Poincaré invariance with a minimal length

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Based on work with:

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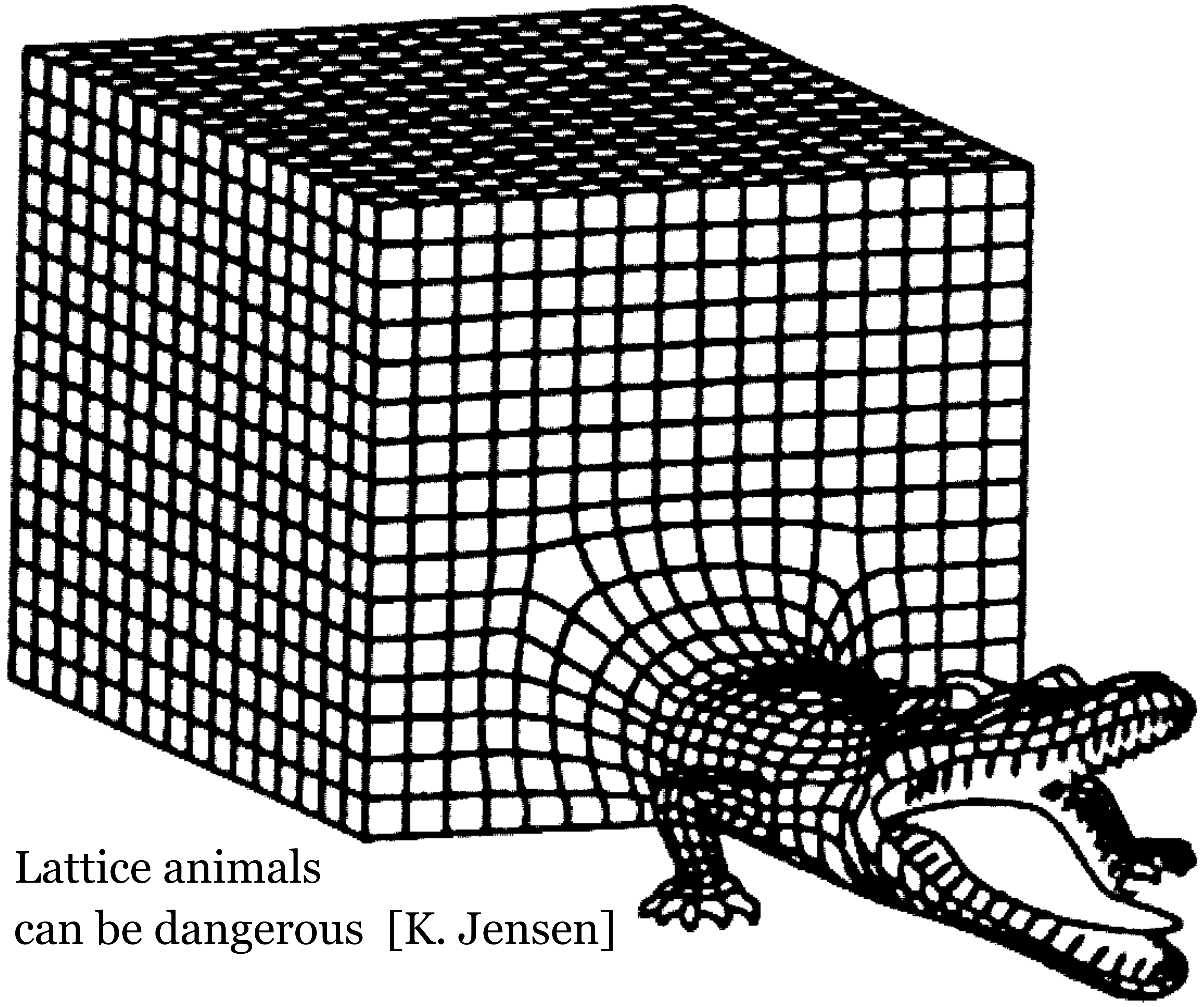


Lattice animals
can be dangerous [K. Jensen]

- Lattice Field Theory:
  - Lattice as cut-off, *maximum momentum*
  - Broken Poincaré invariance, to be recovered

- Loop Quantum Gravity:
  - discrete structure of spacetime at the Planck scale
  - reconstructing Minkowski and long-range correlations
  - how is Poincaré invariance recovered?

- Doubly Special Relativity:
  observer-independent c and *maximum momentum*

This talk: $\Longrightarrow$ Rethinking Lattice Observables and Brillouin Fields

Continuum: Scalar Field in 1+1 (Classical)

- Poincaré invariant action: $S[\phi] = \int dt\, dx \left(-\frac{1}{2}\partial_\mu\phi\partial^\mu\phi - \frac{1}{2}M^2\phi^2\right)$

- Phase space: $\{\phi(x,t), \Pi(y,t)\} = \delta(x-y)$ with $\Pi(x,t) = \partial_t\phi(x,t)$

- Poincaré generators:

$$H = \int_{-\infty}^{+\infty} dx\ \tfrac{1}{2}\left(\Pi^2 + (\partial_x\phi)^2 + M^2\phi^2\right)$$

$$P = \int_{-\infty}^{+\infty} dx\ \tfrac{1}{2}\left(\phi\,\partial_x\Pi - \Pi\,\partial_x\phi\right)$$

$$K = \int_{-\infty}^{+\infty} dx\ \tfrac{1}{2}\,x\left(\Pi^2 + (\partial_x\phi)^2 + M^2\phi^2\right)$$

- Poincaré algebra in phase space:

$$\{H,P\} = 0, \qquad \{K,P\} = H, \qquad \{K,H\} = P$$

Continuum: Scalar Field in 1+1 (*Quantum*)

- Field commutators: $[\phi(x,t),\Pi(y,t)] = \mathrm{i}\,\delta(x-y)$ with $\Pi(x,t) = \partial_t\phi(x,t)$

- Poincaré generators:

$$H = \int_{-\infty}^{+\infty} dx\ \tfrac{1}{2}\left(\Pi^2 + (\partial_x\phi)^2 + M^2\phi^2\right)$$

$$P = \int_{-\infty}^{+\infty} dx\ \tfrac{1}{2}\left(\phi\,\partial_x\Pi - \Pi\,\partial_x\phi\right)$$

$$K = \int_{-\infty}^{+\infty} dx\ \tfrac{1}{2}\,x\left(\Pi^2 + (\partial_x\phi)^2 + M^2\phi^2\right)$$

- Mode expansion that puts the Hamiltonian in normal form $H = \int_{-\infty}^{+\infty} \frac{dp}{2\pi}\,\omega(p)\ a^\dagger(p)a(p)$

$$\phi(x,t) = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\left(\frac{1}{\sqrt{2\omega(p)}}\,\mathrm{e}^{-\mathrm{i}(\omega(p)t-px)}\,a(p) + \frac{1}{\sqrt{2\omega(p)}}\,\mathrm{e}^{+\mathrm{i}(\omega(p)t-px)}\,a^\dagger(p)\right) \quad \text{with} \quad \omega(p) = \sqrt{p^2+M^2}$$

- Poincaré algebra: Representation on Fock space $\mathcal{F}$ built over Poincaré invariant vacuum $|0\rangle$

$$[H,P] = 0\,, \qquad [K,P] = \mathrm{i}H\,, \qquad [K,H] = \mathrm{i}P$$

- Poincaré invariant vacuum: $a(p)|0\rangle = 0\,, \quad \forall\, p\in\mathbb{R}$

Lattice discretization: Scalar Field in 1+1

- Lattice: infinite one-dimensional lattice with fixed spacing ℓ

$$x_n = n\,\ell\,, \qquad \text{with} \quad n = \ldots, -2, -1, 0, +1, +2, \ldots$$

- Lattice discretization of the spatial derivative: $\partial_x \phi \longrightarrow D_{\_}\,\phi_n = \dfrac{\phi_n - \phi_{n-1}}{\ell}$

- Lattice discretization of the Hamiltonian $H_L = \ell \sum_{n=-\infty}^{+\infty} \frac{1}{2}\left(\frac{1}{\ell^2}\,\Pi_n^2 + \left(\frac{\phi_n - \phi_{n-1}}{\ell}\right)^2 + M^2 \phi_n^2\right)$

- Mode expansion that puts the Hamiltonian in normal form $H_L = \int_{-\pi/\ell}^{+\pi/\ell} \dfrac{dp}{2\pi}\, \omega_L(p)\, a_L^\dagger(p) a_L(p)$

$$\phi_n(t) = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi} \left(\frac{1}{\sqrt{2\omega_L(p)}}\, \mathrm{e}^{-\mathrm{i}(\omega_L(p)t - p\ell n)}\, a_L(p) \;+\; \frac{1}{\sqrt{2\omega_L(p)}}\, \mathrm{e}^{+\mathrm{i}(\omega_L(p)t - p\ell n)}\, a_L^\dagger(p) \right)$$

with frequency

$$\omega_L(p) = \sqrt{|D_{\_}(p)|^2 + M^2} \;=\; \sqrt{\frac{\sin^2(p\,\ell/2)}{(\ell/2)^2} + M^2}$$

- Lattice Fock space $\mathcal{F}_L$ built over vacuum $|0_L\rangle$

$$a_L(p)|0_L\rangle = 0\,, \quad \text{with} \quad p \in [-\frac{\pi}{\ell}, +\frac{\pi}{\ell})$$

Lattice discretization: Scalar Field in 1+1 and dispersion relations

- Lattice translations: unitary operator on $\mathcal{F}_L$

$$U_L\,\phi_n(t)U_L^{-1} = \phi_{n+1}(t)$$

Note: $U_L = \mathrm{e}^{\mathrm{i}\ell P_L}$, $P_L = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi}\, p\, a_L^\dagger(p) a_L(p)$

- One-particle states $|1,p\rangle_L = a_L^\dagger(p)|0\rangle_L$ and dispersion relations:

$$H_L\,|1_L,p\rangle = \omega_L(p)\,|1_L,p\rangle\,,$$
$$P_L\,|1_L,p\rangle = p\,|1_L,p\rangle\,.$$

$$\omega_L(p)^2 = p^2 + M^2 - \frac{\ell^2}{12}p^4 + O(\ell^4)$$

- At low momentum in lattice frame: Lorentz violating

- At high momentum in lattice frame $|p| \gtrsim \pi/\ell$: quasi-particle and physics of the Brillouin zone

- Note: Bonechi, Celeghini, Giachetti, Sorace and Tarlini (PRL 1992)

  $H_L, U_L, \tilde{K}_L$ quantum-deformed 1+1 Poincaré algebra $E_q(1,1)$

Lattice discretization: Scalar Field in 1+1 and broken Poincaré algebra

- Discretization of $H\,,\ P\,,\ K$ with the same discrete derivative

$$\partial_x \phi \longrightarrow D_{\_}\phi_n = \frac{\phi_n - \phi_{n-1}}{\ell}$$

$$H_L = \ell \sum_{n=-\infty}^{+\infty} \tfrac{1}{2}\left(\tfrac{1}{\ell^2}\,\Pi_n^2 + \left(\tfrac{\phi_n - \phi_{n-1}}{\ell}\right)^2 + M^2\phi_n^2\right)$$

$$\tilde{P}_L = \sum_{n=-\infty}^{+\infty} \tfrac{1}{2}\left(\phi_n\, D_{\_}\Pi_n - \Pi_n\, D_{\_}\phi_n\right) = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi}\,\frac{\sin(p\ell)}{\ell}\; a_L^\dagger(p) a_L(p)$$

$$K_L = \ell \sum_{n=-\infty}^{+\infty} n\ell\, \tfrac{1}{2}\left(\tfrac{1}{\ell^2}\,\Pi_n^2 + (D_{\_}\phi_n)^2 + M^2\phi_n^2\right)$$

- Broken Poincaré algebra (in a different way in both proposed discretizationso

$$[H_L, \tilde{P}_L] = 0\,, \qquad [K_L, \tilde{P}_L] \neq \mathrm{i} H_L\,, \qquad [K_L, H_L] = \mathrm{i}\tilde{P}_L\,.$$

$$[H_L, P_L] = 0\,, \qquad [K_L, P_L] = \mathrm{i} H_L\,, \qquad [K_L, H_L] \neq \mathrm{i} P_L\,.$$

Brillouin fields on a 1+1 lattice

- Strategy: determine discretization of the derivative ∂_x
  with requirement of exact rep in Fock space of Poincaré algebra

- Discretization $\boxed{\partial_x \phi \longrightarrow D\phi_n}$ with properties

  (i) Linearity: $D\phi_n = \sum_{m=-\infty}^{+\infty} D_{nm}\,\phi_m$

  (ii) Covariance under lattice translations $\phi_n \to \phi_{n+1}$

  (iii) Vanishing discrete derivative of a constant

- Discretization of Poincaré generators:

$$H_B = \ell \sum_{n=-\infty}^{+\infty} \tfrac{1}{2}\big(\tfrac{1}{\ell^2}\,\Pi_n^2 + (D\phi_n)^2 + M^2\phi_n^2\big)\,,$$

$$P_B = \sum_{n=-\infty}^{+\infty} \tfrac{1}{2}\big(\phi_n\, D\Pi_n - \Pi_n\, D\phi_n\big)\,,$$

$$K_B = \ell \sum_{n=-\infty}^{+\infty} n\ell\, \tfrac{1}{2}\big(\tfrac{1}{\ell^2}\,\Pi_n^2 + (D\phi_n)^2 + M^2\phi_n^2\big)\,.$$

- Imposition of exact Poincaré algebra:

$$\boxed{[H_B, P_B] = 0\,, \qquad [K_B, P_B] = \mathrm{i}H_B\,, \qquad [K_B, H_B] = \mathrm{i}P_B}$$

- Does a solution for the matrix D_{nm} exist? Is it unique?

Brillouin fields on a 1+1 lattice

- Strategy: determine discretization of the derivative $\partial_x\phi \longrightarrow D\phi_n$
  with requirement of exact rep in Fock space of Poincaré algebra

$$[H_B, P_B] = 0\,, \qquad [K_B, P_B] = \mathrm{i}H_B\,, \qquad [K_B, H_B] = \mathrm{i}P_B$$

- Unique solution:

$$D\phi_n = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi}\, D(p)\, \phi(p)\, \mathrm{e}^{\mathrm{i}p\ell n} = \sum_{m=-\infty}^{+\infty}{}' \, \frac{1}{\ell}\frac{(-1)^{n-m}}{n-m}\phi_m\,,$$

with $D(p) = \mathrm{i}\, p\,.$

- Generators:

$$H_B = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi}\, \sqrt{p^2+M^2}\;\, a_B^\dagger(p)a_B(p)\,,$$

$$P_B = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi}\, p\, a_B^\dagger(p)a_B(p)\,,$$

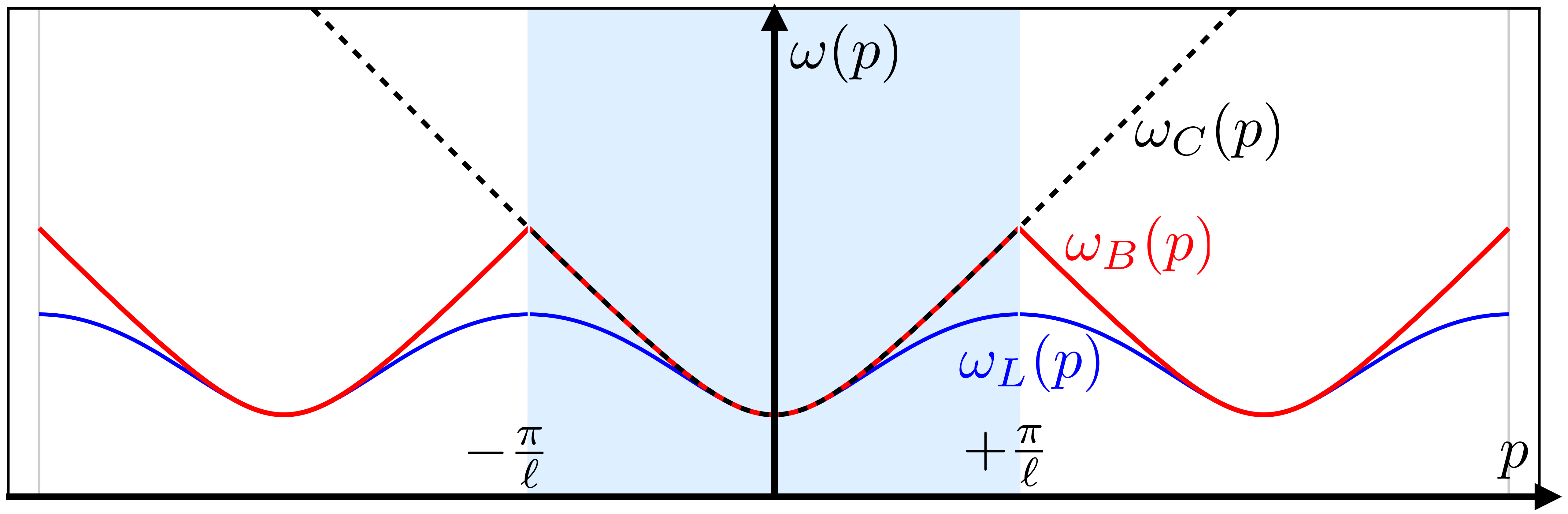
$$K_B = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi}\, \sqrt{p^2+M^2}\; \tfrac{1}{2\mathrm{i}}\Big(\big(\partial_p a_B^\dagger(p)\big)\, a_B(p) - a_B^\dagger(p)\, \partial_p a_B(p)\Big)\,.$$

Brillouin vacuum and dispersion relations

- Brillouin vacuum: invariant under Poincaré transformation

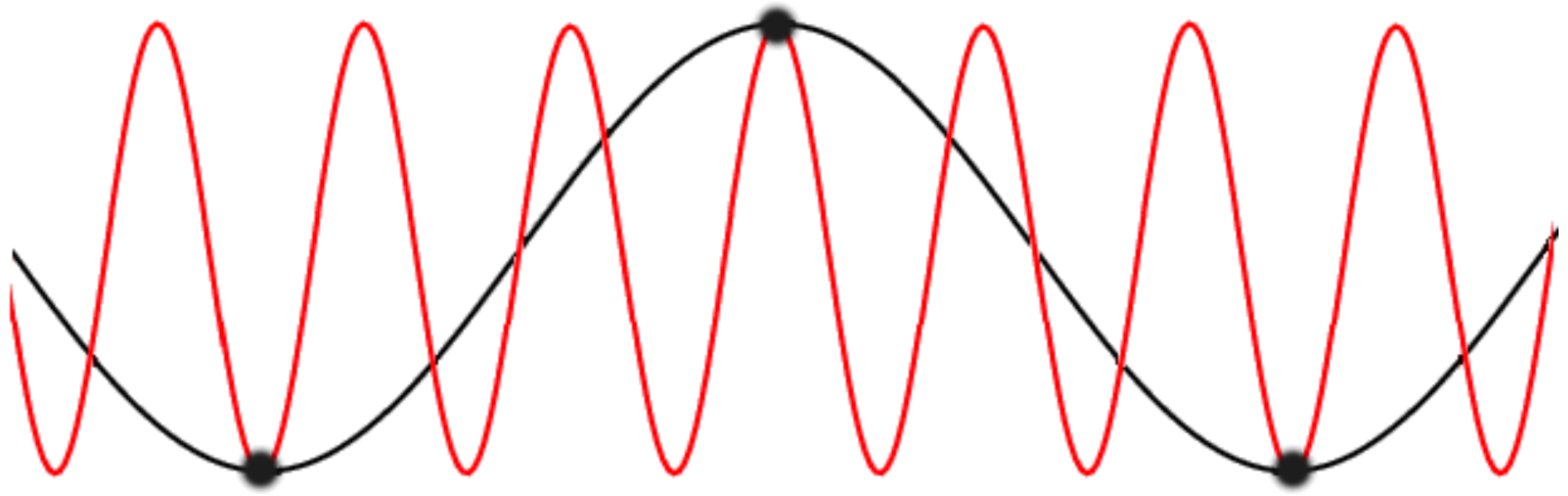
$$e^{-iH_B t}|0_B\rangle = |0_B\rangle\,, \qquad e^{-iP_B x}|0_B\rangle = |0_B\rangle\,, \qquad e^{-iK_B \eta}|0_B\rangle = |0_B\rangle\,.$$

- One-particle state $|1_B, p\rangle = a_B^\dagger(p)|0_B\rangle$ with definite energy and momentum



- Brillouin zone identification in Fock space

$$|1_B, (p + \tfrac{2\pi m}{\ell})\rangle \equiv |1_B, p\rangle\,, \qquad \text{with} \quad m \in \mathbb{Z} \quad \text{and} \quad p \in \mathbb{R}$$



- Boosts and absence of a preferred rest-frame

$$e^{-iK_B \eta}|1_B, p\rangle = |1_B, p_\eta\rangle \qquad\qquad p_\eta = \cosh(\eta)\, p - \sinh(\eta)\, \omega_B(p)$$

Brillouin algebra of observables and continuum fields

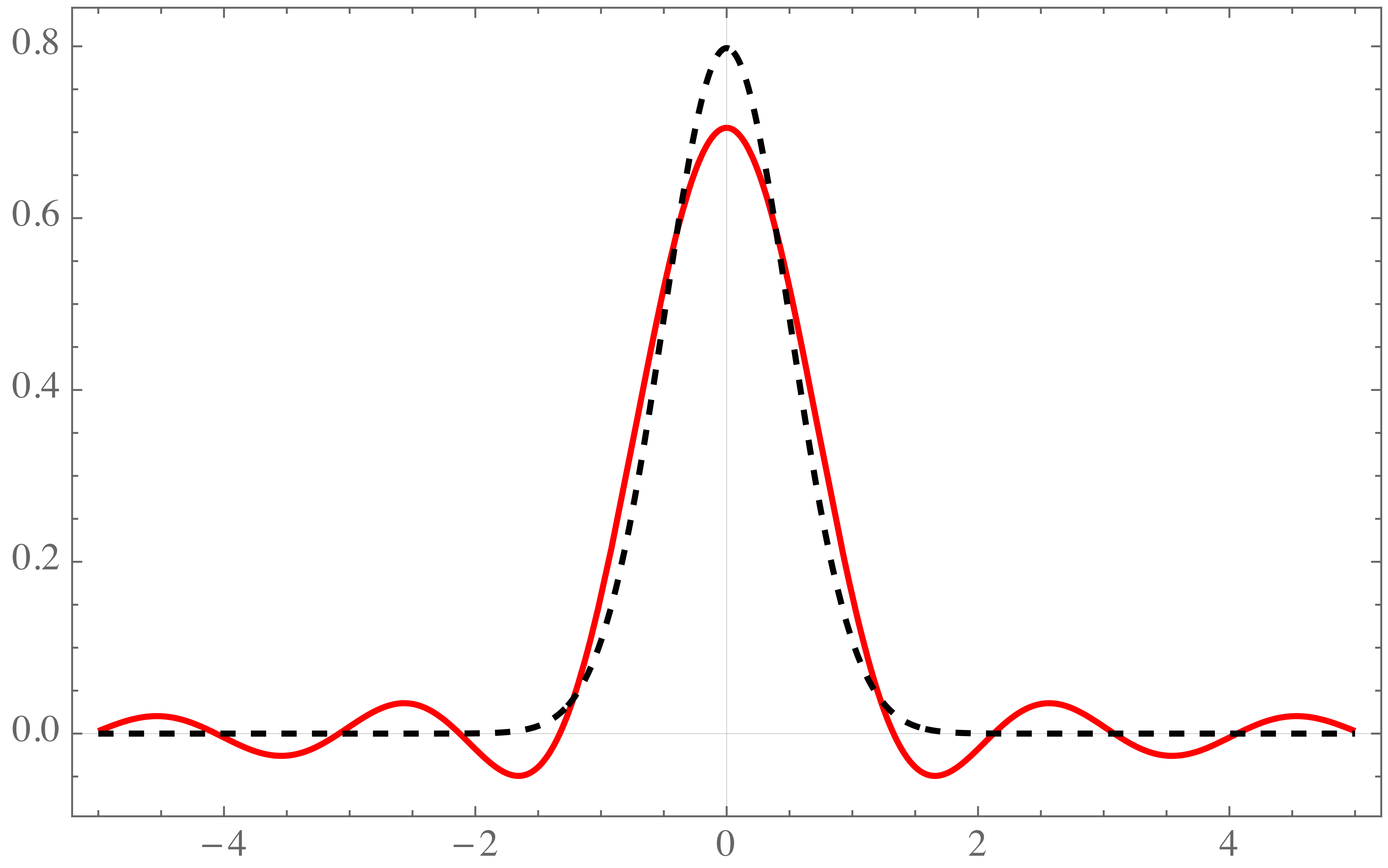
- Localization, probes and smearing in the Brillouin zone

$$\Phi[f] = \sum_{n=-\infty}^{+\infty} f_n(t)\, \phi_n(t) = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi} \tilde{f}(p)\, \phi(p,t)$$

- Compare to band-limited smearing functions in the continuum [A. Kempf 2000]

$$\Phi[f] = \int_{-\infty}^{+\infty} dx\, f(x,t)\, \phi(x,t) = \int_{-\pi/\ell}^{+\pi/\ell} \frac{dp}{2\pi} \tilde{f}(p)\, \phi(p,t)$$

  * Shannon sampling + Brillouin periodicity



- Compare to perfect lattice action [K. Symanzik 1980]

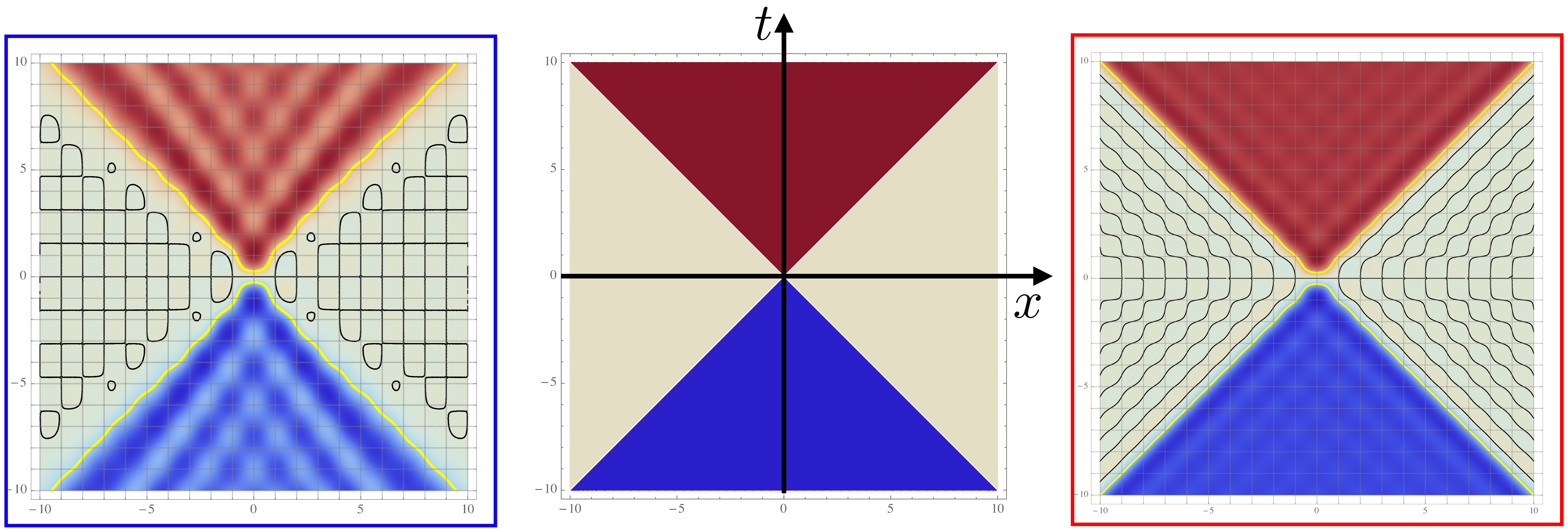
$$V_{nm} = \int_{-\infty}^{+\infty} dx\, \partial_x K(x, n\ell) \partial_x K(x, m\ell)$$

$$K(x,y) = \frac{\sin\left(\frac{\pi}{\ell}(x-y)\right)}{\frac{\pi}{\ell}(x-y)}$$

Causal structure from field commutators

$$\Delta(x,t) \equiv [\phi(0,0), \phi(x,t)]$$

- Comparing to lattice degrees of freedom $\phi_n(t)$

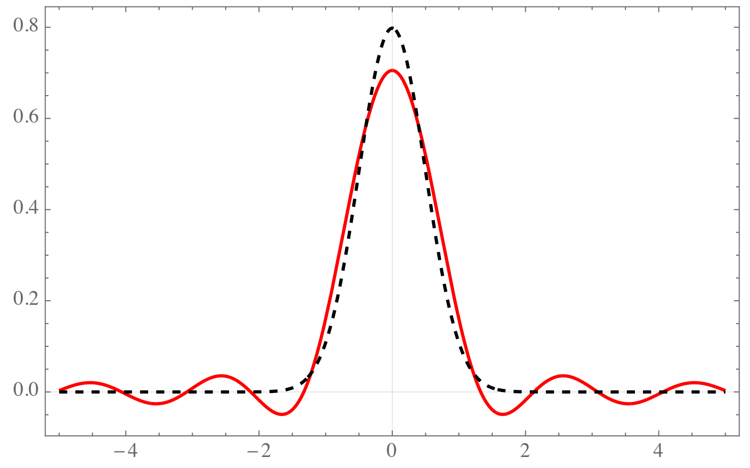


Standard lattice discretiz. | Continuum $\omega_C(p)$ | Brillouin-Poincaré $\omega_B(p)$

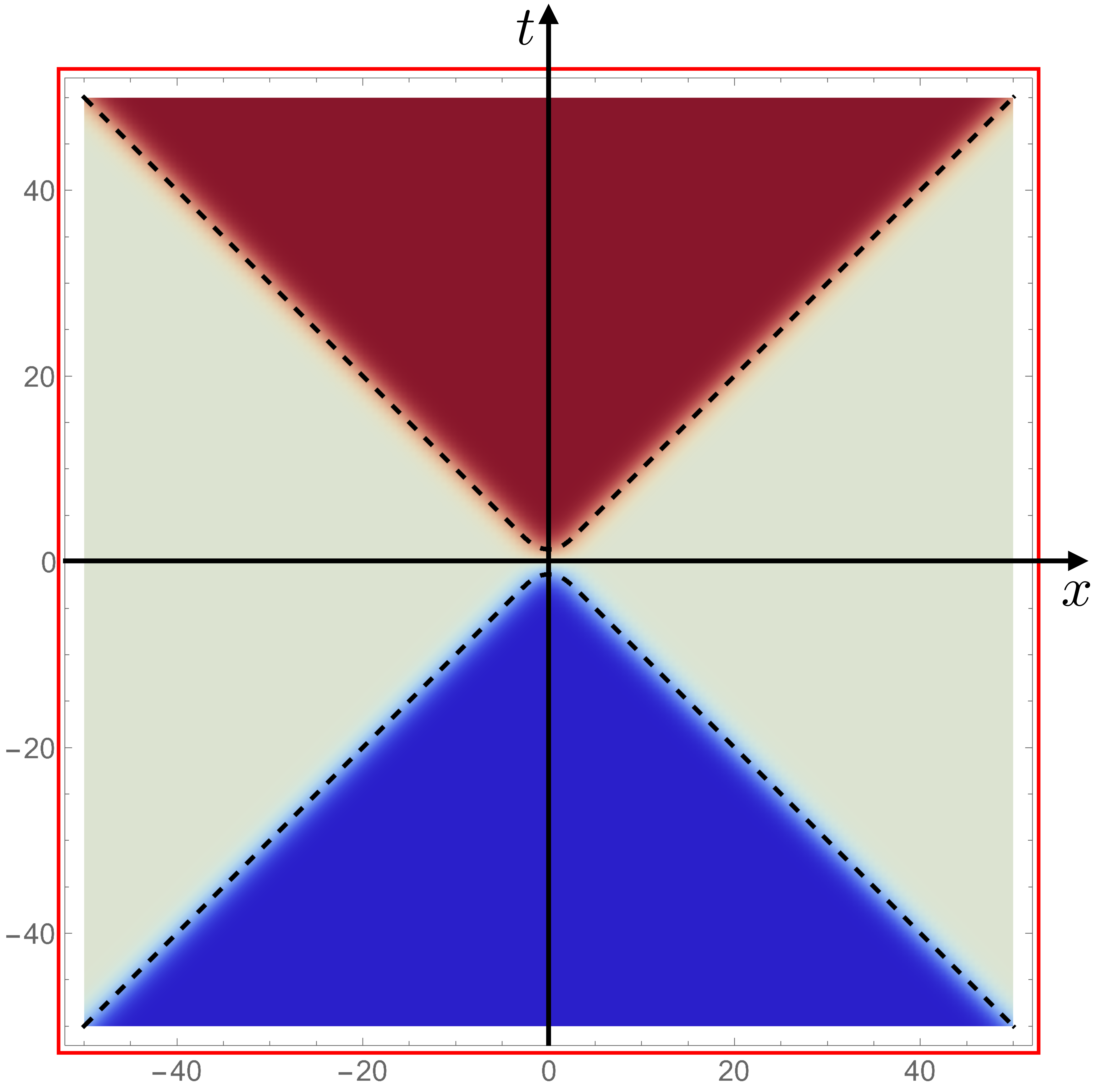
Causal structure from field commutators

$$\Delta(x,t) \equiv [\phi(0,0), \phi(x,t)]$$

- Brillouin observables
  - peaked at (x_0, t_0)
  - band-limited periodic

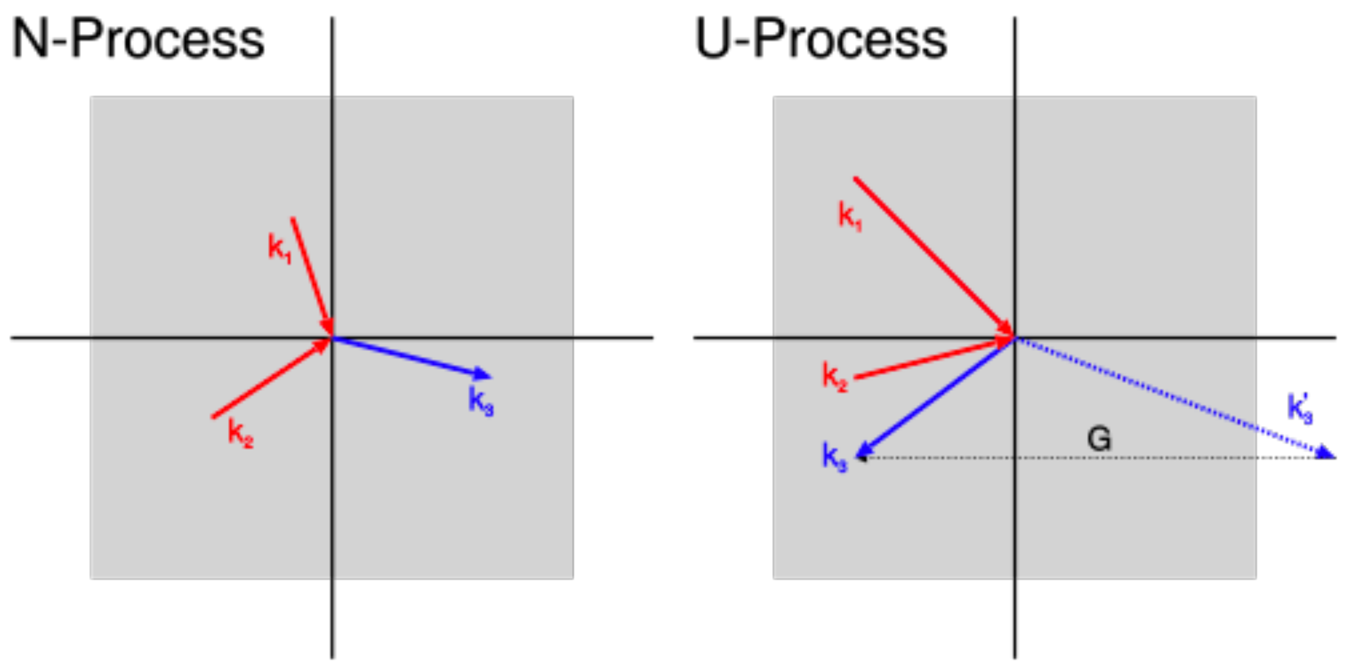


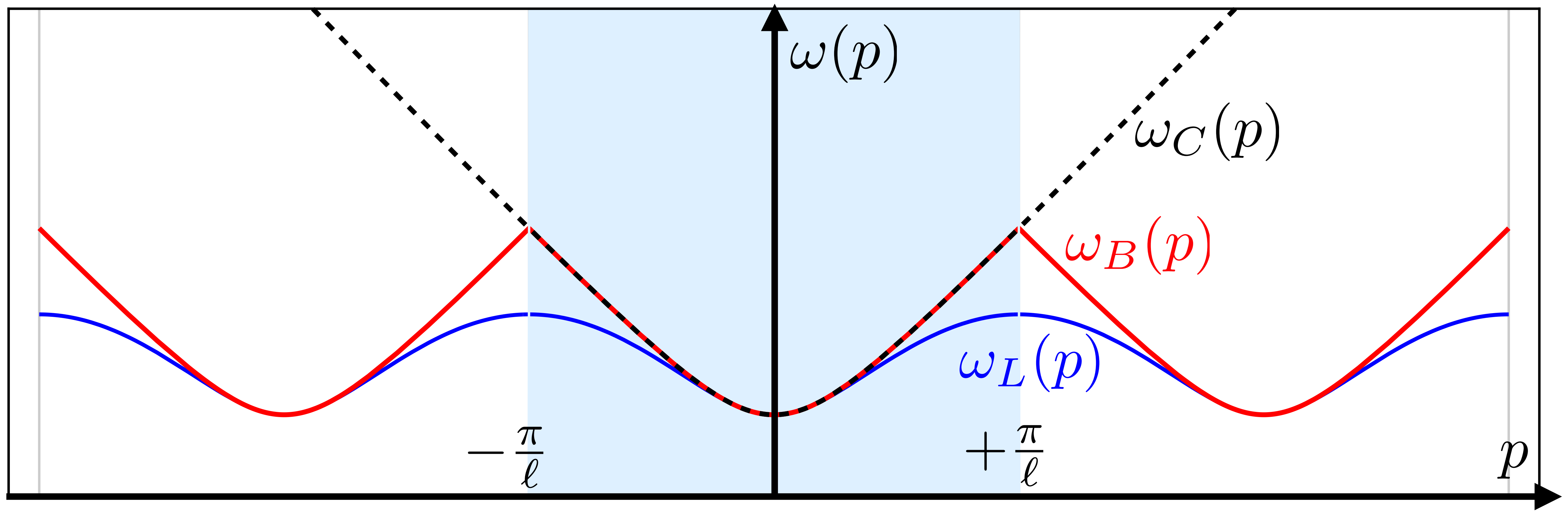
- Identical commutator $[\Phi[f_{(0,0)}], \Phi[f_{(x,t)}]]$ for continuum and Brillouin lattice

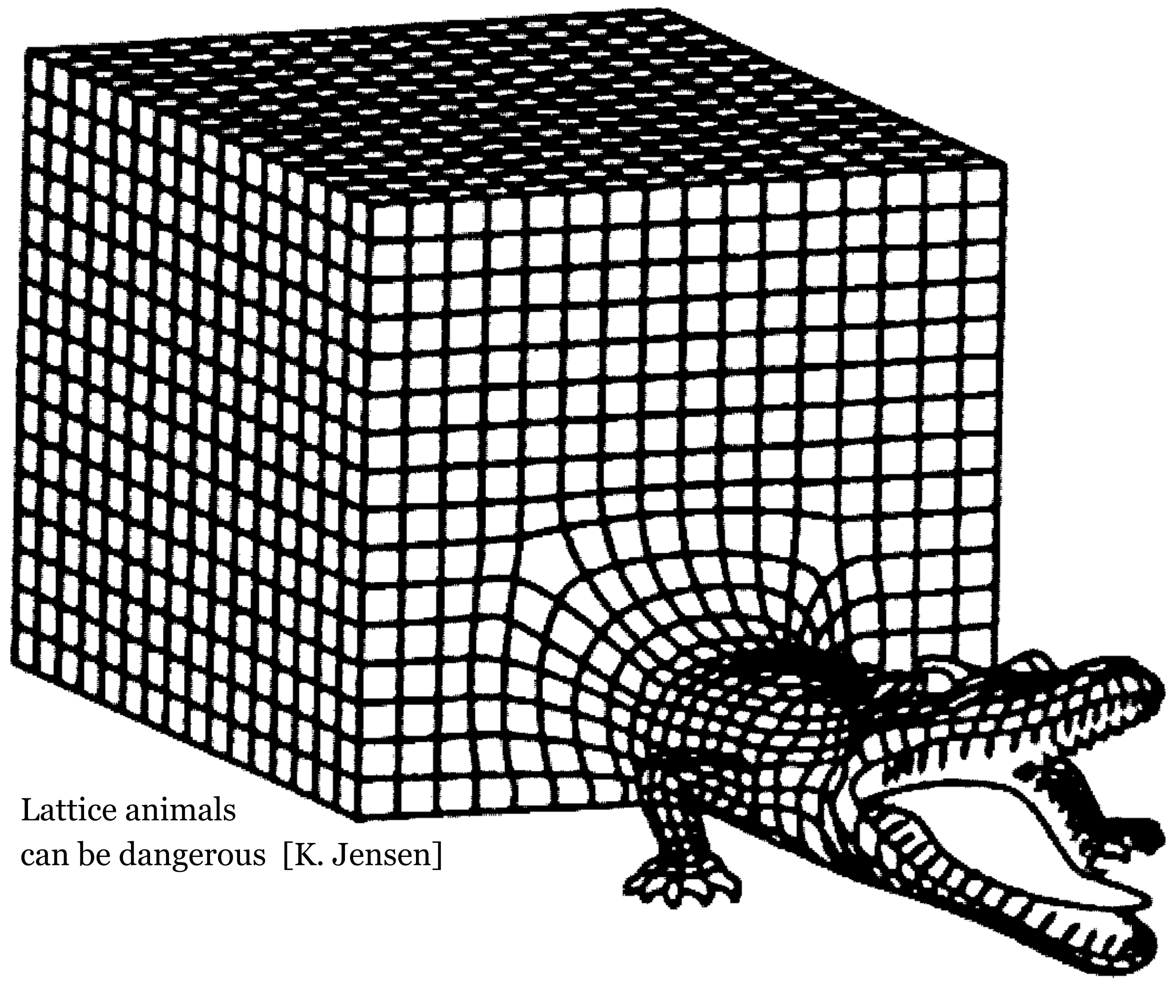


Poincaré invariance on a lattice and Brillouin fields

- Toy model: lattice theory, with non-observable lattice
- Features: non-local discretization, exact Poincaré algebra
  sub-algebra of observables with maximum momentum and periodicity
- Message for QG-theory?
  Reconstruction of Minkowski may require both a state with long-range correlations and and sub-algebra of observable that is insensitive to the microscopic structure
- Message for QG-phen?
  - Lorentz invariant dispersion relations with periodicity
  - Backward (*Umklapp*) scattering







Lattice animals can be dangerous [K. Jensen]