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Polarization Rotation In Geometric Optics Approximation And Its Subleading Order Correction

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This work was done with A/Prof. Daniel Terno

Motivation

- Quantum communications in weakly relativistic regime
- Effects of gravitational field analogous to dielectric medium (gravitational Faraday effect)
- Propagation of electromagnetic/gravitational waves of large but finite frequencies in curved spacetimes
- Test of General Relativity

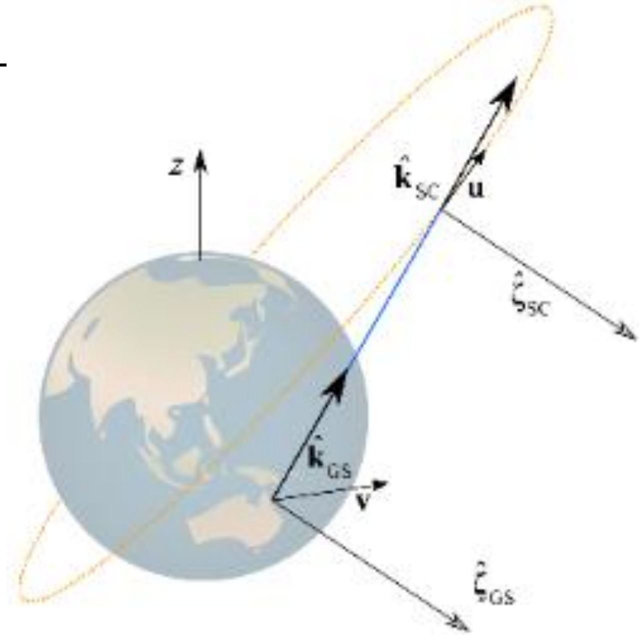
Table of content

Setup

A simple estimation of the emitter/observer dependent phase and its explicit form in several settings.

These phenomena (both due to special and general relativistic effects) can be mitigated, and in the limit of sharply defined frequencies completely removed by a simple encoding procedure.

The problem is analyzed in the geocentric (earth centered inertial) system. Velocity **v** of GS lies in the xy-plane.



Schematics. Vectors are indicated in the Earth-centred inertial frame. In the flat spacetime approximation the unit tangent vectors to the ray $\hat{\mathbf{k}}_{GS}$ and $\hat{\mathbf{k}}_{SC}$, as well as the direction to an infinitely distant guide star, $\hat{\zeta}_{GS}$ and $\hat{\zeta}_{SC}$ respectively coincide. Velocity of the GS is **v** and velocity of the SC is **u.**

Setup

The parametrized post-Newtonian approximation (Poisson & Will 2018) is a systematic method of obtaining corrections to the Newtonian motions of system of slowly moving bodies bound together by weak gravitational forces. For metric theory in question, corrections are organized by the powers of GM/c^2r and v/c. By estimating $\epsilon^2 = GM/c^2R_\oplus$, where $R_\oplus$ is the radius of the earth, we find that typical velocities are of the order ϵ, and upper bound on gravitational potential is of the order ϵ^2.

Special relativistic effects

Unitary transformation U of states of elementary particles are obtained via the induced representation of the Poincare group. Under arbitrary Lorentz transformation, states transform via

$$U(\Lambda)|k,\pm\rangle = U\big(\Lambda L(k)\big)|k_s,\pm\rangle$$
$$= U\big(L(\Lambda\, k)\big)U(W)|k_s,\pm\rangle,$$

where the Wigner transformation

$$W = L^{-1}(\Lambda\, k)\Lambda L(k)$$

leaves the standard four momentum k_s invariant.

For massless particles, Wigner little group W could be decomposed as

$$W = SR_z(\varpi),$$

where S is a translation in the xy-plane and $R_z(\varpi)$ is the rotation. Translation S does not contribute to the physical degrees of freedom:

$$U(\Lambda)|k,\pm\rangle = e^{\pm i\varpi}|\Lambda k,\pm\rangle.$$

Special relativistic effects

The momenta in the GS and SC frames are related by $\boldsymbol{k}_{SC} = B(\boldsymbol{u})B(-\boldsymbol{v})\boldsymbol{k}_{GS}$.

In global frame, velocities of the GS and SC are

$$\boldsymbol{v} = (v\cos\alpha\,, v\sin\alpha\,, 0),$$

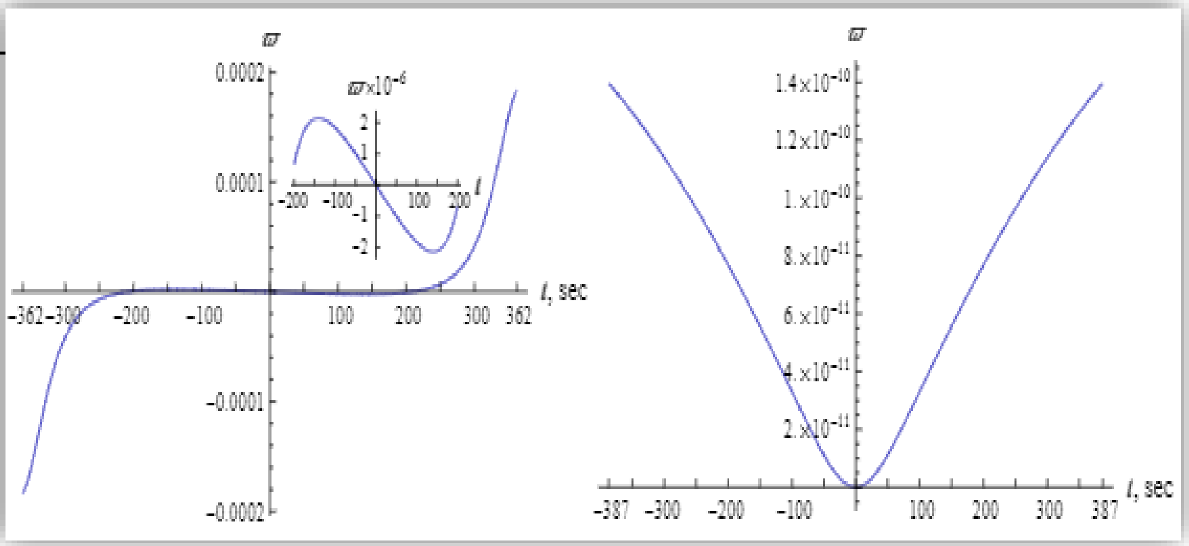
$$\boldsymbol{u} = u(\sin\vartheta\cos\varphi\,, \sin\vartheta\sin\varphi\,, \cos\vartheta).$$

The propagation vector is

$$\hat{\boldsymbol{k}} = (\sin\theta\cos\phi\,, \sin\theta\sin\phi\,, \cos\theta).$$

The Wigner phase up to the first order correction in $\epsilon = v/c$ is (Dahal & Terno 2021)

$$\varpi = u\cot\theta\sin\vartheta\sin(\phi-\varphi) + v\cot\theta\sin(\phi-\alpha)$$



Wigner phase vs the passage time (Dahal & Terno 2021). The GS is located on the equator and the SC has a circular orbit ($r_{SC} = 7.00\times10^6$m). The time origin corresponds to the SC in the zenith. *(left)* The SC moves on a constant celestial meridian and crosses the equator at $t = 0$. The inset shows the Wigner phase when the SC is close the zenith. *(right)* The SC rotates in the equatorial plane.

Faraday Rotation

Geometric optics approximation

$$A^{\mu} = a^{\mu} e^{iS};$$

a^{μ} is complex amplitude, S is real phase. Wave vector $k_{\mu} = S_{;\mu}$, square amplitude $a = \left(\widetilde{a^{\mu}} a_{\mu}\right)^{1/2}$ and polarization vector $f^{\mu} = a^{\mu}/a$. Wave vector k_{μ} satisfies (Misner *et al.* 1973)

$$k^{\mu}k_{\mu} = 0, \quad k^{\mu}\nabla_{\mu}k^{\nu} = 0.$$

Polarization vector f^{μ} satisfies

$$f^{\mu}f_{\mu} = 0, \qquad k^{\mu}\nabla_{\mu}f^{\nu} = 0.$$

Stationary spacetimes, can be written as

$$ds^2 = -h(dx^0 - g_m dx^m)^2 + dl^2,$$

where $h = -g_{00}$, $g_m = -g_{0m}/g_{00}$, and $dl^2 = \gamma_{mn}dx^m dx^n$; $\gamma_{mn} = g_{mn} - \frac{g_{0m}g_{0n}}{g_{00}}$.

In terms of three dimensional covariant derivatives (Fayos & Llosa 1982)

$$\frac{D\hat{\boldsymbol{k}}}{D\sigma} = \boldsymbol{\Omega} \times \hat{\boldsymbol{k}}, \qquad \frac{D\hat{\boldsymbol{f}}}{D\sigma} = \boldsymbol{\Omega} \times \hat{\boldsymbol{f}},$$

$$\boldsymbol{\Omega} = 2\boldsymbol{\omega} - \left(\boldsymbol{\omega}.\hat{\boldsymbol{k}}\right)\hat{\boldsymbol{k}} - \boldsymbol{E}_g \times \hat{\boldsymbol{k}},$$

Faraday Rotation

In the absence of global reference direction, the standard polarization triad $(\hat{\boldsymbol{b}}_1, \hat{\boldsymbol{b}}_2, \hat{\boldsymbol{k}})$is different at every locations. Parallel propagate the initial polarization $f_{in}(x_{in})$, and use the decomposition

$$f_{fin}(x_{fin}) = \cos\chi\hat{\boldsymbol{b}}_1 + \sin\chi\hat{\boldsymbol{b}}_2 .$$

For Schwarzschild spacetime, $\boldsymbol{\omega} = 0$, and thus differentiation gives

$$\frac{d\chi}{d\sigma} = \frac{1}{\hat{\boldsymbol{f}}.\hat{\boldsymbol{b}}_1}\hat{\boldsymbol{f}}.\frac{D\hat{\boldsymbol{b}}_2}{D\sigma}.$$

The post-Newtonian expansion, up to ϵ^3 is

$$ds^2 = -V^2(r)c^2dt^2 + \vec{R}\cdot d\vec{x}cdt + W^2(r)d\vec{x}\cdot d\vec{x},$$

where

$$V(r) = 1 - \epsilon^2\frac{U}{c^2}, \qquad W(r) = 1 + \epsilon^2\frac{U}{c^2}.$$

Gravitational potential $-U \simeq -GM/r$ and $\vec{R} \sim O(\epsilon^3)$. Photon trajectories up to ϵ^2 is (Will 2018)

$$\vec{k} = \vec{n} - \epsilon^2\frac{2M}{r(t)}\vec{n} - \epsilon^2\frac{2M\vec{d}}{d^2}\left(\frac{\vec{x}(t)\cdot\vec{n}}{r(t)} - \frac{\vec{x_0}\cdot\vec{n}}{r_0}\right),$$

where $\vec{d} = \vec{x_0} - (\vec{x_0}\cdot\vec{n})\vec{n}$

General relativistic phase

We take $z = 0$, the plane where the ray from GS to SC lies, set their velocities zero and consider polarization vector

$$\vec{f} = (0,0,1) = \text{const}.$$

The reference direction $\hat{\boldsymbol{\zeta}}_i$ (unit vector pointing to the distant star), i= GS, SC are obtained from the tangents to the rays from the fixed guide star that arrive to the GS and SC respectively. We assume the reference star to be infinitely far.

$$\vec{\zeta_i} = \vec{l}\left(1 - \epsilon^2 \frac{2M}{r_i}\right) - \epsilon^2 \frac{2M\overrightarrow{d_i}}{d_i^2}\left(\frac{\vec{x_i} \cdot \vec{l}}{r_i} - 1\right),$$

where $\vec{l} \cdot \vec{l} = 1$; $-\vec{l}$ is the flat spacetime direction from the infinitely distant star to the observers. Standard polarization vectors:

$$\hat{\boldsymbol{b}}_{2i} = \frac{\vec{\zeta_i} \times \vec{k}}{\left|\vec{\zeta_i} \times \vec{k}\right|}, \qquad \hat{\boldsymbol{b}}_{1i} = \hat{\boldsymbol{b}}_{2i} \times \hat{\boldsymbol{k}}.$$

So, we get

$$\Delta\chi = \frac{1}{\hat{\boldsymbol{f}}_{GS} \cdot \hat{\boldsymbol{b}}_{1GS}} \left(\hat{\boldsymbol{f}}_{SC} \cdot \hat{\boldsymbol{b}}_{2SC} - \hat{\boldsymbol{f}}_{GS} \cdot \hat{\boldsymbol{b}}_{2GS}\right).$$

General relativistic phase

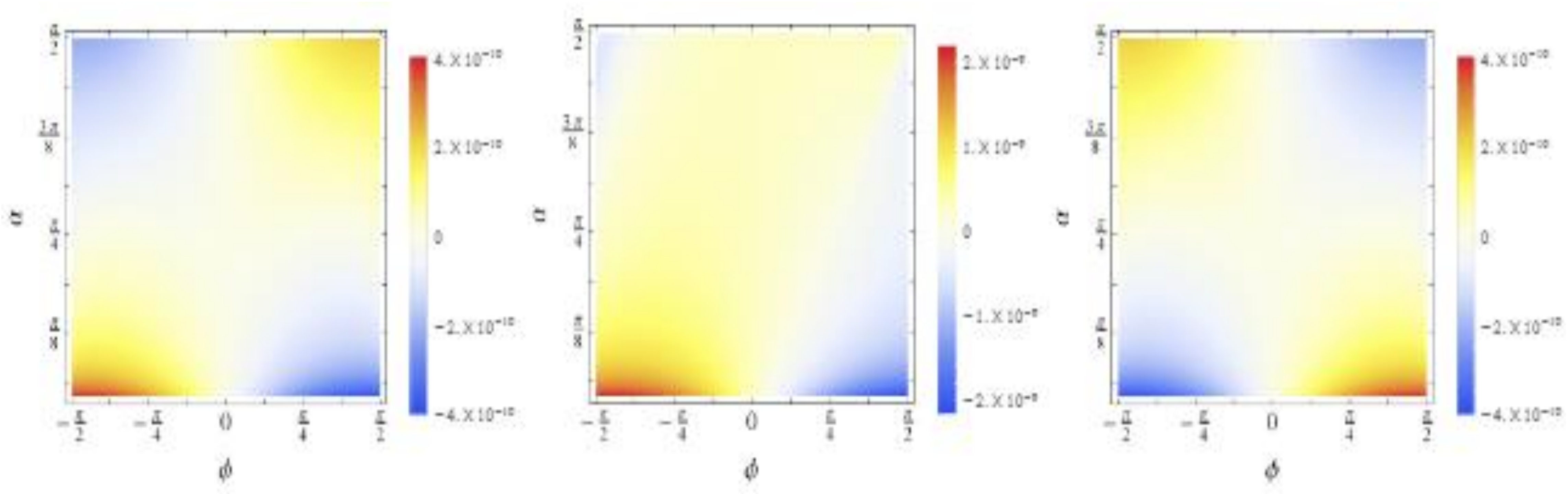
If we choose the x-axis to pass through GS, then

$$\vec{n} = (\cos\phi, \sin\phi, 0), \qquad -\frac{1}{2}\pi < \phi < \frac{1}{2}\pi,$$

$$\vec{l} = (\cos\alpha, \sin\alpha\sin\beta, \sin\alpha\cos\beta), \qquad 0 \leq \alpha \leq \frac{1}{2}\pi,$$

where the altitude of the guided star is $\pi/2 - \alpha$. When the reference direction $\vec{\zeta}$ and the propagation direction $\vec{k}$ are collinear, then the standard polarization directions are undefined. If $\vec{\zeta}$ lies in the plane determined by GS, SC and centre of earth, then the Faraday phase is zero. Moreover, the post-Newtonian phase fails in the limit of $\alpha = 0$ or $\beta = \pi/2, 3\pi/2$.

General relativistic phase



Faraday phase depends on the choice of the reference direction $\vec{\zeta}(\alpha, \beta)$ and the initial propagation direction $\vec{n}\,(\phi)$ (Dahal & Terno 2021). *(left)* $\beta = 0$. *(centre)* $\beta = \frac{7\pi}{16}$. *(right)* $\beta = \pi$.

Mitigation

Consider two entangled, well separated and therefore distinguishable wave packets, with same momentum profile centred on $\vec{k}$, example: two photon states

$$|\Psi_{\mathrm{k}}^{\pm}\rangle = \frac{1}{\sqrt{2}}\left(|\mathrm{k},+\rangle_1 \, |\mathrm{k},-\rangle_2 \pm |\mathrm{k},-\rangle_1 \, |\mathrm{k},+\rangle_2 \right)$$

These states are states of zero helicity and thus insensitive to Wigner and Faraday rotations. General and special relativistic phases are opposite for opposite helicities and thus for the states with sharply defined momenta, phase errors can be easily encountered (Barlett & Terno 2005). So, one logical qubits can be encoded with two physical qubits/photons using states $|\Psi_{\mathrm{k}}^{\pm}\rangle$.

Beyond geometric optics

Again, use the WKB ansatz, $A^{\alpha} = a^{\alpha} e^{iS}$.

Then, expand trajectory and polarization vectors in terms of the characteristic frequency ω

$$k^{\alpha} = k_0^{\alpha} + \frac{k_1^{\alpha}}{\omega} + \frac{k_2^{\alpha}}{\omega^2} + \cdots,$$

$$f^{\alpha} = f_0^{\alpha} + \frac{f_1^{\alpha}}{\omega} + \frac{f_2^{\alpha}}{\omega^2} + \cdots.$$

Higher order phase factors like $S_1(\lambda)$ can not be absorbed into the complex amplitude f_0^{α}, because no transformation freedoms of null tetrad would be available.

Left hand circularly polarized waves could be obtained by the substitution $\omega \rightarrow -\omega$ and $f^{\alpha} \rightarrow \widetilde{f^{\alpha}}$. Beyond geometric optics, this sensitivity of ω with the polarization state causes spin-orbit coupling.

Orbital motion could be due to the particle's rotation itself or due to the rotation of the spacetime.

Results trajectory to deviate from the geodesic path.

Conclusion and future works

- The leading order correction that takes into account the effects of both special and general relativity include the terms of up to the order of ϵ^2. In this approximation, gravitational field of the earth is spherically symmetric and that the perceived polarization rotation results only due to the choice of reference frames. At this level of precision, effects of special and general relativity can be treated separately; leading mixed terms are of order ϵ^3.

- In the leading order geometric optics, the trajectory is geodesic. However, this is no longer true in the subleading order correction. As gravitons/photons still travel at the speed of light, it takes longer time for them to reach the observer from the source.

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