Conformal Dilaton Gravity, Antipodal Mapping and Black Hole Physics on a Warped Spacetime

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An exact time-dependent solution of a black hole is found in a conformally invariant gravity model on a warped Randall-Sundrum spacetime, by writing the metric $g_{\mu\nu} = \omega^{\frac{4}{n-2}} \tilde{g}_{\mu\nu}$. Here $\tilde{g}_{\mu\nu}$ represents the "un-physical" spacetime and ω the dilaton field, which will be treated on equal footing as any renormalizable scalar field. It is remarkable that the 5D and 4D effective field equations for the metric components and dilaton fields can be written in general dimension $n = 4, 5$. The location of the horizon(s) are determined by a quintic polynomial. This polynomial is related to the symmetry group of the icosahedron, isomorphic with the Galois group A_5. We applied the antipodal mapping on the axially symmetric black hole spacetime and make some connection with the information and firewall paradoxes. The dilaton field can be used to describe the different notion the in-going and outside observers have of the Hawking radiation by using different conformal gauge freedom. The disagreement about the interior of the black hole is explained by the antipodal map of points on the horizon. The free parameters of the solution can be chosen in such a way that $\bar{g}_{\mu\nu}$ is singular-free and topologically regular, even for $\omega \to 0$.

Keywords: Conformal Invariance; Dilaton Field; Black Holes; Brane World Models; Antipodal Map.

1. Introduction

One of the greatest challenges of modern physics will be the construction of a quantum-gravity model. In the vicinity of a horizon of a black hole and in the very early universe, these quantum gravity effects will come into play. The quantum features of a black hole were investigated, decades ago, by Hawking in his epic work on radiation effects of a black hole[1] by vacuum polarization. This radiation would be thermal and would contain no information. The black hole will eventually evaporate, and one could say that information is lost, because the anti-particles will fall into the black hole. This would violate quantum mechanics (QM). Related to this issue, is the holographic principle[2], which states that the interior volume of spacetime of a black hole containing the information of the in-going particles is dual to the surface of the horizon. Could it be that the information is still at the horizon? The idea was extended to the well-known Anti-de Sitter/Conformal Field Theory (AdS/CFT) correspondence: is some way, the information must be present in the Hawking radiation. This model relies heavily on string theory, but would solve the information paradox, by introduction the notion of complementarity of the in- and outside of the black hole. The in-going and out-going particles are entangled

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and the information of the in-going particle is also reflected back. However, this viewpoint conflict causality[3]. The previously emitted Hawking radiation and the corresponding in-going particles are independent systems and at the same time indirect entangled. Another solution for the information paradox, which don't rely on string theory, is the introduction of a firewall[4]. The entanglement between the in-going and out-going particles is broken by a high energetic shield. The freely in-falling observer encounters high-energy particles at the horizon. This viewpoint conflicts general relativity, i.e., violation of the equivalence principle. Free falling observers, when falling through the horizon, perceive spacetime as Minkowski, so will not notice the horizon at all. A fundamental issue which is omitted in all the treatments as described above, is the time-dependency of the spacetime structure near the horizon. The emitted Hawking particle will have a back-reaction effect on the spacetime[5,6]. Could it be possible, that the topology of the black hole must be revised? It is well know, that quantum field theory on a curved spacetime opens the possibility that a field theory can have different vacuum states. It can have intrinsic statistical features from a change in topology and not from from a priori statistical description of the matter fields. A spacetime with a given local geometry admits in principle, different possible global topologies. One can consider the modification of the spacetime topology of the form $\widehat{\mathcal{M}}/\Gamma$, where Γ is a discrete subgroup of isometries of $\mathcal{M}$[7–10], without fixed points. $\widehat{\mathcal{M}}$ is non-singular and is obtained from its universal covering $\mathcal{M}$ by identifying points equivalent under Γ. A particular interesting case is obtained, when Γ is the antipodal transformation on $\mathcal{M}$

$$J: \quad P(X) \to \widehat{P}(\widehat{X}). \tag{1}$$

where the light-cone of the antipode of $P(X)$ intersects the light-cone of $P(X)$ only in two point (at the boundary of the spacetime). This is the so-called "elliptic interpretation"[11] of spacetime, where antipodal points represents in fact the same world-point or event. The future and past event horizon intersect each other as a projected cylinder $\mathbb{R}_1 \times \mathbb{S}_1/\mathbb{Z}_2$[a]. At the intersection one then identifies antipodal points. One must realize that the antipodal map is a boundary condition at the horizon, only observable by the outside observer. On a black hole spacetime, the inside is removed. So nothing can escape the interior, since there is no interior. The field theories formulated on $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are globally different, while locally $\mathcal{M}$ and $\widehat{\mathcal{M}}$ are identically. The emitted radiation is only locally thermal. Antipodal identification, however, destroys the thermal features in the Fock space construction. In the construction, one needs unitary evolution operators for the in-going and out-going particles.[5,12]. In order to avoid wormhole constellations or demanding "an other universe" in the construction of the Penrose diagram, it is essential that the asymptotic domain of $\mathcal{M}$ maps one-to-one onto the ordinary spacetime in order to preserve the metric. In fact, one deals with one black hole. A consequence

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[a]We work here in polar coordinates, because the spinning black hole we will consider, has a preferred spin axis. The antipodal identification is then $(U, V, z, \varphi) \to (-U, -V, -z, \pi + \varphi)$.

is that time-inversion take place in region II of the Penrose diagram, so interchange of the creation and annihilation operators and entangling positive energy particles at the horizon with positive energy antiparticles at the antipodes. So the antipodal identification is not in conflict with the general CPT invariance of our world. Further, for the outside observer, the thermodynamically mixed state is replaced by a pure state. So the Hawking particles at opposite sides of the black hole are entangled. The former representation that observers has no access to the inside of the black hole is no longer valid. One arrives by this new geometrical description at pure quantum states for the black hole. It will solve, moreover, the information paradox and firewall problem as well[b]. The gravitational back-reaction as proposed by 't Hooft [13,14], suggests a cut-off of high momenta, which avoids the firewall. The in-going particle has a back-reaction on the other particles, leading to an unitary S-matrix. The gravitational interaction between the in-going and out-going particles will be strong, because we are dealing here with a strongly curved spacetime near the horizon. Using a "cut-and-paste" procedure, one replaces the high-energy particles ("hard"), i.e., mass or momentum of the order of the Planck mass, by low-energy ("soft") particles far away. These hard particles just caused the firewall problem. Hard particles will also influence the local spacetime (to become non-Schwarzschild) and causes the Shapiro effect. The interaction with the soft particles is described by the Shapiro delay. Effectively, all hard particles are quantum clones of all soft particles. By this "firewall-transformation", we look only at the soft particle clones. They define the Hilbert space and leads to a unitary scattering matrix. The net result is that the black hole is actually in a pure state, invalidating the entanglement arguments in the firewall paradox. The entanglement issue can be reformulated by considering the two regions I and II in the maximally extended Penrose diagram of the black hole, as representing two "hemispheres" of the same black hole. It turns out that the antipodal identification keeps the wave functions pure [6] and the central $r = 0$ singularity has disappeared. This gravitational deformation will cause transitions from region I to II in the Penrose diagram. The fundamental construction then consist of the exchange of the position operator with the momentum operator of the in-going particles, which turn them into out-particles. Hereby, 't Hooft expands the moment distributions and position variables in partial waves in (θ, φ) [5]. So the Hawking particles emerging from I are entangled with the particles emerging from II. An important new aspect is the way particles transmit the information they carry across the horizon. In the new model, the Hawking particles emerging from I are maximally entangled with the particles emerging from II. The particles form a pure state, which solves the information paradox. In order to describe the more realistic black holes, such as the axially symmetric Kerr black hole, it is not possible to ignore the dynamics of the horizon. Moreover, one must incorporate gravitation waves. There is another reason to consider axially symmetry. A spherical symmetric

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[b]The technical aspects in constructing the unitary S-matrix can be found in the literature, as provided by the references.

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system cannot emit gravitation waves [15]. Astronomers conjecture that most of the black holes in the center of galaxies are of the Kerr type. A linear approximation is, of course, inadequate in high-curvature situations. In the linear approximation, the waves don't carry enough energy and momentum to affect their own propagation. The notion of the "classical" Hartle-Hawking vacuum thermal state, with a temperature $T \sim \frac{1}{M} \sim \kappa$ and the luminosity $\frac{dM}{dt} \sim -\frac{1}{M^2}$ must also be revised when the mass reaches the order of the Planck mass. On the Kerr black hole spacetime no analog of the Hartle-Hawking vacuum state exists. The Killing field ξ^μ generates a bifurcate Killing horizon ($\xi^\mu \xi_\mu = -1$ at infinity) and possesses spacelike orbits near infinity [16]. Another aspect of the huge curvature in the vicinity of the horizon, will be the problem of constructing a renormalizable (and maintaining unitarity) quantum gravity model of the Standard Model fields, which must be incorporated in the Lagrangian. Up till now, no convincing theory of quantum gravity is available. Many attempts were made in order to make a renormalizable and unitary quantum gravity model. One also can try to construct a renormalizable model, by adding fourth order derivative terms of the curvature tensor (Euler-term). However, one looses unitarity. Also the "old" effective field theory (EFT) has its problems. One ignores what is going on at high energy. In order to solve the anomalies one encounters in calculating the effective action, one can apply the so-called conformal dilaton gravity (CDG) model [5,6,17]. CDG is a promising route to tackle the problems arising in quantum gravity model, such as the loss of unitarity close to the horizon. One assumes local conformal symmetry, which is spontaneously broken (for example by a quartic self-coupling of the Higgs field). Changing the symmetry of the action was also successful in the past, i.e., in the SM of particle physics. A numerical investigation of a black hole solution of a non-vacuum CDG model, was recently performed [18]. The key feature in CDG, is the splitting of the metric tensor $g_{\mu\nu} = \omega^{\frac{4}{n-2}} \tilde{g}_{\mu\nu}$, with ω the dilaton field. Applying perturbation techniques (and renormalization/dimensional regularization), in order to find the effective action and its divergencies, one first integrate over ω (shifted to the complex contour), considered as a conventional renormalizable scalar field and afterwards over $\tilde{g}_{\mu\nu}$ and matter fields. The dilaton field is locally unobservable. It is fixed when we choose the global spacetime and coordinate system. If one applies this principle to a black hole spacetime, then the energy-momentum tensor of ω influences the Hawking radiation. When $\tilde{g}_{\mu\nu}$ is flat, then the handling of the anomalies simplifies considerably [14]. When $\tilde{g}_{\mu\nu}$ is non-flat, the problems are more deep-seated. It is well known, that the antipodal transformation, or inversion, is part of the conformal group [19]. So conformal invariant gravity models could fit very well the models of antipodal mapping as described above. In this context, the modification of GRT by an additional spacetime dimension could be an alternative compromise, because Einstein gravity on the brane will be modified by the very embedding itself and

opens up a possible new way to address the dark energy problem[20c]. These models can be applied to the standard Friedmann-Lemaître-Robertson-Walker (FLRW) spacetime and the modification on the Friedmann equations can be investigated[21]. Recently, Maldacena, et al.[22], applies the RS model to two black hole spacetimes and could construct a traversable macroscopic wormhole solution by adding only a 5D U(1) gauge field (see also Maldacena[23]). However, an empty bulk would be preferable. In stead, one can investigate the contribution of the projected 5D Weyl tensor on the 4D brane. It carries information of the gravitational field outside the brane. If one writes the 5D Einstein equation in CDG setting, it could be possible that an effective theory can be constructed without an UV cutoff, because the fundamental scale M_5 can be much less than the effective scale M_{Pl} due to the warp factor. The physical scale is therefore not determined by M_{Pl}. In this manuscript we will apply the antipodal map on a spinning black hole spacetime in conformal dilaton gravity applied to a warped 5D spacetime.

2. Conformal Transformations and Antipodal Mapping.

2.1. *The origin of the antipodal mapping*

The antipodal map originates from the so-called "elliptic" interpretation[11]. If one considers the hyperboloid H, $-t^2+x^2+y^2+z^2+w^2 = R^2$, then the space-like sections through the origin are ellipses and the time-like sections are hyperbola branches. Since the de Sitter spacetime can be isometrically embedded as a hyperboloid in $\mathbb{R}^5$, one can take $R^2 = -\frac{3}{\Lambda}$. If one suppresses the coordinates (z, w), we have the $\mathbb{R}^3$ Minkowski metric. Lorentz transformations (LT's) around the origin transforms H into itself. Circles on H represent space at different epochs. The bottle-neck parallel is a spatial geodesic, while the others are not. Further, the circumferences contract from $z = -\infty$ to $z = 0$ and then expand. A LT of $\mathbb{R}^3$ turns the bottle-neck into an ellipse, cut out of H with an angle $< 45^o$ with the (x, y)-plane. See figure 1. All the ellipses are equivalent space-like geodesics since each of them is transferred by a suitable automorphism into the bottle-neck, which is one of them. One defines the antipodal map

$$J : P(t, x, y) \rightarrow \widehat{P}(-t, -x, -y), \qquad (2)$$

on H. The antipodicity is Lorentz invariant. When the angle approaches 45°, then the ellipses degenerate into a couple of parallel generators (g_1, g_2) (null geodesics). The other plane of 45° delivers the set (g_3, g_4). The sets (g_1, g_4) and (g_2, g_3) form, for example, the light-cones at the points M and $\widehat{M}$. If one moves upwards along t, the inner angles of the light-cones decrease. Note that the light-cones at P and $\widehat{P}$ has no point in common and the antipodes are joined by a space-like geodesic. Now Schrödinger proposed to identify P and $\widehat{P}$ with the same physical world-point or

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[c]There is another argument in favor of a (warped) 5D spacetime. It turns out, as we shall see, that a surface in 4D can be immersed to 5D, like a Klein bottle.
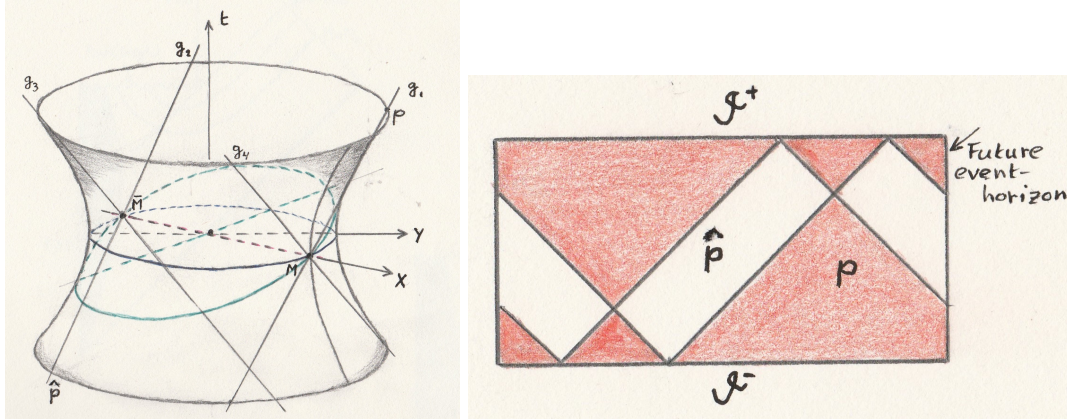
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Fig. 1. Hyperboloid H representing the $\mathbb{R}^3$ spacetime of the compactified de Sitter universe $\mathbb{R}^5$ ((z, w) suppressed). Right: the Penrose diagram. The antipodal points P and $J(P)$ are spacelike separated. An observer moving in de Sitter spacetime cannot meet both P and $J(P)$. He cannot receive a message from P, $J(P)$. Moreover, he cannot receive a message from P and send a message to $J(P)$.

event. One half of H, containing no antipodal points, represents the "whole world". Thereafter, Schrödinger argues in a clever way that the total potential of experiences of any observer is complete and embraces the same events for any two observers, whatever their world lines be. But there is a price we have to pay for[d]. The direction of the arrow of time is lost (or the distinction between the "fore-cone" and "after-cone" is lost). The allotment of past and future is undecidable. The elliptic model is time-reversible. This can open perspective to the general CPT invariance of our world. The real problems arise, when one considers thermodynamical systems, as is the case for the Hawking effect in the vicinity of the horizon of a black hole. Then the entropy comes into play. Note, quoting Schrödinger, "the irreversible laws of thermodynamics can only be based on the statistical microscopically reversible systems on condition that statistical theory be autonomous in defining the arrow of time. If any other law of nature determines this arrow, the statistical theory collapses."

In a pseudo-polar frame $(\chi, T, \theta, \varphi)$ we can write the line element

$$ds^2 = -R^2 dT^2 + R^2 \cosh^2 T \Big[d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\varphi^2) \Big], \tag{3}$$

where $0 < \chi < 2\pi$. The antipodal map becomes now

$$J : (T, \chi, \theta, \varphi) \rightarrow (-T, \pi + \chi, \pi - \theta, \pi + \varphi). \tag{4}$$

We already mentioned that de Sitter can be embedded as a hyperboloid in 5D

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[d]This price is worth paying in the black hole situation, when the information paradox will be solved by the antipodal map. The antipodal half is not time orientable. There is a breakdown of the global distinction between past and future in the interior of the black hole.

Minkowski. We then say that $J : X^\mu \to -X^\mu$ is an inversion $^{\text{e}}$. There exist another coordinate system (introduced by de Sitter himself) in which the line element is written as

$$ds^2 = -(1 - \frac{\rho^2}{R^2})dT'^2 + \frac{1}{(1 - \frac{\rho^2}{R^2})}d\rho^2 + \rho^2(d\theta'^2 + \sin^2\theta' d\varphi'^2), \qquad (5)$$

where we have taken the velocity of the LT $\tanh T = \frac{t}{y}$. This is the static de Sitter and the spaces of constant time are all equivalent. There are singularities for $x = \pm R$ ($\chi = \pm 90^o$), i.e., the points $(M, \widehat{M})$. However, as also observed by Schrödinger, this static model is not adequate for applying the antipodal map. In order to apply the antipodal map on a black hole spacetime in a more general setting, one needs a time dependent spacetime.

2.2. *The "classical" Hawking effect and its problems*

The famous result of Hawking states, that a black hole will radiate at "sufficiently" late times like a black body at a temperature

$$kT \sim \frac{\kappa}{2\pi} = \frac{\hbar c^3}{8\pi GM}, \qquad (6)$$

with κ the surface gravity and M the mass. The entropy should then be $S_{bh} = \frac{kc^3}{4\hbar}A$, with A the area of the horizon. However, one runs into problems by the back-reaction effect of the particle creation, which will alter the area. It is questionable if the ordinary laws of thermodynamics can be applied to a black hole. It is clear that these laws must be constrained to form quantum states with orthonormality and unitarity conditions. Suppose that an isolated black hole completely evaporate within a finite time. Loss of quantum coherence should then occur i.e., an initially pure quantum state should evolve to a mixed state. In general, in the classical picture, a black hole cannot causally influence its exterior, so it is hard to understand the mechanism by which thermal equilibrium could be achieved. Observe that the state of the field at late times in the region I of the Penrose diagram (and so the particles flux reaching infinity) is described by a density matrix by the S-matrix analysis. The particles present in region I are strongly correlated with the particles which entered the black hole at earlier times. Consider now in figure 2 the evolution of two Cauchy surfaces ("time" Σ_1 to "time" Σ_2). When the black hole disappears from the spacetime, then at late times, the entire state of the field is mixed. If one takes the "out" Hilbert space to be the Fock space of the particles propagating out to infinity at late times, one cannot describe particle creation and scattering by an ordinary S-matrix. The initial pure state will evolve to a final density matrix. So we have a breakdown of quantum theory. The antipodal model, however, could "repair" this breakdown.

$^{\text{e}}$The inversion $X_\mu \to -\frac{X_\mu}{X^2}$ (as well as the dilatations) is part of the conformal group [19]. We shall see in the next sections that in general the conformal group is a projective group from 5D. The fifth "degree of freedom" is a sort of gauge space.
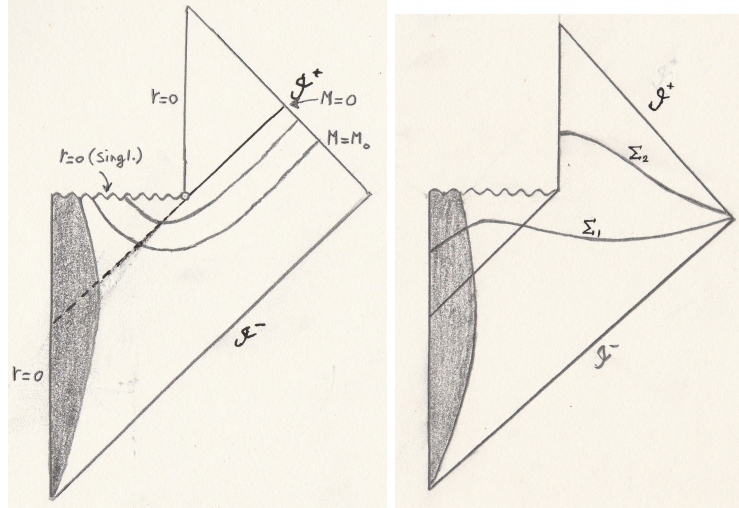
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Fig. 2. Left: formation and evaporation of the Schwarschild black hole. The contour $M = 0$ lies at the retarded time corresponding to the final evaporation. The geometry is flat above this contour. Right: Loss of quantum coherence: evolution from a pure state to a mixed state[16].

3. The black hole solution on a 5D warped spacetime in conformal dilaton gravity

3.1. The 5D warped spacetime

Let us consider the 5D spacetime warped spacetime[21,24]

$$ds^2 = \omega(t,r,y)^2\Big[-N(t,r)^2dt^2+\frac{1}{N(t,r)^2}dr^2+dz^2+r^2(d\varphi+N^\varphi(t,r)dt)^2+dy^2\Big], \quad (7)$$

where y is the extra dimension and ω a warp factor in the formulation of Randall-Sundrum's (RS) 5D warped spacetime with one large extra dimension and negative bulk tension Λ_5. The Standard Model (SM) fields are confined to the 4D brane, while gravity acts also in the fifth dimension. Originally, the RS model was applied to a 5-dimensional anti-de Sitter (AdS) spacetime with a positive brane tension. This is the so-called RS-1 model, with one brane. The RS-2 model treats two branes with $\mathbb{Z}_2$ symmetry. However, the effective cosmological constant on the brane can be zero by fine tuning with the negative Λ_5. In the RS model there is a bound state of the graviton confined to the wall as well as a continuum of Kaluza-Klein (KK) states. Four dimensional gravity is then recovered on the brane and the hierarchy problem seems to be solved. Since the pioneering publication of RS, many investigation were done in diverge domains. In particular, Shiromizu et.al.[24,25], extended the RS model to a fully covariant curvature formalism. It this extended model, an effective Einstein equation is found on the brane, with on the right-hand side a contribution from the 5D Weyl tensor which carries information of the gravitational field outside the brane. So the brane world observer may be

subject to influences from the bulk. The field equations are (were we took an empty bulk)[18]

$$^{(5)}G_{\mu\nu} = -\Lambda_5{}^{(5)}g_{\mu\nu}, \tag{8}$$

$$^{(4)}G_{\mu\nu} = -\Lambda_{eff}{}^{(4)}g_{\mu\nu} + \kappa_4^2{}^{(4)}T_{\mu\nu} + \kappa_5^4 \mathcal{S}_{\mu\nu} - \mathcal{E}_{\mu\nu}, \tag{9}$$

where we have written

$$^{(5)}g_{\mu\nu} = {}^{(4)}g_{\mu\nu} + n_\mu n_\nu, \tag{10}$$

with n^μ the unit normal to the brane. Here $^{(4)}T_{\mu\nu}$ is the energy-momentum tensor on the brane and $\mathcal{S}_{\mu\nu}$ the quadratic contribution of the energy-momentum tensor $^{(4)}T_{\mu\nu}$ arising from the extrinsic curvature terms in the projected Einstein tensor. Further,

$$\mathcal{E}_{\mu\nu} = {}^{(5)}C_{\beta\rho\sigma}^{\alpha} n_\alpha n^{\rho(4)} g_\mu^{\beta(4)} g_\nu^\sigma, \tag{11}$$

represents the projection of the bulk Weyl tensor orthogonal to n^μ. The effective gravitational filed equations on the brane are not closed. One must solve at the same time the 5D gravitation field in the bulk.

3.2. *The conformal dilaton gravity (CDG) model on a 5D warped spacetime*

One can distinguish several possible "routes" to the unification of GR and QFT. One can start, for example, with a given classical theory and applies heuristic quantization rules. One then can make a division in canonical and covariant approaches, i.e., uses a Hamiltonial formalism or employs covariance at some stage. The CDG model we consider here, is part of the covariant approach to quantum gravity. The key feature in CDG, is the splitting of the metric tensor[5,17]

$$g_{\mu\nu} = \omega^{\frac{4}{n-2}} \tilde{g}_{\mu\nu}, \tag{12}$$

with ω the dilaton field and $\tilde{g}_{\mu\nu}$ the ""un-physical" spacetime. At high energy, ω will be treated as a (renormalizable) quantum field. One can prove that the action (without matter terms for the time being)

$$S = \int d^n x \sqrt{-\tilde{g}} \Big[\frac{1}{2} \xi \omega^2 \tilde{R} + \frac{1}{2} \tilde{g}^{\mu\nu} \partial_\mu \omega \partial_\nu \omega + \Lambda \kappa^{\frac{4}{n-2}} \xi^{\frac{n}{n-2}} \omega^{\frac{2n}{n-2}} \Big], \tag{13}$$

is conformal invariant under

$$\tilde{g}_{\mu\nu} \to \Omega^{\frac{4}{n-2}} \tilde{g}_{\mu\nu}, \quad \omega \to \Omega^{-\frac{n-2}{2}} \omega. \tag{14}$$

The covariant derivative is taken with respect to $\tilde{g}_{\mu\nu}$. For details, see Slagter[18]. Now we implement the 5D warped spacetime Eq.(7). So

$$^{(5)}g_{\mu\nu} = \omega^{4/3(5)} \tilde{g}_{\mu\nu}, \qquad {}^{(5)}\tilde{g}_{\mu\nu} = {}^{(4)}\tilde{g}_{\mu\nu} + n_\mu n_\nu, \tag{15}$$

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and write again

$$^{(4)}\tilde{g}_{\mu\nu} = \bar{\omega}^2 \bar{g}_{\mu\nu}. \tag{16}$$

Variation of the action leads to the field equations

$$\xi\omega\tilde{R} - \tilde{g}^{\mu\nu}\tilde{\nabla}_\mu\tilde{\nabla}_\nu\omega - \frac{2n}{n-2}\Lambda\kappa^{\frac{4}{n-2}}\xi^{\frac{n}{n-2}}\omega^{\frac{n+2}{n-2}} = 0 \tag{17}$$

and

$$\omega^2\tilde{G}_{\mu\nu} = T^\omega_{\mu\nu} - \Lambda\tilde{g}_{\mu\nu}\kappa^{\frac{4}{n-2}}\xi^{\frac{2}{n-2}}\omega^{\frac{2n}{n-2}}, \tag{18}$$

with

$$T^\omega_{\mu\nu} = \tilde{\nabla}_\mu\tilde{\nabla}_\nu\omega^2 - \tilde{g}_{\mu\nu}\tilde{\nabla}^2\omega^2 + \frac{1}{\xi}\Big(\frac{1}{2}\tilde{g}_{\alpha\beta}\tilde{g}_{\mu\nu} - \tilde{g}_{\mu\alpha}\tilde{g}_{\nu\beta}\Big)\partial^\alpha\omega\partial^\beta\omega. \tag{19}$$

From the 5D Einstein equations Eq.(8) one obtains $\omega(t,r,y) = \omega_1(t,r)\omega_2(y)$, with $\omega_2(y) = l$=constant (the length scale of the extra dimension). The dilaton equations Eq.(17) is superfluous. Note that the effective Einstein equations Eq.(9) contains the $\mathcal{E}_{\mu\nu}$, while $T_{\mu\nu}$ and $\mathcal{S}_{\mu\nu}$ are taken zero in our case. The dilaton equation is again superfluous.

It turns out that one can write the field equations for ω and N in the form (n=4,5)

$$\ddot{\omega} = -N^4\omega'' + \frac{n}{\omega(n-2)}\Big(N^4\omega'^2 + \dot{\omega}^2\Big), \tag{20}$$

$$\ddot{N} = \frac{3\dot{N}^2}{N} - N^4\Big(N'' + \frac{3N'}{r} + \frac{N'^2}{N}\Big)$$
$$-\frac{n-1}{(n-3)\omega}\Big[N^5\Big(\omega'' + \frac{\omega'}{r} + \frac{n}{2-n}\frac{\omega'^2}{\omega}\Big) + N^4\omega'N' + \dot{\omega}\dot{N}\Big]. \tag{21}$$

One can solve these equations exact (we took $\Lambda_{eff} = 0$):

$$\omega = \Big(\frac{a_1}{(r+a_2)t + a_3 r + a_4}\Big)^{\frac{1}{2}n-1},$$
$$N^2 = \frac{1}{5r^2}\frac{10a_2^3 r^2 + 20a_2^2 r^3 + 15a_2 r^4 + 4r^5 + C_1}{C_2(a_3+t)^4 + C_3}, \tag{22}$$

with a_i some constants. There is a constraint equation

$$\bar{\omega}'' = -\frac{2n}{n-2}\frac{\Lambda l\kappa^{\frac{4}{(n-2)}}\xi^{\frac{n-2}{4(n-1)}}\bar{\omega}^{\frac{n+2}{n-2}}}{N^2} - \frac{\omega'N'}{N} - \frac{\omega'}{2r} + \frac{4}{n-2}\frac{\dot{\bar{\omega}}^2}{\bar{\omega}N^4} - \frac{\dot{\bar{\omega}}\dot{N}}{N^5}, \tag{23}$$

which l the dimension of y. The solution for the two dilaton fields ω and $\bar{\omega}$ differs only by the different exponent $\frac{3}{2}$ and 1 respectively. The solution For the metric component is the same (apart from the constants). The solution for the angular momentum component is

$$N^\varphi = F_n(t) + \int \frac{1}{r^3\bar{\omega}^{\frac{n-1}{n-3}}}dr. \tag{24}$$

The Ricci scalar for $\bar{g}_{\mu\nu}$ $(\Lambda = 0)$ is given by

$$\bar{R} = \frac{12}{N^2}\Big[\dot{\bar{\omega}}^2 - N^4\bar{\omega}'^2\Big], \tag{25}$$

with is consistent with the null condition for the two-dimensional (t, r) line element, when $\bar{R} = 0$. One can easily check that the trace of the Einstein equations is zero. Note that N^2 can be written as

$$N^2 = \frac{4\int r(r + a_2)^3 dr}{C_2(a_3 + t)^4 + C_3 r^2}. \tag{26}$$

So the spacetime seems to have two poles. However, the $r = 0$ is questionable. The conservation equations become

$$\bar{\nabla}^\mu \mathcal{E}_{\mu\nu} = \bar{\nabla}^\mu\Big[\frac{1}{\bar{\omega}^2}\Big(-\Lambda\kappa^{2(4)}\bar{g}_{\mu\nu}\bar{\omega}^4 + {}^{(4)}T_{\mu\nu}{}^{(\bar{\omega})}\Big)\Big], \tag{27}$$

which yields differential equations for $\ddot{N}'$ and $\dot{N}$ as boundary conditions at the brane. It can be described as the non-local conservation equation. In the high energy case close to the horizon, one must include the $\mathcal{S}_{\mu\nu}$ term. So the divergence of $\mathcal{E}_{\mu\nu}$ is constrained. In the non-conformal case, Eq.(27) contains on the right hand side also the quadratic correction $\mathcal{S}_{\mu\nu}$ of the matter fields on the brane. The effective field equations, Eq.(9), are then not a closed system. One needs the Bianchi equations. In fact, $\mathcal{E}_{\mu\nu}$ encodes corrections from the 5D graviton effects and are for the brane observer non-local. In our model under consideration, we have only the $T_{\mu\nu}^{(\omega)}$ term and no source terms (only the 5D Λ_5). But it still sources the KK modes. The dilaton ω plays the role of a "scalar field". But we don't need the 5D equations themselves, because the solution for N is the same! It is only the $\omega^{4/3}$ which represents the 5D contribution. There is no exchange of energy-momentum between the bulk and brane. If one applies the model to a FLRW model[?], then the evolution equations are very complicated. Inhomogeneous and anisotropic effects from the 4D matter radiation distribution on the brane are sources for the 5D Weyl tensor $\mathcal{E}_{\mu\nu}$ and cause non-local back-reaction on the brane. One needs an approximation scheme in order to find the missing evolution equation for $\mathcal{E}_{\mu\nu}$.
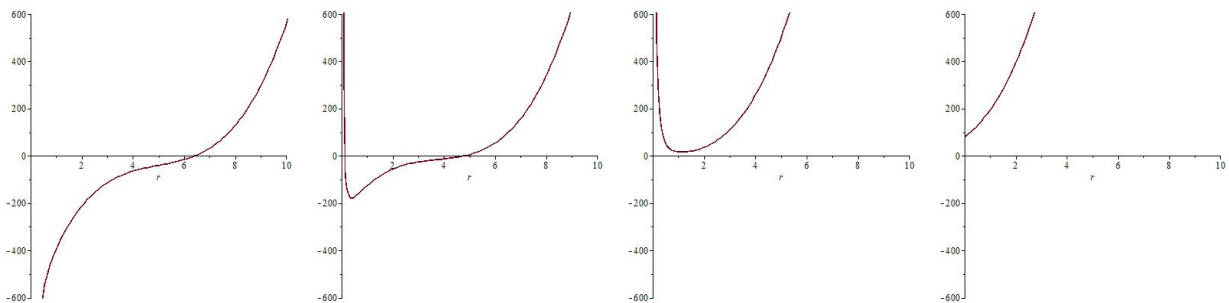

Fig. 3.   Four possible plots of N^2 as function of r.

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The locations of the horizon's and ergo-spheres are found by solving $N^2 = 0$ and $\bar{g}_{tt} = 0$ respectively. N^2 becomes singular at coordinate time $t = t_H = -b_3 + \sqrt[4]{-\frac{C_3}{C_2}}$. However, $\bar{g}_{\mu\nu}$ can be made regular everywhere and singular free by suitable choices of the parameters b_i, c_i and C_i. For $C_1 = 0$, $\bar{g}_{\mu\nu}$ has one real zero $r_H = \sim |1.606 b_2|$ and two complex zero's $\sim (0.178 \pm 0.638 I) b_2$. In figure 3 we plotted the possible graphs. If one ignores the contribution from the bulk, then N^2 has for $C_1 = 0$ no real roots, so only naked singularities. The contribution from the bulk then generates at least one horizon.

3.3. *Penrose diagram*

If we define the coordinates, $dr^* \equiv \frac{1}{N_1(r)^2} dr$ and $dt^* \equiv N_2(t)^2 dt$, then our induced spacetime can be written as

$$ds^2 = \omega^{4/3} \bar{\omega}^2 \Big[\frac{N_1^2}{N_2^2} \Big(-dt^{*2} + dr^{*2} \Big) + dz^2 + r^2 (d\varphi + \frac{N^\varphi}{N_2^2} dt^*)^2 \Big], \qquad (28)$$

with

$$N_1^2 = \frac{10 b_2^3 r^2 + 20 b_2^2 r^3 + 15 b_2 r^4 + 4 r^5 + C_1}{5 r^2}, N_2^2 = \frac{1}{C_2 (t + b_3)^4 + C_3} \qquad (29)$$

and

$$r^* = \frac{1}{4} \sum_{r_i^H} \frac{r_i^H \log(r - r_i^H)}{(r_i^H + b_2)^3}, \qquad t^* = \frac{1}{4 C_2} \sum_{t_i^H} \frac{\log(t - t_i^H)}{(t_i^H + b_3)^3}. \qquad (30)$$

The sum it taken over the roots of $(10 b_2^3 r^2 + 20 b_2^2 r^3 + 15 b_2 r^4 + 4 r^5 + C_1)$ and $C_2 (t + b_3)^4 + C_3$, i. e., r_i^H and t_i^H. This polynomial in r defining the roots of N_1^2, is a quintic equation, which has some interesting connection with Klein's icosahedral solution (see appendix). Further, one can define the azimuthal angular coordinate $d\varphi^* \equiv (d\varphi + \frac{N^\varphi}{N_2^2} dt^*)$, which can be used when an incoming null geodesic falls into the event horizon. φ^* is the azimuthal angle in a coordinate system rotating about the z-axis relative to the Boyer-Lindquist coordinates. Next, we define the coordinates [27] (in the case of $C_1 = C_3 = 0$ and 1 horizon, for the time being)

$$\begin{aligned} U_+ = e^{\kappa(r^* - t^*)}, \quad V_+ = e^{\kappa(r^* + t^*)} \qquad r > r_H \\ U_- = -e^{\kappa(r^* - t^*)}, \quad V_- = -e^{\kappa(r^* + t^*)} \qquad r < r_H, \end{aligned} \qquad (31)$$

with κ a constant. The spacetime becomes

$$ds^2 = \omega^{4/3} \bar{\omega}^2 \Big[\frac{N_1^2}{N_2^2} \log \Big(UV \Big)^{\frac{1}{2\kappa}} dU dV + dz^2 + r^2 d\varphi^{*2} \Big]. \qquad (32)$$

In figure 4 we plotted the Penrose diagram (left). The antipodal points $P(X)$ and $\bar{P}(\bar{X})$ are physically identified. If we compactify the coordinates,

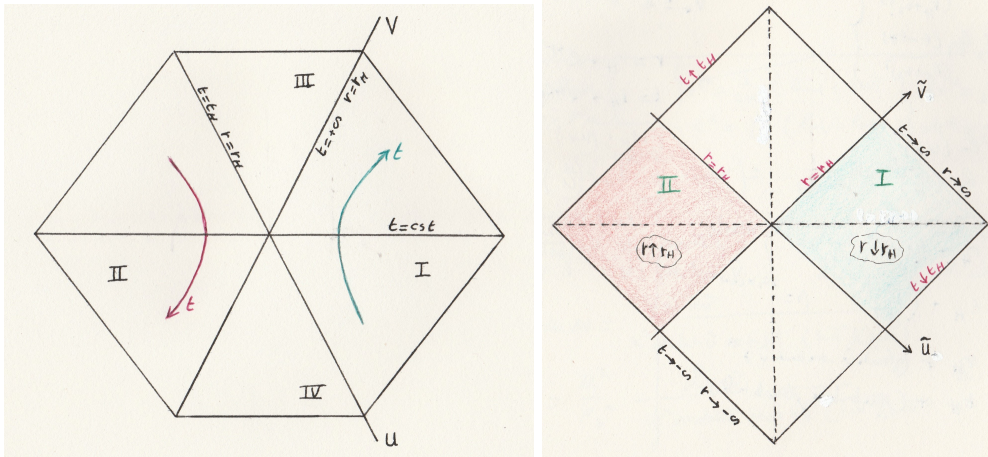$$\tilde{U} = \tanh U, \qquad \tilde{V} = \tanh V, \qquad (33)$$

Fig. 4. Left: Kruskal diagram for $\bar{g}_{\mu\nu}$ in (U, V)- coordinates. If one approaches the horizon from the outside and passes the horizon, one approaches from the "otherside" the horizon. Right: Kruskal diagram for $\bar{g}_{\mu\nu}$ in $(\tilde{U}, \tilde{V})$- coordinates.

then the spacetime can be written as

$$ds^2 = \omega^{4/3}\bar{\omega}^2\Big[H(\tilde{U},\tilde{V})d\tilde{U}d\tilde{V} + dz^2 + r^2 d\varphi^{*2}\Big], \tag{34}$$

with

$$H = \frac{N_1^2}{N_2^2}\frac{1}{\kappa^2 \operatorname{arctanh}\tilde{U} \operatorname{arctanh}\tilde{V}(1-\tilde{U}^2)(1-\tilde{V}^2)}. \tag{35}$$

We can write r and t as

$$r = r_H + \Big(\operatorname{arctanh}\tilde{U} \operatorname{arctanh}\tilde{V}\Big)^{\frac{1}{2\kappa\alpha}}, \quad t = t_H + \Big(\frac{\operatorname{arctanh}\tilde{V}}{\operatorname{arctanh}\tilde{U}}\Big)^{\frac{1}{2\kappa\beta}}, \tag{36}$$

with

$$\alpha = \frac{r_H}{4(r_H+b_2)^3}, \qquad \beta = \frac{1}{4C_2(t_H+b_3)^3}. \tag{37}$$

Observe that N_1 and N_2 can be expressed in $(\tilde{U}, \tilde{V})$. The Penrose diagram is drawn in figure 4 (right). Note that ds^2 and H are invariant under $\tilde{U} \to -\tilde{U}$ and $\tilde{U} \to -\tilde{U}$. $\tilde{g}_{\mu\nu}$ is regular everywhere and conformally flat. The "scale-term" H is consistent with the features of the Penrose diagram. Now we have still the φ dependency. We assume no z-dependency. It is expected that the differential equation for ω can be separated in a (U, V) part and a φ part. The method of 't Hooft can then be applied. In the next sections, we will briefly come back to this issue.

4. Related issues of the new black hole solution

4.1. *Treatment of the quantum fields*

The physical identification in the de Sitter spacetime of $P(X)$ and $\widehat{P}(\widehat{X})$ are considered as different representations in Kruskal space of one and the same Schwarzschild

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event. There is only one world with one singularity and one exterior region. Fields which are symmetric under J are identified as

$$\Psi_{JS} = \frac{1}{2}\Big[\Psi(X) + \widehat{\Psi}(\widehat{X})\Big]. \tag{38}$$

One then builds these fields from fields with arguments specified in I^8. Each of these fields, positive or negative frequency in I, can be extended to global spacetime surfaces. However, due to the time reversal, the inner product on the full Hilbert space have zero norm for the symmetric fields. One then defines negative frequency functions $\Psi^{\uparrow}_{(-)}(X) = \Psi^{\downarrow}_{(+)}(JX)$ and $\Psi^{\downarrow}_{(-)}(X) = \Psi^{\uparrow}_{(+)}(JX)$, where the arrows stands for the solutions on the future/past singularity. The symmetric (anti-) solutions ($\epsilon = \pm 1$) are then

$$\Psi^{(1)}_{JS}(X) = \frac{1}{2}\Big[\Psi^{\uparrow}_{(+)}(X) + \epsilon\Psi^{\downarrow}_{(-)}(X)\Big], \quad \Psi^{(2)}_{JS}(X) = \frac{1}{2}\Big[\Psi^{\downarrow}_{(+)}(X) + \epsilon\Psi^{\uparrow}_{(-)}(X)\Big]. \tag{39}$$

Introducing then reflection and transmission coefficients, one can construct a wave function regular at the singularities, $\Psi^{(r)}_{JS} = \frac{\epsilon K}{(K+\epsilon)^2}\Big[\Psi^{(1)}_{JS} + \epsilon\Psi^{(2)}_{JS}\Big]$, with $K = e^{\pi\omega/\kappa}, \kappa = 1/4M$. Thereafter, one constructs hermitian field operators for the Fock space. Next, one needs the renormalized expectation value of the stress-energy tensor $< T_{\mu\nu} >$ in the "semiclassical" equations of Einstein $G_{\mu\nu} = 8\pi G < T_{\mu\nu} >$. If one assumes that there is a $r = 0$ singularity, then back-reaction will be small in the vicinity of the horizon (at least for massless fields). The spacetime can then be approximated by Schwarzschild geometry. The mass will decrease slowly with time and evaporates. In a flat spacetime, this is easily done, because the vacuum is well defined. One can calculate the zero-energy state and can construct finite quantum operators. In curved spacetime, the vacuum state is dependent of the boundary condition for the propagators (positive frequency modes). In principle, we can follow the method of Sanchez (for the de Sitter spacetime) for the dilaton field and our "un-physical" spacetime $\bar{g}_{\mu\nu}$ ($\Lambda = 0$),

$$< \bar{\omega}^2 > \bar{G}_{\mu\nu} = < T^{(\omega)}_{\mu\nu}(\bar{\omega}, \bar{g}_{\mu\nu}) > - < \bar{\omega}^2 > \mathcal{E}_{\mu\nu}, \tag{40}$$

where $T^{(\omega)}_{\mu\nu}$ depends on the geometry and boundary conditions (see Eq.(19)). Further, $< T^{(\omega)} > = - < \bar{\omega}^2 > \bar{R}$, because $\mathcal{E}_{\mu\nu}$ is traceless. We have now contributions from the antipode:

$$< T^{(\omega)}_{\mu\nu} > \rightarrow < T^{(\omega)}_{\mu\nu} > \pm < \widehat{T}^{(\omega)}_{\mu\nu} >, \quad < \bar{\omega}^2 > \rightarrow < \bar{\omega}^2 > \pm < \widehat{\bar{\omega}^2} > . \tag{41}$$

In the simplified de Sitter space, one then easily construct Green functions [9]

$$G_{\alpha JS}(X, X') = e^{2\alpha}\Big[G(X.X') + G(X, JX')\Big],$$
$$G_{\alpha JA}(X, X') = e^{2\alpha}\Big[G(X.X') - G(X, JX')\Big], \tag{42}$$

with α labels the one parameter family of the de Sitter vacua. The expectation values for a scalar field and the energy momentum tensor can then be calculated.

One obtain, for example[8],

$$< \widehat{\Phi^2_{JS,JA}} >= \frac{1}{16\pi \cos \pi\nu}[m^2 + (\xi - \frac{1}{6})R], \qquad (43)$$

with $\nu = (9/4 - M^2/H^2)^{1/2}, M^2 = m^2 + \xi R$, m the mass of the field and $H \sim \Lambda$. In our case we have no scalar field, but instead ω. The expression for $T_{\tilde{U}\tilde{U}}$ becomes[29]

$$T_{UU}^{(\omega)} = \frac{c_1 e^{-2c_1 U}}{c_3^2 c_4^2(c_2\rho + c_3)^2}(c_2 c_3 \rho^2 F(U) - c_1^2 c_2^2 \rho^2 + c_1 c_3^2), \qquad (44)$$

which can be used to evaluate the expectation value. In order to apply the full antipodal map, one includes the φ-dependency in the dilaton equation. The relevant operator (d'Alembertian) can be separated in the used coordinate system. The relevant φ contribution comes from periodic Mathieu functions (in variable φ). They converge uniformly on all compact sets in the z-plane. Next, one applies the method of 't Hooft, by expanding the position variables $u^\pm(z,\varphi)$ and momentum distributions $p^\pm(z,\varphi)$ in the partial waves of Mathieu functions[f]. Further, one then calculates the gravitational shift $\delta\tilde{U}(z,\varphi)$, in order to carry a particle over from I to II, or back[6], using the Shapiro delay.

4.2. *The surface gravity and the conformal gauge*

Since we have now the description of the antipodal map in our black hole spacetime, we will look more closely at the conformal invariance. First of all, one should rely in the dynamical situation on (conformal) Killing vectors in order to describe the spacetime symmetries. Our Lagrangian is conformal invariant under Eq.(14), so we can use the freedom of the conformal factor Ω. Remember, different ω means different notion of the vacuum state for the in-going and outside observer, so they will use different conformal gauge freedom. It is desirable that for the out-going observer, the surface gravity of the horizon is conformal invariant. Further, conformal transformations must preserve affinely parameterized null geodesics. This will deliver Ω for the in-going observer. We can define out-going and in-going null normals[18] for $\bar{g}_{\mu\nu}$

$$\bar{l}^\mu = (1, N\sqrt{N^2 - r^2 N^{\varphi 2}}, 0, 0),$$
$$\bar{m}^\mu = \left(-\frac{1}{2r^2 N^{\varphi 2} - 2N^2}, -\frac{N}{2\sqrt{N^2 - r^2 N^{\varphi 2}}}, 0, 0\right). \qquad (45)$$

with $\bar{l}^\mu \bar{l}_\mu = \bar{m}^\mu \bar{m}_\mu = 0, \bar{l}^\mu \bar{m}_\mu = -1$. The surface gravity then becomes

$$\kappa = 2N\left(\partial_r(\sqrt{N^2 - r^2 N^{\varphi 2}}) + \partial_t\left(\frac{1}{N}\right)\right) = 2N\left(\partial_t \sqrt{\bar{g}_{rr}} - \partial_r \sqrt{\bar{g}_{tt}}\right). \qquad (46)$$

This is consistent with the metric definition of κ.

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[f]So the spherical harmonics are replaced by the Mathieu harmonics

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4.3. *The meaning of the warped spacetime*

Let us now return to $g_{\mu\nu} = \omega^{4/3}\bar{\omega}^2\bar{g}_{\mu\nu}$. In the CDG setting, the evaporation of the black hole is also determined by the complementarity transformation of ω between the in-going and outside observer. Our spacetime is now ($b_4 = b_3 b_2$)

$$ds^2 = \omega^{4/3}\bar{\omega}^2\Big[(C_2(t+b_3)^4 + C_3)\frac{(10b_2^3 r^2 + 20b_2^2 r^3 + 15b_2 r^4 + 4r^5 + C_1)}{5r^2}\cdot$$
$$\Big(-dt^{*2} + dr^{*2}\Big) + dz^2 + r^2(d\varphi + \frac{N^\varphi}{N_2^2}dt^*)^2\Big], \quad (47)$$

with

$$\omega^{4/3}\bar{\omega}^2 = \frac{1}{(r+c_2)^2(t+c_3)^2(r+b_2)^2(t+b_3)^2}, \quad (48)$$

We observe that $\omega^{4/3}\bar{\omega}^2$ approaches zero for coordinate time $t \to \infty$, so $g_{\mu\nu}$ shrinks to zero, so the distant observer sees a gradually shrinking black hole when the metric time runs to infinity. Further, the only contribution from the 5D spacetime is the $\omega^{4/3}$. Remarkable, the projected Weyl component is necessary in order to obtain the same form of N^2 and to avoid naked singularities. So $\omega^{4/3} = [(r+c_2)^2(t+c_3)^2]^{-1}$ is the "scale" term from the 5D warped spacetime (the warpfactor in the RS model is the product of y-dependent part and ω part). Suppose one wants combine the conformal transformation with an internal symmetry transformation, i.e., a spacetime transformation. In particular, the scale transformations. One can proof in that case, $\Box \log \Omega = 0$ [19], which is consistent with our 2D null hypersurface of Eq.(32). Further, in dimension $n \neq 4$ only the scale-invariant theories based upon scalar fields (so ω from 5D) are conformally invariant. Conclusion: $\bar{\omega}$ of our $\bar{g}_{\mu\nu}$ can be used in non-vacuum models. An additional advantage of the warped spacetime in connection with cosmology and hierarchy problem, was already mentioned in the introduction. A new aspect will be the embedding of the 5D in the 4D spacetime and the relation with the 3D BTZ blackhole solution.

4.4. *The relation with the 3D Baňados-Teitelboim-Zanelli black hole*

In the spacetime under consideration, the dz^2 term can be omitted. One obtains then the 3D Baňados-Teitelboim-Zanelli (BTZ) black hole spacetime. It solves the Einstein equations with a negative cosmological constant [28]. The BTZ solution is related to the AdS/CFT correspondence and intensively studies in connection with black hole entropy issues. However, we should like to take the cosmological constant zero. In a former study [29], an exact solution was found in a CDG setting in Eddington-Finkelstein retarded coordinates (U, ρ) (or advanced V) where the antipodal map $(U, V, \varphi) \to (-U, -V, \varphi + \pi)$ is applicable:

$$ds^2 = \frac{e^{-2c_1 U}}{(c_2\rho + c_3)^2}\Big[\pm\frac{c_1(c_3^2 - c_2^2\rho^2)}{c_2 c_3}dU^2 - 2dU d\rho + dz^2 + \rho^2\Big(d\xi + F(U)dU\Big)^2\Big], \quad (49)$$

which is Ricci flat, while $\tilde{R}^{(4)} = \frac{6c_1c_2}{c_3}$. The function $F(U)$ will be fixed when matter terms are incorporated (i.e. for example, a scalar gauge field). The metric Eq.(49) will then contain a term $b(U,\rho)^2 d\varphi^2$ and a relation like $(N^\xi)' = \frac{b}{\eta^2 X^2 + \omega^2}$ will be obtained. It has no curvature singularity. The location of the apparent horizon in U:

$$\rho_{AH} = \pm \frac{c_3}{\sqrt{c_2(c_2 + \frac{c_3}{c_1}F(U)^2)}}, \tag{50}$$

with

$$\frac{d\rho}{dU} = \frac{1}{2}e^{-2c_1U} \cdot \begin{cases} -\frac{c_1}{c_2c_3} & \ldots\ldots\ldots\rho \to 0 \\ \frac{c_3F(U)^2 + c_1c_2}{c_2^2c_3} & \ldots\,\rho \to \infty \\ 0 & \ldots\ldots\ldots\ldots\rho = \rho_{AH} \end{cases} \tag{51}$$

which is independent of ω. Here C_i are constants and $F(U)$ a function determined by the non-diagonal contribution. Further, we have

$$\lim_{\rho \to 0} g_{UU} \to \pm \frac{c_1}{c_2c_3e^{2c_1U}}, \tag{52}$$

So when the evaporation speeds up, it approaches zero. We are dealing here with null-radiation in the (ρ, z)-plane. One could compare this solution with that found by Chan[31] in standard GR of a spinning black hole. They also find a solution for $F(U)$ which is determined by an energy-momentum tensor of null spinning dust. It is again curious that the "uplifted" BTZ has the same solution, comparable with the "up-lifting" 5D[29]

5. Metric fluctuation and Hawking radiation

In the original deviation of the Hawking radiation, one uses the propagation of a linear quantized field in a classical background metric. However, near the horizon, high-frequencies metric fluctuations can contribute to the vacuum polarization and the impact of gravitational back reactions can be large. These zero-point fluctuations result in a modification of the Hawking radiation by gravitational waves[30]. One could question what the effect is of these waves in our CDG model, where we have instead the dilaton field. Of course, one should need a quantum gravitational approach, which is not available yet. So need some approximation. However, effect of the scattering of these quanta at the horizon can be investigated in the context of the antipodal mapping considered here. [g] Without the contribution of the metric fluctuations, the mean number of quanta reaching $\mathcal{J}^+$ takes the form

$$< \bar{n}_\lambda >_0 \sim \frac{1}{e^{2\pi E_+/\kappa} - 1}, \tag{53}$$

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[g]A suitable approximation is the high-frequency approximation applied to a Vaidya spacetime, where the not-flat background spacetime is distorted by the gravitational waves[32]. A recent application was provided by Slagter[33,34].

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with E_+ the energy measured at $\mathcal{J}^+$ for the out modes. This is the Planck distribution with temperature $T = \kappa/2\pi = (8\pi M)^{-1}$. The correction terms can then be calculated by using the s-modes of a quantum massless scalar field and by using the fact that the in-going and out-going modes decouple[30]. One makes use of the mean energy flux, by calculating $dE/dU = 4\pi r^2 < T_{UU} >_{ren}$, where the renormalized surface gravity is used. However, in this approximation, the reflection conditions are at r=0, with in our antipodal map must be revised (we have no inside). We can use the (U, U) energy-momentum component of our model and can apply Eq(41) for the antipodal contribution.

Notice that the meaning of the local dilaton $\bar{\omega}$, is twofold. First, it determines the metric fluctuations (one also must incorporate in the dilaton equation the φ-dependency). Secondly, The in-going observer will use a different conformal gauge freedom Ω on $\bar{\omega}$ to describe the vacuum. Further, $\bar{\omega}$ is locally unobservable, unless we include metric fluctuations (gravitational waves. It will be necessary to compare this with the usual contribution using the Bunch-Davies method (and to taken into count the antipodal contribution). Note that the outside observer will use a different gauge and he/she experiences a mass $\sim \omega^2 N^2$ and Hawking radiation $\sim \partial_U(\omega^2 N^2)$, while for the in-going observer it is part of his vacuum. On the other hand, the outside observer is not aware of the antipodal identification. One could also say that they disagree about the observed scales. Or differently stated, they disagree about the back reaction from the Hawking radiation.

6. Conclusion and outlook

We find an exact time dependent solution in the conformal dilaton gravity model on a warped 5D spacetime. The spacetime is written as $^{(5)}g_{\mu\nu} = \omega^{4/3(5)}\tilde{g}_{\mu\nu}$ and $^{(4)}\tilde{g}_{\mu\nu} = \bar{\omega}^{2(4)}\bar{g}_{\mu\nu}$. In our model, ω can be seen as the contribution from the bulk, while $\bar{\omega}$ is the brane component. It is conjectured that the different conformal gauge freedom, Ω, the in-going and outside observers possess, can be calculated by demanding a conformal invariant surface gravity and the preservation of affinely parameterized null geodesics. This means that the complementarity is expressed by the different notion of the vacuum state. The solution guarantees regularity of the action when $\omega \to 0$. We don't need a Weyl term in the action (generates negative metric states). In stead, we have a contribution from the bulk, i. e., the electric part of the 5D Weyl tensor. It is remarkable that the 5D field equations and the effective 4D equations can be written for general dimension n, with $n = 4, 5$. The energy-momentum tensor of the time-dependent dilaton, determining also the Hawking radiation, can be calculated exactly. By suitable choice of the parameters, the spacetime $\bar{g}_{\mu\nu}$ can be regular and singular free. This exact solution, nonetheless without mass terms, can be used to tackle the deep-seated problem of the black hole complementarity: the infalling and outside observer experience different ω by the choice of Ω. The next task is to incorporate mass into our model and investigate the dilaton-scalar field interaction. The conformal invariance will then spontaneously

be broken.

Appendix: The Quintic Horizon Equation and Related Issues

Our quintic polynomial, determining the horizons,

$$f = r^5 + \frac{15}{4}b_2 r^4 + 5b_2^2 r^3 + \frac{5}{2}b_2^3 r^2 + \frac{C_1}{4} = 0, \tag{54}$$

can be written by a, so-called Tschirnhaus transformation, in the form

$$r^5 - \frac{15b_2}{16}(C_1 + b_2^5)r^2 - \frac{125b_2^3}{256}(C_1 + b_2^5)r - \frac{1}{16}(C_1 + b_2^5)^2 = 0 \tag{55}$$

By scaling, this form can be reduced to the Bring-Jerrard form $r^5 + r - c$, with c a function of b_2 and C_1[26]. There is an interesting relation between the symmetry group of the icosahedron and our quintic equation. The symmetry group is isomorphic with the Galois group A_5 (of an irreducible quintic polynomial). The icosahedron is dual to the dodecahedron, i.e., their symmetries are isomorphic. The A_5 is interesting in physics, because it is a simple group having no invariant subgroups. It has three orbits, which are invariant under the antipodal map. So the connection with the Möbius group is clear. For details, we refer to Toth[26]. It is conjectured that our quintic polynomial (Eq.(54)) has a deep-seated relation with the 5D spacetime solution. Further, it is remarkable that the resulting quintic equation is independent of the dimension of our manifold ($n = 4, 5$). Moreover, the nice fitting of the antipodal map in our model cannot be a coincident. From Eq.(26) we observe that the derivative of f is $5r(r + b_2)^3$. So it is expected that our quintic equation results from a immersion[h] of a closed surface S in $\mathbb{R}^3$ into $\mathbb{R}^4$. This is currently under investigation by the author.

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[h] An immersion is a differentiable function between differentiable manifold whose derivative is everywhere injective. It is also a topological embedding.

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