Hydrodynamic representation for fermions in Curved Space-times

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Introduction

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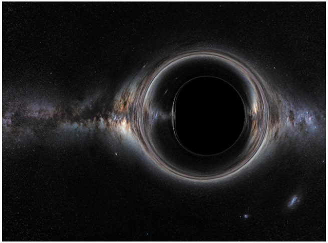

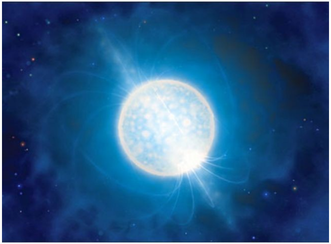

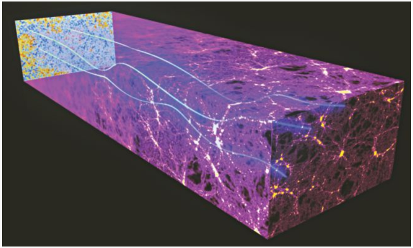

Energy Balance of a Bose Gas in a Curved Spacetime

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We derive a general energy balance equation for a self-interacting boson gas at vanishing temperature in a curved spacetime. This represents a first step towards a formulation of the first law of thermodynamics for a scalar field in general relativity. By using a $3+1$ foliation of the spacetime and performing a Madelung transformation, we rewrite the Klein-Gordon-Maxwell equations in a general curved spacetime into its hydrodynamic version where we can identify the different energy contributions of the system and separate them into kinetic, quantum, electromagnetic, and gravitational.

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Introduction. One of the most interesting open problems in general relativity is to identify different energy contributions in some relativistic objects, like neutron stars or black holes, since the standard laws of thermodynamics are not applicable in these cases. Essentially the problem stems from the fact that the spacetime metric framework are needed to model phenomena beyond the Minkowskian threshold. In this work, we are interested in solutions of the KG equation as models of objects at large scales like compact stars and dark matter halos.

The first attempt to describe astronomical objects as macroscopic bosonic states was made by Wheeler [2] who

Field equations

Geometric description

We use the tetrad formalism and the canonical expansion of the space-time in a 3+1 ADM decomposition

$$\mathrm{d}s^2 = N^2 c^2 \mathrm{d}t^2 - h_{ij}\left(\mathrm{d}x^i + N^i c\,\mathrm{d}t\right)\left(\mathrm{d}x^j + N^j c\,\mathrm{d}t\right), \quad (1)$$

We write eq.(1) in the tetrad formalism as $\mathrm{d}s^2 = \eta_{ab} e^a{}_\mu e^b{}_\nu dx^\mu dx^\nu$, where η_{ab} =diag$(1, -1, -1, -1)$.

Here $e^a = e^a{}_\mu dx^\mu$ is the set of one-forms base of the cotangent space at the space-time manifold given by

$$\begin{aligned} e^0 &= N\mathrm{d}t, \\ e^k &= \hat{e}^k{}_i \left(\mathrm{d}x^i + N^i c\,\mathrm{d}t\right), \end{aligned} \tag{2}$$

whose inverse is given by

$$\begin{aligned} e_0 &= \frac{1}{N}\left(\frac{\partial}{c\,\partial t} - N^j \frac{\partial}{\partial x^j}\right), \\ e_k &= \hat{e}_k{}^j \frac{\partial}{\partial x^j}. \end{aligned} \tag{3}$$

We can also define the set of vectors base of the tangent-space to the space-time as $e_a = e_a{}^\mu \partial_\mu$, such that $e^a e_b = \delta^a{}_b$.

For fermions the action S is given by $S\left[\psi, \partial_\mu \psi\right] = \int \mathcal{L}\left(\psi, \partial_\mu \psi(x^\mu)\right) d^4x$, where $\mathcal{L} = \mathcal{L}\left(\psi, \partial_\mu \psi(x^\mu)\right)$ is the Lagrangian density of the Dirac equation in curved space-times coupling to an electromagnetic field A_μ [1] [2] [3]

$$\mathcal{L} = \sqrt{-g}\frac{i\hbar c}{2}\left[\psi^\dagger B\gamma^\mu \left(D_\mu\psi\right) - \left(D_\mu\psi\right)^\dagger B\gamma^\mu\psi + \frac{2imc}{\hbar}\psi^\dagger B\psi\right], \qquad (4)$$

we denote $D_\mu = \partial_\mu + \Gamma_\mu + \frac{iq}{\hbar c}A_\mu = \nabla_\mu + \frac{iq}{\hbar c}A_\mu$, where the covariant derivative is given by $\nabla_\mu = \partial_\mu + \Gamma_\mu$, and Γ_μ is the space-time connection [4] [5].

From (4) it is possible to obtain the Dirac equation in curved space-times coupled with an electromagnetic A_μ. This equation is given by

$$[i\hbar\gamma^\mu(\nabla_\mu + iqA_\mu) - mc]\,\psi = 0. \tag{5}$$

Since the gamma matrices are $\gamma^\mu = e^\mu{}_a\tilde{\gamma}^a$, we can identify $\tilde{\gamma}^a$ as the gamma matrices in flat space-time, they are well-know from Quantum Field Theory (QFT) [6] [7] [8], such that

$$\begin{aligned} \gamma^0 &= N\tilde{\gamma}^0, \\ \gamma^k &= \hat{e}^k{}_j(\tilde{\gamma}^j + N^j\tilde{\gamma}^0). \end{aligned} \tag{6}$$

In general, these matrices fulfill the following anti-commutation relation [5] [9]

$$\{\gamma^\mu, \gamma^\nu\} = \gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \tag{7}$$

where $g_{\mu\nu}$ represents the metric that describes the space-time geometry as in General Relativity (GR). In general, the gamma matrices obey the following relation [1] [2] [3] [10]

$$(\gamma^\mu)^\dagger = B\gamma^\mu B^{-1}, \tag{8}$$

where B is the hermitian matrix, i.e. $B = B^\dagger$, which is uniquely determined by the gamma matrices γ^μ.

From the action of the Dirac equation we can find the transpose conjugated spinor making the infinitesimal variation of this action with respect to ψ. It is possible to obtain the transpose conjugated equation of Dirac equation in curved space-time as follows

$$i\left(\nabla_\mu \bar{\psi}\right)\gamma^\mu - i\psi^\dagger \nabla_\mu \left(B\gamma^\mu\right) + i\bar{\psi}\nabla_\mu \gamma^\mu + \bar{\psi}A_\mu \gamma^\mu + m\bar{\psi} = 0. \tag{9}$$

We consider that $(\nabla_\mu \psi)^\dagger = \nabla_\mu \psi^\dagger$, and we denote the adjoint spinor as $\bar{\psi} = \psi^\dagger B$.

In an arbitrary space-time $\nabla_\mu \gamma^\mu$ is distinct from zero, since $\gamma^\mu = e^\mu{}_a \tilde{\gamma}^a$, that is in general $\nabla_\mu e^\mu{}_a$ is non-zero.

In addition, we can get the conserved charge from the Noether theorem [11], which we define as the Dirac current density J^{μ}, having the following form

$$J^{\mu} = \bar{\psi}\gamma^{\mu}\psi = \psi^{\dagger}B\gamma^{\mu}\psi. \tag{10}$$

To obtain the continuity equation for the Dirac current density, we obtain the covariant derivative of eq.(10), This is

$$\nabla_{\mu}J^{\mu} = (\nabla_{\mu}\bar{\psi})\gamma^{\mu}\psi + \bar{\psi}\left(\nabla_{\mu}\gamma^{\mu}\right)\psi + \bar{\psi}\gamma^{\mu}\nabla_{\mu}\psi. \tag{11}$$

We require that the continuity equation is fulfilled, i.e., that the number of particles is conserved, then we need that $\nabla_\mu (B\gamma^\mu) = 0$, or equivalently that

$$(\nabla_\mu B)\gamma^\mu = -B\nabla_\mu\gamma^\mu. \tag{12}$$

Equation (12) gives a method for obtaining the matrices B for a corresponding metric (1), we can see that

$$\left(\nabla_0(BN) + \nabla_j(B\hat{e}_i^j N^i)\right)\tilde{\gamma}^0 - \nabla_j(B\hat{e}_i^j)\tilde{\gamma}^i = 0, \tag{13}$$

Therefore, with this differential condition (12), it is possible to rewrite the transpose conjugated Dirac equation as

$$i\left(\nabla_\mu\bar{\psi}\right)\gamma^\mu + i\bar{\psi}\nabla_\mu\gamma^\mu + \bar{\psi}A_\mu\gamma^\mu + m\bar{\psi} = 0. \tag{14}$$

Dirac hydrodynamic representation

Hydrodynamic representation

Analogously to the hydrodynamic representation for the Schrödinger equation, which was introduced by Madelung [12], we derive the hydrodynamic representation for the Dirac equation through the following generalized Madelung transformation for each component of the spinor $\psi = \psi(x^\mu)$ as follows

$$\psi = \exp(i\theta)R, \tag{15}$$

where R is a spinor and θ is a matrix which is diagonal $\theta = \theta_{\dot{\nu}}\delta^{\dot{\nu}}_{\dot{\mu}}$ such that the spinor ψ reads

$$\psi = \begin{pmatrix} R_{\dot{1}}\exp(i\theta_{\dot{1}}) \\ R_{\dot{2}}\exp(i\theta_{\dot{2}}) \\ R_{\dot{3}}\exp(i\theta_{\dot{3}}) \\ R_{\dot{4}}\exp(i\theta_{\dot{4}}) \end{pmatrix} = R_{\dot{\mu}}\exp\left(i\theta_{\dot{\nu}}\delta^{\dot{\nu}}_{\dot{\mu}}\right), \tag{16}$$

where we use the notation $\dot{\mu}, \dot{\nu}, ...= \dot{1}, \cdots, \dot{4}$ for spinor indices, such that

$$R = \begin{pmatrix} R_{\dot{1}} \\ R_{\dot{2}} \\ R_{\dot{3}} \\ R_{\dot{4}} \end{pmatrix}. \tag{17}$$

Moreover, it is convenient to take $R_{\dot{\mu}} = \sqrt{n_{\dot{\mu}}}$, where $n_{\dot{\mu}}$ is the corresponding number density and $\theta_{\dot{\mu}}$ is its phase, both are real components. The electron-like spinor is represented and it is associated with the $SO(1,3)$ group, this idea can straightforwardly extend for more general groups, as $SO(1, n-1)$.

We assume θ is an arbitrary diagonal matrix as in (16). Thus, the Dirac equation in terms of the variables R and θ read

$$\exp(i\theta)\gamma^{\mu}\left(i\nabla_{\mu}R - R\nabla_{\mu}\theta - qA_{\mu}R - \frac{m}{4}\gamma_{\mu}R\right) = 0. \tag{18}$$

Then, transforming with (16) the equation (11), the continuity equation with these new variables is

$$\left(\nabla_{\mu}R^{T}\right)K^{\mu}R + R^{T}K^{\mu}\left(\nabla_{\mu}R\right) = 0, \tag{19}$$

for each component, where $K^{\mu} = \exp(-i\theta)B\gamma^{\mu}\exp\left(i\theta^{T}\right)$. Observe that K^{μ} is hermitian, that is $K^{\mu\dagger} = K^{\mu}$. Additionally $K^{\mu} = B\gamma^{\mu}$ since matrix $\exp(i\theta)$ is a diagonal one.

We introduce the notation for any vector B_μ, namely, $\not{B} = \gamma^\mu B_\mu$. Furthermore, the equation (18) can be also transformed into

$$\gamma^\mu \left[i\nabla_\mu R - R\nabla_\mu \theta - qA_\mu R - \frac{m}{4}\gamma_\mu R \right] = 0. \tag{20}$$

For R and θ it follows that equation (20) transforms into

$$\gamma^\mu \left[i\nabla_\mu R_{\dot{\nu}} - R_{\dot{\nu}}\nabla_\mu \theta_{\dot{\alpha}} \delta^{\dot{\alpha}}{}_{\dot{\nu}} - qA_\mu R_{\dot{\nu}} - \frac{m}{4}\gamma_\mu R_{\dot{\nu}} \right] = 0, \tag{21}$$

where we write the equation in spinorial components.

Since in the equations (19) and (21), in general both the spinor and conjugate spinor are different from zero. We can rewrite the components of either spinor or its adjoint one as follows

$$\frac{i}{2}\not\nabla \ln\left(n_{\dot{\nu}}\right) - \not\nabla\theta_{\dot{\nu}} - q\not{A} - m = 0, \tag{22}$$

$$\frac{i}{2}\nabla_{\mu} \ln\left(n^{\dot{\nu}}\right)\gamma^{\mu} + \nabla_{\mu}\theta^{\dot{\nu}}\gamma^{\mu} + qA_{\mu}\gamma^{\mu} + m = 0, \tag{23}$$

nevertheless, $R_{\dot{\alpha}} = R^{\dot{\alpha}}$ and the same for θ due to that is a diagonal matrix and it is fulfilled $\theta = \theta^{T}$. We use the notation for defining $\left(R^{T}\right)_{\dot{\alpha}} = R^{\dot{\alpha}} = (\sqrt{n})^{\dot{\alpha}} = \sqrt{n^{\dot{\alpha}}}$.

In analogous way as in the Klein-Gordon equation [13], we define the 4-velocity v_μ as follows

$$mv_{\mu\dot{\nu}} = \nabla_\mu S_{\dot{\nu}} + qA_\mu. \tag{24}$$

We further define $S_{\dot{\alpha}} = \theta_{\dot{\nu}}\delta^{\dot{\nu}}{}_{\dot{\alpha}} - \omega_{\dot{\nu}}\delta^{\dot{\nu}}{}_{\dot{\alpha}}t$, where $S(x^\mu)$ is a phase and $\omega_{\dot{\nu}}$ are constants that can be related to the mass of the fermion particle as $\omega_{\dot{\nu}} = mc^2/\hbar$, such that we can write

$$\nabla_\mu\theta_{\dot{\alpha}}\delta^{\dot{\alpha}}{}_{\dot{\nu}} = (mv_{\mu\dot{\alpha}} - \omega_{\dot{\alpha}}\delta^0{}_\mu)\delta^{\dot{\alpha}}{}_{\dot{\nu}} - qA_\mu. \tag{25}$$

Thus, we can write the Dirac equations (5) and (9) with these new variables, which are given by

$$\frac{i}{2}\not\nabla \ln\left(n_{\dot{\nu}}\right) - m\not{v}_{\dot{\nu}} - \omega_{\dot{\nu}}\not\nabla t - m = 0, \tag{26}$$

$$\left(\frac{i}{2}\nabla_{\mu}\ln\left(n^{\dot{\nu}}\right) + \left(m{v_{\mu}}^{\dot{\nu}} + \omega^{\dot{\nu}}\nabla_{\mu}t\right) + \frac{m}{4}\gamma_{\mu}\right)\gamma^{\mu} = 0. \tag{27}$$

As we will see bellow, equation (26) is the corresponding first integral of the Bernoulli equation for fermions in an arbitrary space-time.

In order to see this, we apply the operator $i\gamma^\mu D_\mu = i\gamma^\mu \nabla_\mu - q\gamma^\mu A_\mu$ to the Dirac equation (5), then we use the Dirac equation and the relation (7) to obtain

$$\begin{aligned} \Box_E \psi + m^2 \psi &+ \\ \frac{i}{2} q \gamma^\mu \gamma^\nu F_{\mu\nu} \psi + \gamma^\mu (\nabla_\mu \gamma^\nu)(D_\nu \psi) &= 0 \end{aligned} \quad (28)$$

where we define the D'Alambertian operator with an electromagnetic field as $\Box_E = (\nabla_\mu + iqA_\mu)(\nabla^\mu + iqA^\mu)$ and the anti-symmetric Faraday tensor as it is usual.

Following [13] if we apply the same transformation (15) to equation (28), we should obtain for the imaginary part, the continuity equation and for the real part, the Bernoulli equation.

$$\begin{aligned} i\left[2(mv^{\mu}-\omega\delta^{\mu}_{0})\nabla_{\mu}R-qA_{\mu}+q\nabla_{\mu}(A^{\mu}R)+\nabla_{\mu}(mv^{\mu}-\omega\delta^{\mu}_{0}-qA^{\mu})R\right] &+ \\ \left(m^{2}v_{\mu}v^{\mu}+2m\omega v^{0}+\frac{\omega^{2}}{N^{2}}+m^{2}\right)R-\Box R &+ \\ \frac{i}{2}q\gamma^{\mu}\gamma^{\nu}F_{\mu\nu}R+\gamma^{\mu}(\nabla_{\mu}\gamma^{\nu})(i(mv_{\nu}+\omega\nabla_{\nu}t)R+D_{\nu}R) &= 0. \end{aligned} \tag{29}$$

Here, we denote $\Box = \nabla^{\nu}\nabla_{\nu}$ and we define the diagonal matrices $v_{\mu} = v_{\mu\dot{\nu}}\delta^{\dot{\nu}}_{\dot{\alpha}}$ and $\omega = \omega_{\dot{\nu}}\delta^{\dot{\nu}}_{\dot{\alpha}}$.

Weyl representation

Weyl representation

The Dirac equation for 1/2-spin particle can be related by the $SO(1,3)$ symmetry group. Nevertheless, we can introduce a new representation just like it is done in standard QFT, since there exists a surjective homomorphism map between $SO(1,3)$ and $SU(2) \otimes SU(2)$ Lie groups.

In terms of the Pauli matrices the 4×4 gamma matrices γ^μ can be written as two 2×2 block matrices as

$$\gamma^0 = N\tilde{\gamma}^0 = N \begin{pmatrix} 0 & \mathbb{I} \\ \mathbb{I} & 0 \end{pmatrix}, \tag{30}$$

$$\gamma^j = \hat{e}^j{}_i(\tilde{\gamma}^i + N^i\tilde{\gamma}^0) = \begin{pmatrix} 0 & -\hat{e}^j{}_i(\tilde{\sigma}^i - N^i\mathbb{I}) \\ \hat{e}^j{}_i(\tilde{\sigma}^i + N^i\mathbb{I}) & 0 \end{pmatrix}, \tag{31}$$

where $\tilde{\sigma}^i$ are the 2×2 Pauli matrices in flat space-time and $\mathbb{I}$ is the 2×2 identity matrix. Observe that the γ^μ matrices fulfill that $\left(\gamma^0\right)^\dagger = \gamma^0$ and $\left(\gamma^j\right)^\dagger = -\gamma^j + 2N^j\gamma^0/N$.

We can write a Dirac fermion as a four-spinor ψ into two spinors of two components each one, instance

$$\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{32}$$

if we write the adjoint spinor $\bar{\psi}$ and we utilize the Weyl representation, it follows that

$$\bar{\psi} = \psi^\dagger B = \left(\psi_R^\dagger, \psi_L^\dagger\right) B, \tag{33}$$

where B is the matrix from (8) and (12).

We can see that the matrix B must have the following form

$$B = \begin{pmatrix} 0 & B_\zeta \\ B_\zeta & 0 \end{pmatrix}, \tag{34}$$

being B_ζ a 2×2 matrix which is a diagonal matrix, that is $B_\zeta = b\mathbb{I}$, where $b = b(x^\mu)$, hence $B = b\tilde{\gamma}^0$. Thus, equation (13) transform into

$$\begin{aligned} \nabla_0(Nb) + \nabla_j(\hat{e}^j_i N^i b) &= 0, & (35) \\ \nabla_j(\hat{e}^j_i b)\tilde{\sigma}^i &= 0. & (36) \end{aligned}$$

Using the definitions of the spinor and its adjoint we can obtain the Dirac current density J^μ from equation (10)

$$J^\mu = \left(\psi_R^\dagger, \psi_L^\dagger\right) B\gamma^\mu \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix}, \tag{37}$$

where the gamma matrices are defined by eq.(30) and in general, B is given by the previously mentioned conditions.

We can obtain the Dirac current (10) as

$$J^0 = Nb(\psi_R^\dagger\psi_R + \psi_L^\dagger\psi_L), \quad (38)$$

$$J^j = b\hat{e}^j{}_i(\psi_R^\dagger(\tilde{\sigma}^i + N^i\mathbb{I})\psi_R - \psi_L^\dagger(\tilde{\sigma}^i - N^i\mathbb{I})\psi_L). \quad (39)$$

In order to simplify the notation, we now define the vectors of 2×2 matrices $\mathbb{S}^a = (\mathbb{I}, \tilde{\sigma}^j + N^j\mathbb{I})$ and $\bar{\mathbb{S}}^a = (-\mathbb{I}, \tilde{\sigma}^j - N^j\mathbb{I})$ in terms of the Pauli matrices, the $\mathbb{S}^a$ and $\bar{\mathbb{S}}^a$ are the (generalized) Pauli matrices in flat space-time, we have defined the 2×2 Pauli matrices in a curved space-time $\sigma^\mu = e^\mu{}_a\mathbb{S}^a$ and $\bar{\sigma}^\mu = e^\mu{}_a\bar{\mathbb{S}}^a$.

With this definition matrices γ^μ and γ^j are read as

$$\gamma^\mu = \hat{e}_a^\mu \begin{pmatrix} 0 & -\bar{\mathbb{S}}^a \\ \mathbb{S}^a & 0 \end{pmatrix}, \quad \gamma^j = \begin{pmatrix} 0 & -\bar{\sigma}^j \\ \sigma^j & 0 \end{pmatrix}. \tag{40}$$

We need to redefine the covariant derivative $\nabla_\mu = \partial_\mu + \Gamma_\mu$ and affine connection $\Gamma_\mu = \frac{1}{4}\bar{\sigma}_\nu \sigma^\nu_{;\mu}$ where $\sigma^\mu_{;\nu} = \partial_\nu \sigma^\mu + \Gamma^\mu_{\alpha\nu}\sigma^\alpha$. Let $\bar{\nabla}_\mu$ and $\tilde{\Gamma}_\mu$ be the bar covariant derivative and the bar spinor affine connection, respectively. They are defined as $\bar{\nabla}_\mu = \partial_\mu + \bar{\Gamma}_\mu$ where $\bar{\Gamma}_\mu = \frac{1}{4}\sigma_\nu \bar{\sigma}^\nu_{;\mu}$.

In terms of these new definitions, the density currents are read as

$$\begin{aligned} J^j &= b\hat{e}^j{}_i(\psi_R^\dagger \mathbb{S}^i \psi_R - \psi_L^\dagger \bar{\mathbb{S}}^i \psi_L) \\ &= b(\psi_R^\dagger \sigma^j \psi_R - \psi_L^\dagger \bar{\sigma}^j \psi_L), \end{aligned} \tag{41}$$

We can rewrite the Dirac equation (5) for a Dirac spinor with four components. We can apply the Weyl representation into the Dirac equation, it reads as

$$\begin{pmatrix} i\sigma^\mu \left(\bar{\nabla}_\mu + iqA_\mu\right)\psi_L - m\psi_R \\ i\bar{\sigma}^\mu \left(\nabla_\mu + iqA_\mu\right)\psi_R - m\psi_L \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{42}$$

If we set $B = b\tilde{\gamma}^0$, the current density is now reads as

$$J^\mu = b\left(\psi_R^\dagger \sigma^\mu \psi_R - \psi_L^\dagger \bar{\sigma}^\mu \psi_L\right). \tag{43}$$

Explicitly, we have

$$J^j = b\hat{e}^j_i \left(\psi_R^\dagger \tilde{\sigma}^i \psi_R - \psi_L^\dagger \tilde{\sigma}^i \psi_L + \frac{N^i}{Nb^2} J^0\right) \tag{44}$$

Observe that

$$\gamma^\mu \gamma^\nu F_{\mu\nu}\psi = \begin{cases} (2NN^k F_{0k} + i\hat{F}_{ij}\epsilon^{ij}{}_k \tilde{\sigma}^k)\psi_R \\ -(2NN^k F_{0k} - i\hat{F}_{ij}\epsilon^{ij}{}_k \tilde{\sigma}^k)\psi_L \end{cases}, \tag{45}$$

and using definition (40) we find that

$$
\begin{aligned}
\gamma^{\mu}(\nabla_{\mu}\gamma^{\nu})(D_{\nu}\psi) &= \begin{cases} -\bar{\mathbb{S}}^{a}\mathbb{S}^{b}(\hat{\nabla}_{a}\hat{e}_{b}^{\nu})(D_{\nu}\psi_{R}) \\ -\mathbb{S}^{a}\bar{\mathbb{S}}^{b}(\hat{\nabla}_{a}\hat{e}_{b}^{\nu})(D_{\nu}\psi_{L}) \end{cases} \\
= & \begin{cases} (N(\nabla_{0}N) - \bar{\sigma}^{j}(\nabla_{j}N))(D_{0}\psi_{R}) + (N(\nabla_{0}\sigma^{i}) - \bar{\sigma}^{j}(\nabla_{j}\sigma^{i}))(D_{i}\psi_{R}) \\ (N(\nabla_{0}N) + \sigma^{j}(\nabla_{j}N))(D_{0}\psi_{L}) - (N(\nabla_{0}\bar{\sigma}^{i}) - \sigma^{j}(\nabla_{j}\bar{\sigma}^{i}))(D_{i}\psi_{L}) \end{cases} \\
= & \begin{cases} (\hat{\nabla}_{0}N - \bar{\mathbb{S}}^{k}(\hat{\nabla}_{k}N))(D_{0}\psi_{R}) + (\mathbb{S}^{k}\hat{\nabla}_{0}\hat{e}_{k}^{i} - \bar{\mathbb{S}}^{k}\mathbb{S}^{l}\hat{\nabla}_{k}\hat{e}_{l}^{i}))(D_{i}\psi_{R}) \\ (\hat{\nabla}_{0}N + \mathbb{S}^{k}(\hat{\nabla}_{k}N))(D_{0}\psi_{L}) - (\bar{\mathbb{S}}^{k}\hat{\nabla}_{0}\hat{e}_{k}^{i} - \mathbb{S}^{k}\bar{\mathbb{S}}^{l}(\hat{\nabla}_{k}\hat{e}_{l}^{i}))(D_{i}\psi_{L}) \end{cases}
\end{aligned} \tag{46}
$$

Weyl hydrodynamic representation

Weyl hydrodynamic representation

We start to propose our Madelung transformation in the Weyl spinor, that is

$$\Psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix} = \begin{pmatrix} e^{i\theta_R} R_R \\ e^{i\theta_L} R_L \end{pmatrix}. \tag{47}$$

The Weyl representation of the adjoint spinor $\bar{\Psi}$ when $B = b\tilde{\gamma}^0$, is

$$\bar{\Psi} = b\left(\psi_R^\dagger, \psi_L^\dagger\right)\tilde{\gamma}^0 = \left(R_R^\dagger e^{-i\theta_R^\dagger}, R_L^\dagger e^{-i\theta_L^\dagger}\right). \tag{48}$$

Note that $\left(e^{i\theta}\right)^\dagger = e^{-i\theta^\dagger} = e^{i\theta}$.

Using the Madelung transformation (47) in the Weyl equations (42) and applying the Lie algebra and the Lie groups, we can get the following expression

$$\begin{pmatrix} -\sigma^{\mu}\left(\bar{\nabla}_{\mu}\theta_L\right)R_L + i\sigma^{\mu}\left(\bar{\nabla}_{\mu}R_L\right) - q\sigma^{\mu}A_{\mu}R_L \\ -\bar{\sigma}^{\mu}\left(\nabla_{\mu}\theta_L\right)R_R + i\bar{\sigma}^{\mu}\left(\nabla_{\mu}R_R\right) - q\bar{\sigma}^{\mu}A_{\mu}R_R \end{pmatrix} = \begin{pmatrix} me^{-i(\theta_L-\theta_R)}R_R \\ me^{i(\theta_L-\theta_R)}R_L \end{pmatrix}, \tag{49}$$

these are the Weyl equations in curved space-time with the Madelung transformation.

In addition, we can apply the Madelung transformation (47) and (48) into the current density (44), we obtain

$$J^{\mu} = b\left(R_R^T\bar{\sigma}^{\mu}R_R + R_L^T\sigma^{\mu}R_L\right). \tag{50}$$

We note that taking the zero component, we obtain the number of particles both right- and left- handed particles.

$$J^0 \quad = \quad Nb(R_R^T R_R + R_L^T R_L) = Nbn, \tag{51}$$

$$J^j \quad = \quad b\left(\hat{e}_3^j(n_{\dot{1}} - n_{\dot{2}} - n_{\dot{3}} + n_{\dot{4}}) + 2\hat{e}_1^j(\sqrt{n_{\dot{1}}n_{\dot{2}}} - \sqrt{n_{\dot{3}}n_{\dot{4}}}) + \hat{e}_i^j N^i n\right), \tag{52}$$

Explicitly, we can write in both right- and left- handed spinors the following expressions $|\psi_R|^2 = \psi_R^{\dagger}\psi_R = R_R^T R_R = n_R$ and $|\psi_L|^2 = \psi_L^{\dagger}\psi_L = R_L^T R_L = n_L$,. Thus, n_R, n_L are the right- and left- handed particle number. Therefore, $n = n_R + n_L$ is the total density number.

Furthermore, from the equation (29), it becomes

$$\begin{aligned}
& i\,[2(mv_R^\mu - \omega_R\delta_0^\mu)\nabla_\mu R_R - qA_\mu + q\nabla_\mu(A^\mu R_R) + \nabla_\mu(mv_R^\mu - \omega_R\delta_0^\mu - qA^\mu)R_R] \\
& + \left(m^2 v_{R\mu}v_R^\mu + 2m\omega_R v_R^0 + \frac{\omega^2}{N^2} + m^2\right) R_R - \Box R_R + \\
& +(2NN^k F_{0k} + i\epsilon^{ij}{}_k \hat{F}_{ij}\tilde{\sigma}^k)R_R + \\
& -(\hat{\nabla}_a \hat{e}_b^\alpha)\bar{\mathbb{S}}^a\mathbb{S}^b((mv_{R\alpha} - \omega_R\delta_\alpha^0)R_R + D_\alpha R_R) = 0.
\end{aligned} \tag{53}$$

and the same equation for the left spinor R_L, changing $R \longrightarrow L$ and $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ in (53).

We simplify the first line in this equation and write it in components, it becomes as follows

$$\begin{aligned}
i\frac{m}{\sqrt{n_{\dot{\nu}}}}\left[-\frac{\omega_{\dot{\nu}}}{m}\nabla_0 n_{\dot{\nu}}+\nabla_\mu(n_{\dot{\nu}}v^\mu_{\dot{\nu}})+\frac{\omega_{\dot{\nu}}}{m}\Box t\right] &\quad + \\
\sqrt{n_{\dot{\nu}}}\left[m^2 v_{\mu\dot{\nu}}v^\mu_{\dot{\nu}}+2m\omega_{\dot{\nu}}v^0_{\dot{\nu}}+\frac{\omega^2}{N^2}+m^2-\frac{\Box\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}\right] &\quad + \\
(2NN^kF_{0k}+i\epsilon^{ij}{}_k\hat{F}_{ij}\tilde{\sigma}^k)R_R &\quad + \\
-(\hat{\nabla}_a\hat{e}^\alpha_b)\bar{\mathbb{S}}^a\mathbb{S}^b((mv_{R\alpha\dot{\nu}}-\omega_{R\dot{\nu}}\delta^0_\alpha)R_R+D_\alpha R_R) &\quad = \quad 0.
\end{aligned} \tag{54}$$

The first line in (54) is the continuity equation of a fluid, in this case, of the fermionic fluid. The second one is the Bernoulli equation, in other words, equation (22) is the first integral of the second line of (54) and the last three lines are the source of the fermionic fluid, something that is not present in the case of bosons.

Energy Balance

In order to simplify the notations, we can re-write equation (54) in terms of the $\dot{\nu}$ coefficients with the understanding that the subindex R refers to each component $R = \dot{1}, \dot{2}$ individually. We get

$$\begin{aligned} i\left[-\omega_{\dot{\nu}}\nabla_0 \ln(n_{\dot{\nu}}) + \frac{m\nabla_\mu(n_{\dot{\nu}}v^\mu_{\dot{\nu}})}{n_{\dot{\nu}}} + \frac{\omega_{\dot{\nu}}}{n_{\dot{\nu}}}\Box t\right] \quad & + \\ 2m^2\left(K_{\dot{\nu}} + \frac{1}{m}\omega_{\dot{\nu}}v^0_{\dot{\nu}} + \frac{1}{2}U^N_{\dot{\nu}} + U^Q_{\dot{\nu}}\right) + E_{\dot{\nu}} + U^S_{\dot{\nu}} \quad & = \quad 0. \end{aligned} \tag{55}$$

This equation is valid for right handed fermions. The result is the same for left handed fermions changing $R_R \longrightarrow R_L$ in the first line, and $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ in the second line.

The first one is the kinetic energy $K_{\dot{\nu}}$ defined as

$$K_{\dot{\nu}} = \frac{1}{2} v_{\dot{\nu}\mu} v^{\mu}_{\dot{\nu}}. \tag{56}$$

The lapse potential $U^{N}_{\dot{\nu}}$ is given by

$$U^{N}_{\dot{\nu}} = \frac{\omega^{2}_{\dot{\nu}}}{m^{2}} \frac{1}{N^{2}} + 1. \tag{57}$$

It represents the energy contribution due to the chosen lapse function N. The quantum potential $U^{Q}_{\dot{\nu}}$ is defined as

$$U^{Q}_{\dot{\nu}} = -\frac{1}{2m^{2}} \frac{\Box\sqrt{n_{\dot{\nu}}}}{\sqrt{n_{\dot{\nu}}}}. \tag{58}$$

The contribution of the electromagnetic interaction $E_{\dot{\nu}}$ is given by

$$\begin{aligned} E_{\dot{\nu}} &= (2NN^k F_{0k} + i\epsilon^{lj}{}_k \hat{F}_{lj}\tilde{\sigma}^k)|_{\dot{\nu}}, \qquad (59) \\ &= 2N(F_{01}N^1 + F_{02}N^2 + F_{03}N^3) - e_{1\dot{\nu}}F_{13}\sqrt{\frac{n_{\ddot{\nu}}}{n_{\dot{\nu}}}} + i\left(e_{1\dot{\nu}}F_{12} + F_{23}\sqrt{\frac{n_{\ddot{\nu}}}{n_{\dot{\nu}}}}\right). \end{aligned}$$

It depends on the Faraday tensor, shift vector and lapse function that are related to the Pauli matrices. This relationship is due to the interaction between the electromagnetic field and the fermionic spin.

The potential $U^S_{\dot\nu}$ describes the interaction between the spin and the geometry of space-time. It is given by

$$U^S_{\dot\nu} \quad = \quad -\left((m\hat{v}_{Rd} - \omega_{\dot\nu}\hat{\delta}^0_d) + \frac{\hat{D}_\alpha\sqrt{n_{\dot\nu}}}{\sqrt{n_{\dot\nu}}}\right)\Gamma^d_{ba}\bar{\mathbb{S}}^a\mathbb{S}^b|_{\dot\nu}, \qquad (60)$$

for $\dot\nu = \dot1, \dot2$, and by making the substitution $\mathbb{S} \longleftrightarrow \bar{\mathbb{S}}$ for $\dot\nu = \dot3, \dot4$. In the foregoing equations, the notation $|_{\dot\nu}$ means that we have to evaluate the quantity at the corresponding $\dot\nu$.

Conclusions

Comparison

The principal differences between the hydrodynamic representation for bosons [13] and for fermions are in Bernoulli equation. The contribution due to the gamma matrices, then after to do the Madelung transformation in KG equation, we can separate into real and complex parts. Instead of, here, for fermion particles we have to work with the complete equations of motion, since the gamma matrices are a representation of $SO(1,3)$ group for electrons.

The origin of the Dirac equation [14] showed that spin is a fundamental quantity, which combine both of special relativity and quantum mechanics to solve the problem of negative probability given by the KG equation, which is also a quantum-relativistic equation. Here, we observe that exists a contribution due to geometry and spin by the generalized gamma and Pauli matrices, namely, these terms arise from endowing a quantum field with a curvature (or geometry), which is given by a metric just like in GR. This contribution is not obtained in a flat space-time, neither in a system without spin as in a scalar field.

With this work we open the possibility of researching in detail how is the behavior of fermions in different frameworks such as stars, near black holes, or in the early universe, to mean a few. Furthermore, we solved the problem of energy balance either bosons or fermions, with this we can compare the result of the hydrodynamic representation for classical and quantum fluids in the diverse geometries mentioned above.

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