt{\frac{8m_g}{m_i}}\omega t}$$





ZARM Drop tower
Bremen (Germany)
140 m

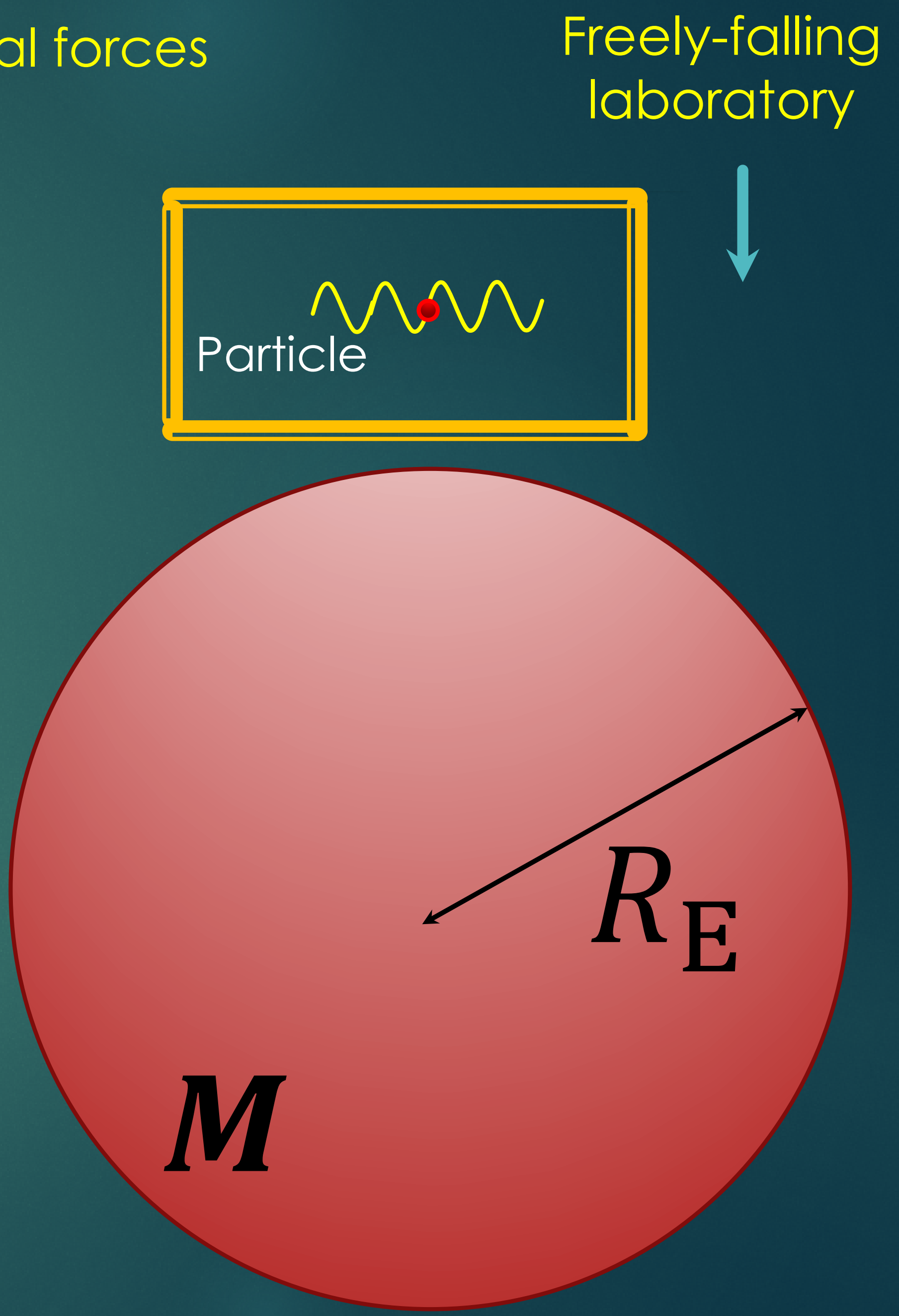


③ Three ways of exploiting gravitational tidal forces

Ⅲ The gravitational potential in the Schrödinger equation in the inertial frame is:

$$mV(r) = \underbrace{\frac{GMm}{2r^3(\tau)}(x^2 + y^2 - 2z^2)}$$

Time-dependent simple harmonic oscillators.

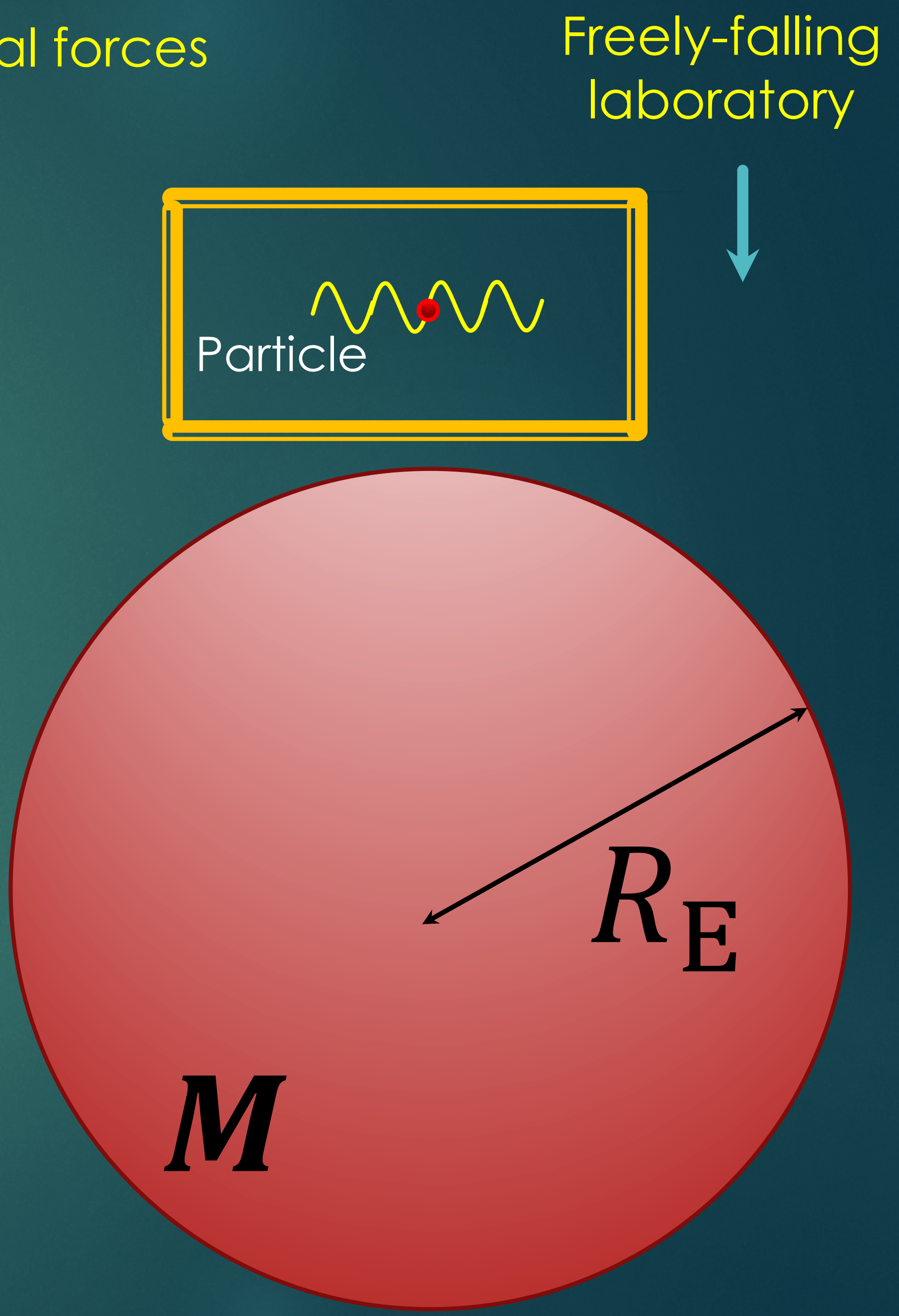


3 Three ways of exploiting gravitational tidal forces

III The gravitational potential in the Schrödinger equation in the inertial frame is:

$$mV(r) = \frac{GMm}{2r^3(\tau)}(x^2 + y^2 - 2z^2)$$

$$r^3(\tau) = \frac{9}{2}GM\left(\sqrt{\frac{2r_0^3}{9GM}} - \tau\right)^2$$



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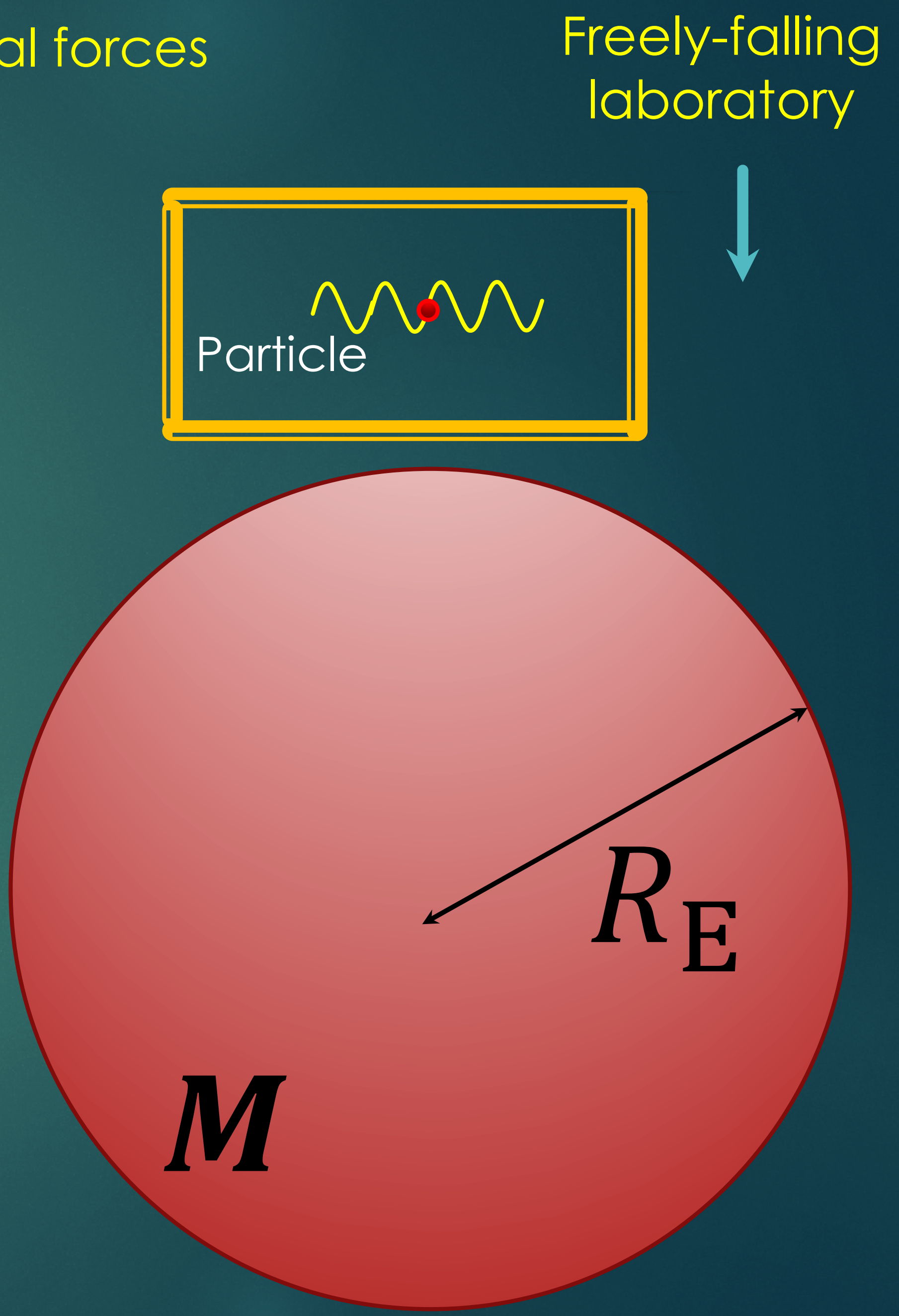
III The gravitational potential in the Schrödinger equation in the inertial frame is:

$$mV(r) = \frac{GMm}{2r^3(\tau)}(x^2 + y^2 - 2z^2)$$

$$t_0 = \sqrt{\frac{2r_0^3}{9GM}}. \qquad t = t_0 - \tau.$$

$$\langle n|H|n\rangle = \frac{\hbar}{\sqrt{2}t_0}\left(n + \tfrac{1}{2}\right) f\left(\frac{t_0}{t}\right)$$



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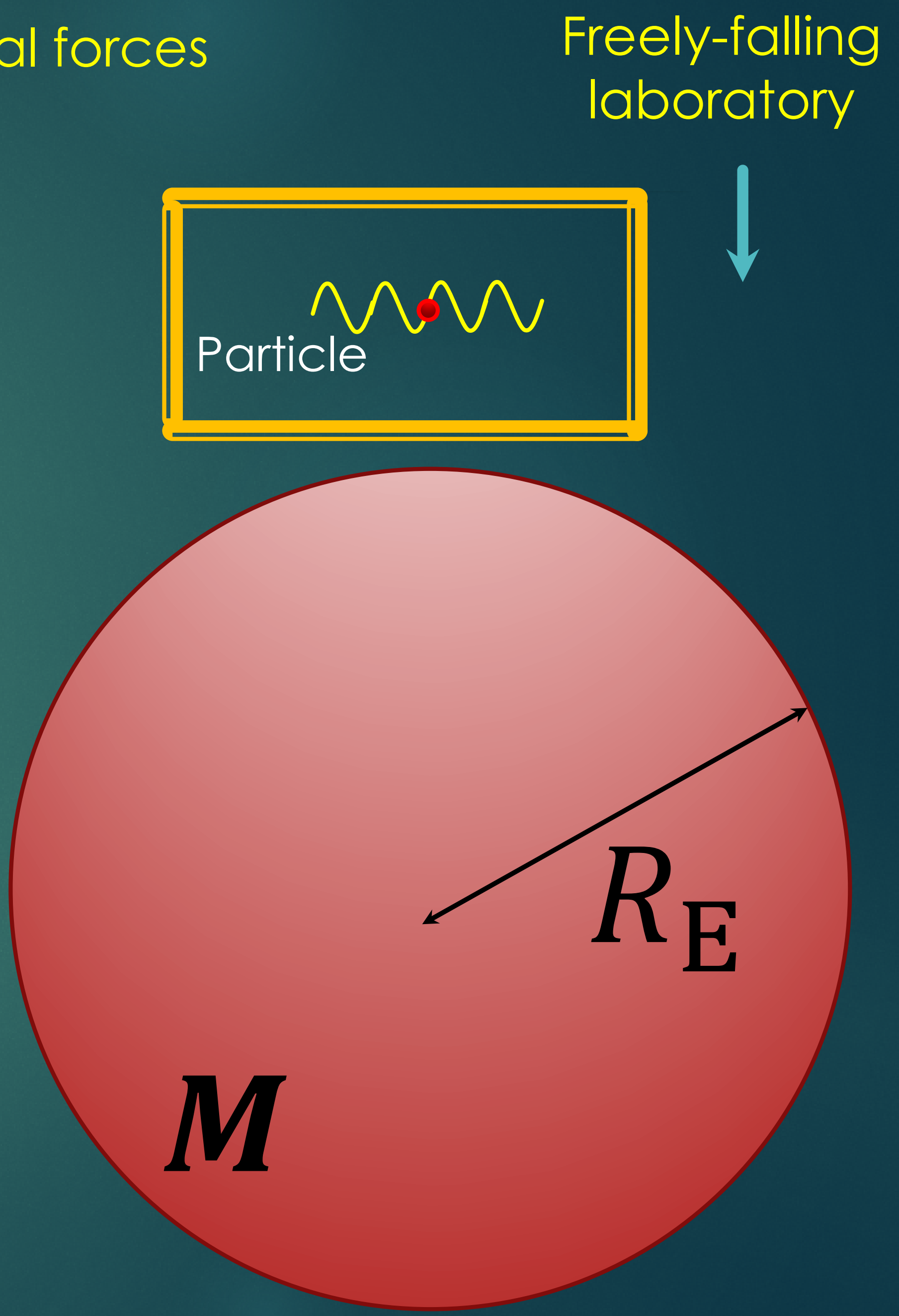
Consider the width of the wave packet:

$$w = \langle x \rangle^2 - \langle x^2 \rangle$$



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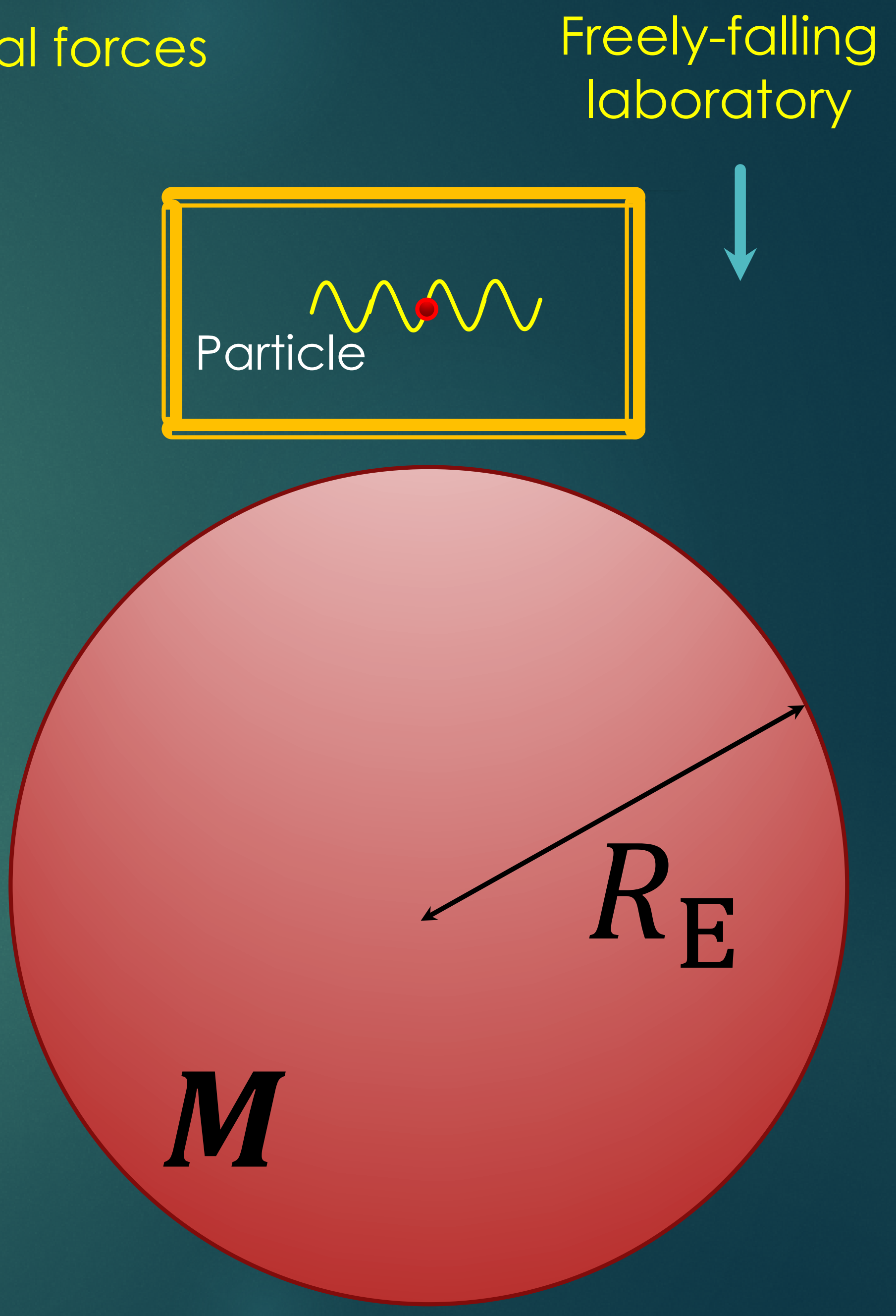
$$\frac{\mathrm{d}w}{w\,\mathrm{d}t} = \frac{12}{t}\, h\left(\frac{t_0}{t}\right)$$



4 The remarkable mass-independence $m_g = m_i$

Non-inertial frame (along x and y)

$$E_{n_x, n_y} = \hbar\omega\sqrt{\frac{m_g}{m_i}}(n_x + n_y + 1)$$ ✓

Non-inertial frame (along z)

$$\psi(\zeta) \sim e^{\pi\eta}\Gamma\left(\tfrac{1}{2} - i\eta\right)\mathcal{D}_{i\eta-\frac{1}{2}}\left(e^{\frac{3\pi i}{4}}2^{\frac{3}{4}}\zeta\left[\frac{m_i m_g g}{\hbar^2 R_E}\right]^{\frac{1}{4}}\right)$$ ✓ ✗

Non-inertial frame (moving boundaries)

$$E_{n_\zeta} = \frac{n^2(s-1)^2\pi^2\hbar^2}{8m_i z_0}e^{-\sqrt{\frac{8m_g}{m_i}}\omega t}$$ ✗ ✓

Inertial frame (along x and y)

$$\langle n|H|n\rangle = \frac{\hbar}{\sqrt{2}t_0}\left(n + \tfrac{1}{2}\right)f\left(\frac{t_0}{t}\right)$$ ✓

$$\frac{\mathrm{d}w}{w\,\mathrm{d}t} = \frac{12}{t}\quad h\left(\frac{t_0}{t}\right)$$ ✓

4 The remarkable mass-independence $m_g = m_i$

The mass-independence is due to the combined spatial extension of the wavefunction and the gravitational tidal forces!

5 What does general relativity say about this?

The Schrödinger equation with the gravitational potential is extracted from the Klein-Gordon equation in curved spacetime

$$\left(g^{\mu\nu}\partial_\mu\partial_\nu - g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\partial_\lambda - \frac{m_i^2 c^2}{\hbar^2}\right)\varphi = 0,$$

by using the ansatz: $\varphi = e^{-i\frac{m_i c^2 t}{\hbar}}\psi.$

For a spherically symmetric and static metric, the equation becomes

$$\frac{g^{00}}{c^2}\partial_t^2\psi + g^{ij}\partial_{ij}\psi - \frac{2m_i g^{00}}{\hbar}\partial_t\psi - \left(g^{00}\Gamma^k_{00} + g^{ij}\Gamma^k_{ij}\right)\partial_k\psi - \frac{m_i^2 c^2}{\hbar^2}(g^{00}+1)\psi = 0$$

There is only the inertial mass in this equation!

5 What does general relativity say about this?

The Schrödinger equation with the gravitational potential is extracted from the Klein-Gordon equation in curved spacetime

$$\left(g^{\mu\nu}\partial_\mu\partial_\nu - g^{\mu\nu}\Gamma^\lambda_{\mu\nu}\partial_\lambda - \frac{m_i^2 c^2}{\hbar^2}\right)\varphi = 0,$$

by using the ansatz: $\varphi = e^{-i\frac{m_i c^2 t}{\hbar}}\psi.$

The universality of gravity, as implemented in general relativity, seems to be due to the universality of quantum mechanics.

6 Experimental challenges

All three setups are doable.
The major hindrance would be to maintain quantum coherence for long times:

- For a freely-falling laboratory, times larger than 38 seconds are required.
- For moving boundaries, at least 2 hours are required.

A Bose-Einstein condensate could be used instead of a single quantum particle!

Use the Gross-Pitaevskii equation: $-\frac{\hbar^2}{2m_i}\nabla^2\psi + m_g V(r)\psi + \frac{4\pi\hbar^2 a_s}{m_i}|\psi|^2\psi = i\hbar\partial_t\psi$

7 Conclusion

- Just as in classical mechanics, it is possible to devise setups for which a quantum phenomenon depends on the ratio $\frac{m_g}{m_i}$ in an isolated form.
- Mass-independent behavior could be witnessed in quantum systems inside a gravitational field whenever the inertial mass is identical to the gravitational mass of the system.
- These results provide, therefore, a novel way for testing the equivalence between the two concepts of mass.

Many thanks for your attention!