How Classical are Gaussian States in Loop Quantum Cosmology?

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Motivation

We do not typically observe fundamentally quantum features of spacetime. If loop quantum gravity is to provide an adequate picture of reality, it must have a semi-classical sector that recovers certain cosmological phenomena in appropriate limits. In particular, loop quantum cosmology must admit semi-classical states.

Classicality

We will say a quantum state is *kinematically* semi-classical if it satisfies the following:[1]

1. It should approximate a state having a definite coordinate x.
2. It should have small quantum fluctuations in x and p.

[1] Martin-Dussaud, Quantum, 2021.

Classicality

When are fluctuations "small"?

In ideal conditions, $\Delta^2 x \Delta^2 p = 0$.

From the uncertainty relation,[2] we know:

$$\Delta^2 x \Delta^2 p \geq \underbrace{\frac{1}{2}\left|\langle\{x,p\}\rangle - \langle x\rangle\langle p\rangle\right|^2 + \left|\frac{1}{2i}\langle[x,p]\rangle\right|^2}_{\text{Robertson-Schrödinger}} \geq \underbrace{\left|\frac{1}{2i}\langle[x,p]\rangle\right|^2}_{\text{Heisenberg}}$$

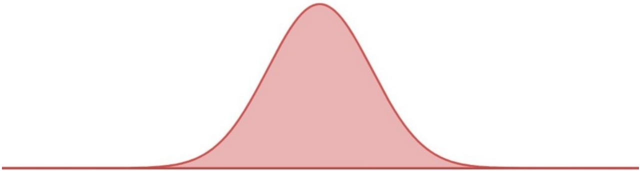
"Small" means the RS inequality is saturated.

[2] Robertson, Phys. Rev., 1929.

Classicality

In traditional quantum mechanics, Gaussian wave functions satisfy these basic criteria:

$$|\psi\rangle = \underbrace{c\int dx e^{-(x-\mu)^2/2\sigma^2}|x\rangle}_{\text{Coordinates become sharply peaked as } \sigma \to 0.}, \qquad \Delta^2 x \Delta^2 p = \underbrace{\left|\frac{1}{2i}\langle[x,p]\rangle\right|^2}_{\text{Fluctuations are minimal.}}$$



Classicality

Question

Are Gaussian states semi-classical in LQC?

Classicality

Foreshadowing

No! Not even at the kinematic level.
(Though they are asymptotically so.)

Basic LQC

Consider an FLRW spacetime with a metric:

$$ds^2 = -dt^2 + a(t)^2[dx^2 + dy^2 + dz^2].$$

Then the spatial volume of a time-like slice $V = a^3(t)$ and the Hubble parameter $\beta = \dot{a}(t)/a(t)$ are canonical variables from which the EFEs may be derived using a Hamiltonian treatment via the Holst action.

LQC is the theory one obtains by applying loop quantization to V and β.

Basic LQC

The Holst action has a divergent symplectic term. Thus, the loop quantization procedure must introduce volume regularization. One therefore considers LQC on a compact fiducial spacetime region with volume V_0.[3]

[3] Ashtekar & Singh, Class. Quant. Grav., 2011; Rovelli & Wilson-Ewing, Phys. Rev. D, 2014; Rovelli & Vidotto, CUP, 2015.

Basic LQC

The "coordinate" observable in LQC is the volume operator $\hat{V}$. However, the loop quantization procedure does not allow for a unique operator $\hat{\beta}$.[4] Instead, the conjugate momentum is chosen to be $\hat{S}_\lambda$, determined by the holonomy operator $\hat{h}_\lambda$:

$$\hat{V}|V\rangle = V|V\rangle, \qquad \hat{h}_\lambda|V\rangle = \left|V + \frac{4\pi G\gamma\hbar\lambda}{V_0}\right\rangle, \qquad \hat{S}_\lambda = \frac{1}{2i\lambda}(\hat{h}_\lambda - \hat{h}^*_\lambda).$$

Henceforth, $\alpha := 4\pi G\gamma\hbar$.

[4] Ashtekar, Bojowald, & Lewandowsky, Adv. Theor. Math. Phys., 2003.

Gaussian States in LQC

The kinematic volume Hilbert space is given by states of the form:

$$|\psi\rangle = \sum_{n\in\mathbb{Z}} f(n)|V_n\rangle, \qquad \sum_{n\in\mathbb{Z}} |f(n)|^2 < \infty$$

for $V_n \in \mathbb{R}$, where $\langle V_n|V_m\rangle = \delta_{nm}$. The sign of V_n indicates the orientation of the spatial slice. Physical states in LQC must have parity symmetry:[5]

$$\psi(V) = \psi(-V)$$

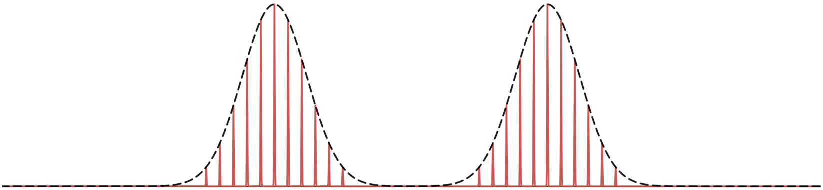
[5] Bentivegna & Pawlowsky, Phys. Rev. D, 2008; Ashtekar & Wilson-Ewing, Phys. Rev. D, 2009; Ashtekar & Singh, Class. Quant. Grav., 2011.

Gaussian States in LQC

Gaussian states centered around $|V| = \mu l_P$ are states of the following form:[6]

$$|\psi\rangle = \frac{c}{\sqrt{2}} \sum_{n\in\mathbb{Z}} \left(e^{-(nl_P - \mu l_P)^2/2\sigma^2} + e^{-(nl_P + \mu l_P)^2/2\sigma^2} \right) |nl_P\rangle$$

where l_P is the Planck length; henceforth $l_P = 1$.



[6] A generalization of Willis, PhD Thesis, 2004.

Fluctuations for Gaussian States

By construction, Gaussian states in LQC are "sharply-peaked" around $|V| = \mu$. Thus, in order for them to be semi-classical kinematic states, we only require that their fluctuations in $\hat{V}$ and $\hat{S}_\lambda$ are minimal.

Fluctuations for Gaussian States

The exact volume-fluctuations are thus:

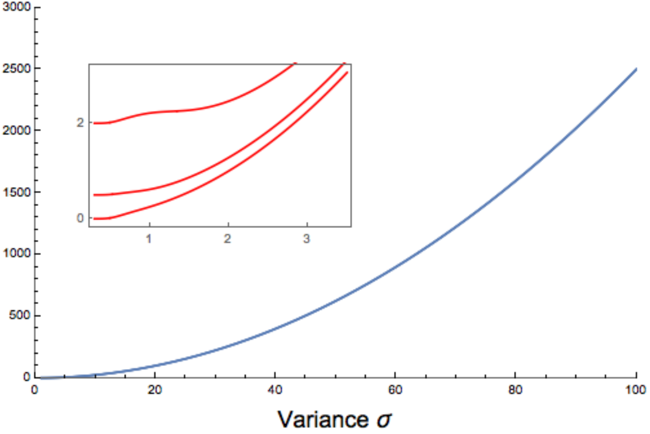
$$\Delta_V^2 = \frac{\sqrt{-\ln v}}{4\sqrt{\pi}\left(1+e^{\mu^2\pi^2/\ln v}\right)\vartheta_3(v^{1/\pi^2\sigma^4})} \times \left\{ \left[2\mu^2 - \frac{\ln v}{\pi^2}\left(1+e^{\mu^2\pi^2/\ln v}\right)\right]\vartheta_3(v) - \left(1+e^{\mu^2\pi^2/\ln v}\right)\left(\frac{2v(\ln v)^2}{\pi^2}\right)\left(\frac{d}{dv}\vartheta_3(v)\right)\right\}$$

where $v := \exp\{-\pi^2\sigma^2\}$ and $\vartheta_3(q) = \vartheta_3(0, q)$ is the Jacobi ϑ-function.

Fluctuations for Gaussian States

$$\vartheta_3(z, q) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{2niz}$$

Fluctuations for Gaussian States



Δ_V^2 as a function of σ for $\mu \in \{0, 1, 2\}$ with $\frac{\alpha\lambda}{V_0} = 1$.

Fluctuations for Gaussian States

The exact holonomy-flux fluctuations have two cases to consider. If $2\alpha\lambda/V_0 \notin \mathbb{Z}$, then:

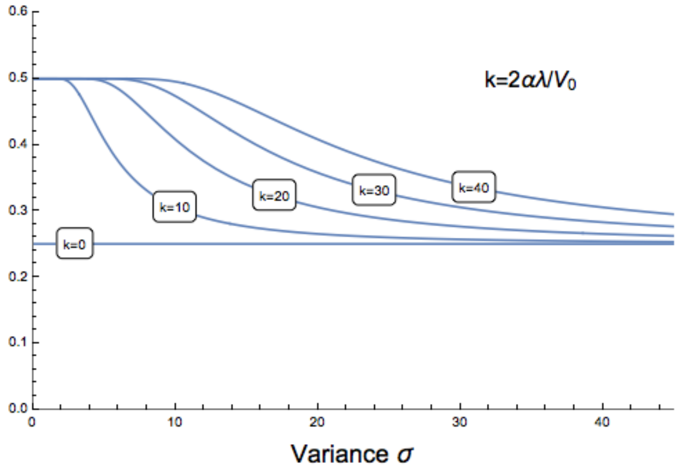
$$\Delta^2_{S_\lambda} = \frac{1}{2\lambda^2}$$

Fluctuations for Gaussian States

If $k = 2\alpha\lambda/V_0$ is an integer, then:

$$\Delta^2_{S_\lambda} = \frac{\alpha^2 4\pi\sigma}{k^2 V_0^2} - \frac{\alpha^2\pi\sigma}{k^2 V_0^2 (1 + e^{-\mu^2/\sigma^2})\vartheta_3\left(e^{-1/\sigma^2}\right)} \times$$
$$\Bigg\{ 2e^{-2(\alpha\lambda/V_0)^2/\sigma^2} \left[\vartheta_3\left(\frac{-\alpha\lambda/V_0}{i\sigma^2}, e^{-1/\sigma^2}\right) + \vartheta_3\left(\frac{\alpha\lambda/V_0}{i\sigma^2}, e^{-1/\sigma^2}\right) \right]$$
$$+ e^{-2(\alpha\lambda/V_0-\mu)^2/\sigma^2} \left[\vartheta_3\left(\frac{-\alpha\lambda/V_0+\mu}{i\sigma^2}, e^{-1/\sigma^2}\right) + \vartheta_3\left(\frac{\alpha\lambda/V_0-\mu}{i\sigma^2}, e^{-1/\sigma^2}\right) \right]$$
$$+ e^{-2(\alpha\lambda/V_0+\mu)^2/\sigma^2} \left[\vartheta_3\left(\frac{-\alpha\lambda/V_0-\mu}{i\sigma^2}, e^{-1/\sigma^2}\right) + \vartheta_3\left(\frac{\alpha\lambda/V_0+\mu}{i\sigma^2}, e^{-1/\sigma^2}\right) \right] \Bigg\}$$

Fluctuations for Gaussian States

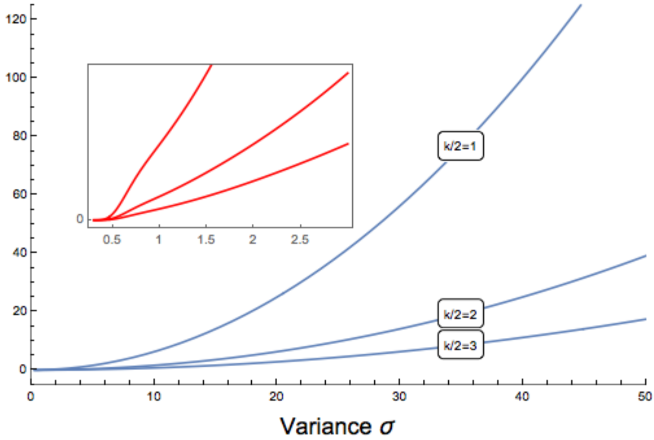


$\Delta^2_{S_\lambda}$ as a function of σ for different λ with $\mu = 0$.

Fluctuations for Gaussian States

The question now is whether or not $\Delta_V^2 \Delta_{S_\lambda}^2$ is equal to the RS lower-bound. There are three different cases that lead to different behaviours, namely, if $\alpha\lambda/V_0 \notin \mathbb{Z}$, or $\alpha\lambda/V_0 \in \mathbb{Z}$ and is even or odd. In all three cases, the fluctuations *never* saturate the inequality (for any values of σ, μ, and λ).

Fluctuations for Gaussian States



$\Delta^2_V \Delta^2_{S_\lambda} - L.B.$ as a function of σ for different k with $\mu = 0$ and $\alpha\lambda/V_0 \in \mathbb{Z}$ even.

Fluctuations for Gaussian States

Conclusion

Unlike in traditional quantum mechanics, Gaussian states in LQC do *not* minimize the uncertainty relation. Thus, they are not strictly semi-classical!

Fluctuations for Gaussian States

Follow-Up

Are the conditions under which they are *essentially* semi-classical?

Fluctuations for Gaussian States

Yes! One can check that in the $V_0 \to \infty$ limit, the fluctuations (and lower bound) vanish.[7] However, V_0 must be fixed within a model of LQC; it is not a parameter that can be varied.

[7] Rovelli & Wilson-Ewing, Phys. Rev. D, 2014.

Summary

1. For a Gaussian state with a fixed width σ and a fixed parity-symmetric mean μ, one can always choose a model of LQC with a sufficiently large fiducial volume V_0 such that the fluctuations of this state become negligible.
2. Within a fixed model of LQC (with a given finite V_0), one can always find a Gaussian state with sufficiently large σ such that its fluctuations become arbitrarily large.
3. The relation between λ and V_0 play an important role in determining the phenomenology of a theory of LQC, namely, they determine whether or not the holonomy operator $\hat{h}_\lambda$ is closed, and the different integer cases affect fluctuations.

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