How Classical are Gaussian States in Loop Quantum Cosmology?

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Motivation

We do not typically observe fundamentally quantum features of spacetime. If loop quantum gravity is to provide an adequate picture of reality, it must have a semi-classical sector that recovers certain cosmological phenomena in appropriate limits. In particular, loop quantum cosmology must admit semi-classical states.
We will say a quantum state is *kinematically* semi-classical if it satisfies the following:\(^1\)

1. It should approximate a state having a definite coordinate \(x\).
2. It should have small quantum fluctuations in \(x\) and \(p\).

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\(^1\)Martin-Dussaud, Quantum, 2021.
Classicality

When are fluctuations “small”?

In ideal conditions, \( \Delta^2 x \Delta^2 p = 0 \).

From the uncertainty relation\(^2\), we know:

\[
\Delta^2 x \Delta^2 p \geq \frac{1}{2} \left| \langle \{ x, p \} \rangle - \langle x \rangle \langle p \rangle \right|^2 + \left| \frac{1}{2i} \langle [x, p] \rangle \right|^2 \geq \left| \frac{1}{2i} \langle [x, p] \rangle \right|^2
\]

“Small” means the RS inequality is saturated.

\(^2\)Robertson, Phys. Rev., 1929.
Classicality

In traditional quantum mechanics, Gaussian wave functions satisfy these basic criteria:

\[ |\psi\rangle = c \int dx e^{-(x-\mu)^2/2\sigma^2} |x\rangle, \quad \Delta^2 x \Delta^2 p = \left| \frac{1}{2i} \langle [x, p] \rangle \right|^2 \]

Coordinates become sharply peaked as \( \sigma \to 0 \).

Fluctuations are minimal.
Question
Are Gaussian states semi-classical in LQC?
Classicality

Foreshadowing

No! Not even at the kinematic level.
(Though they are asymptotically so.)
Consider an FLRW spacetime with a metric:

$$ds^2 = -dt^2 + a(t)^2[dx^2 + dy^2 + dz^2].$$

Then the spatial volume of a time-like slice $V = a^3(t)$ and the Hubble parameter $\beta = \dot{a}(t)/a(t)$ are canonical variables from which the EFEs may be derived using a Hamiltonian treatment via the Holst action.

LQC is the theory one obtains by applying loop quantization to $V$ and $\beta$. 
The Holst action has a divergent symplectic term. Thus, the loop quantization procedure must introduce volume regularization. One therefore considers LQC on a compact fiducial spacetime region with volume $V_0$.\(^3\)

The “coordinate” observable in LQC is the volume operator $\hat{V}$. However, the loop quantization procedure does not allow for a unique operator $\hat{\beta}$. Instead, the conjugate momentum is chosen to be $\hat{S}_\lambda$, determined by the holonomy operator $\hat{h}_\lambda$:

\[
\hat{V} | V \rangle = V | V \rangle, \quad \hat{h}_\lambda | V \rangle = \left| V + \frac{4\pi G \gamma \bar{h}_\lambda}{V_0} \right\rangle, \quad \hat{S}_\lambda = \frac{1}{2i\lambda} (\hat{h}_\lambda - \hat{h}_\lambda^*).
\]

Henceforth, $\alpha := 4\pi G \gamma \bar{h}$.

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The kinematic volume Hilbert space is given by states of the form:

$$|\psi\rangle = \sum_{n \in \mathbb{Z}} f(n)|V_n\rangle, \quad \sum_{n \in \mathbb{Z}} |f(n)|^2 < \infty$$

for $V_n \in \mathbb{R}$, where $\langle V_n | V_m \rangle = \delta_{nm}$. The sign of $V_n$ indicates the orientation of the spatial slice. Physical states in LQC must have parity symmetry:\(^5\)

$$\psi(V) = \psi(-V)$$

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Gaussian states centered around $|V| = \mu l_P$ are states of the following form:\(^6\)

$$|\psi\rangle = \frac{c}{\sqrt{2}} \sum_{n \in \mathbb{Z}} \left( e^{-\left(n l_P - \mu l_P\right)^2/2\sigma^2} + e^{-\left(n l_P + \mu l_P\right)^2/2\sigma^2} \right) |n l_P\rangle$$

where $l_P$ is the Planck length; henceforth $l_P = 1$.

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By construction, Gaussian states in LQC are “sharply-peaked” around $|V| = \mu$. Thus, in order for them to be semi-classical kinematic states, we only require that their fluctuations in $\hat{V}$ and $\hat{S}_\lambda$ are minimal.
The exact volume-fluctuations are thus:

\[
\Delta^2_V = \frac{\sqrt{-\ln v}}{4\sqrt{\pi} \left(1 + e^{\mu^2 \pi^2 / \ln v}\right) \vartheta_3 \left(v^{1/\pi^2 \sigma^4}\right)} \\
\times \left\{ 2\mu^2 - \frac{\ln v}{\pi^2} \left(1 + e^{\mu^2 \pi^2 / \ln v}\right) \right\} \vartheta_3(v) \\
- \left(1 + e^{\mu^2 \pi^2 / \ln v}\right) \left(\frac{2v (\ln v)^2}{\pi^2}\right) \left(\frac{d}{dv} \vartheta_3(v)\right)
\]

where \(v := \exp\{-\pi^2 \sigma^2\}\) and \(\vartheta_3(q) = \vartheta_3(0, q)\) is the Jacobi \(\vartheta\)-function.
Fluctuations for Gaussian States

$$\vartheta_3(z, q) = \sum_{n \in \mathbb{Z}} q^{n^2} e^{2niz}$$
\( \Delta_V^2 \) as a function of \( \sigma \) for \( \mu \in \{0, 1, 2\} \) with \( \frac{\alpha \lambda}{V_0} = 1 \).
The exact holonomy-flux fluctuations have two cases to consider. If \(2\alpha\lambda/V_0 \notin \mathbb{Z}\), then:

\[
\Delta_{S\lambda}^2 = \frac{1}{2\lambda^2}
\]
If \( k = 2\alpha \lambda / V_0 \) is an integer, then:

\[
\Delta^2 S_\lambda = \frac{\alpha^2 4\pi \sigma}{k^2 V_0^2} - \frac{\alpha^2 \pi \sigma}{k^2 V_0^2 (1 + e^{-\mu^2/\sigma^2})} \vartheta_3 \left( e^{-1/\sigma^2} \right) \times
\]

\[
\left\{ 2 e^{-2(\alpha \lambda / V_0)^2 / \sigma^2} \left[ \vartheta_3 \left( \frac{-\alpha \lambda / V_0}{i \sigma^2}, e^{-1/\sigma^2} \right) + \vartheta_3 \left( \frac{\alpha \lambda / V_0}{i \sigma^2}, e^{-1/\sigma^2} \right) \right] +
\right.
\]

\[
+ e^{-2(\alpha \lambda / V_0 - \mu)^2 / \sigma^2} \left[ \vartheta_3 \left( \frac{-\alpha \lambda / V_0 + \mu}{i \sigma^2}, e^{-1/\sigma^2} \right) + \vartheta_3 \left( \frac{\alpha \lambda / V_0 - \mu}{i \sigma^2}, e^{-1/\sigma^2} \right) \right] +
\]

\[
+ e^{-2(\alpha \lambda / V_0 + \mu)^2 / \sigma^2} \left[ \vartheta_3 \left( \frac{-\alpha \lambda / V_0 - \mu}{i \sigma^2}, e^{-1/\sigma^2} \right) + \vartheta_3 \left( \frac{\alpha \lambda / V_0 + \mu}{i \sigma^2}, e^{-1/\sigma^2} \right) \right] \right\}
\]
\[ \Delta^2_{S_{\lambda}} \] as a function of \( \sigma \) for different \( \lambda \) with \( \mu = 0 \).
The question now is whether or not $\Delta^2_V \Delta^2_{S\lambda}$ is equal to the RS lower-bound. There are three different cases that lead to different behaviours, namely, if $\alpha \lambda / V_0 \not\in \mathbb{Z}$, or $\alpha \lambda / V_0 \in \mathbb{Z}$ and is even or odd. In all three cases, the fluctuations never saturate the inequality (for any values of $\sigma$, $\mu$, and $\lambda$).
\[ \Delta_V^2 \Delta_{S^2}^2 - L.B. \text{ as a function of } \sigma \text{ for different } k \text{ with } \mu = 0 \text{ and } \alpha \lambda / V_0 \in \mathbb{Z} \text{ even.} \]
Conclusion

Unlike in traditional quantum mechanics, Gaussian states in LQC do not minimize the uncertainty relation. Thus, they are not strictly semi-classical!
Follow-Up

Are the conditions under which they are *essentially* semi-classical?
Yes! One can check that in the $V_0 \rightarrow \infty$ limit, the fluctuations (and lower bound) vanish.$^7$ However, $V_0$ must be fixed within a model of LQC; it is not a parameter that can be varied.

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1. For a Gaussian state with a fixed width $\sigma$ and a fixed parity-symmetric mean $\mu$, one can always choose a model of LQC with a sufficiently large fiducial volume $V_0$ such that the fluctuations of this state become negligible.

2. Within a fixed model of LQC (with a given finite $V_0$), one can always find a Gaussian state with sufficiently large $\sigma$ such that its fluctuations become arbitrarily large.

3. The relation between $\lambda$ and $V_0$ play an important role in determining the phenomenology of a theory of LQC, namely, they determine whether or not the holonomy operator $\hat{h}_\lambda$ is closed, and the different integer cases affect fluctuations.
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