General Relativistic Evolution Equations for Density Perturbations in Closed, Flat and Open FLRW Universes

Structure Formation in the Early Universe

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Outline

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Cosmological Perturbation Theory

Open, Flat and Closed FLRW Universes

- Energy momentum tensor for a perfect fluid:

$$T^{\mu\nu} = (\varepsilon + p)u^{\mu}u^{\nu} - pg^{\mu\nu}$$

- Equations of state from Thermodynamics:

$$\varepsilon = \varepsilon(n, T), \quad p = p(n, T) \quad \Rightarrow \quad p = p(n, \varepsilon)$$

- Background equations for a FLRW universe:
  - Global evolution of the universe
- Linearized Einstein equations and conservation laws:
  - Local evolution of diabatic density perturbations
    - Growth of structure in a FLRW universe

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General Covariance

- General Relativity is invariant under general coordinate transformations

$$x^\mu \to x'^\mu(x^\nu)$$

- Linearized Einstein equations and conservation laws are invariant under gauge transformations

$$x^\mu \to x'^\mu = x^\mu - \xi^\mu(x^\nu)$$

- General covariance $\Rightarrow$ no preferred systems of reference
- Linearity of the perturbation equations creates the gauge problem of cosmology, which must be solved first

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Gauge Problem of Cosmology

FLRW universes are characterized by three scalars:

- Expansion scalar $\theta_{(0)} = 3H$
- Energy density $\varepsilon_{(0)}$
- Particle number density $n_{(0)}$
- The perturbations to these three scalars transform under a gauge transformation $x^\mu \to x'^\mu = x^\mu - \xi^\mu(x^\nu)$ according to
  - $S'_{(1)} = S_{(1)} + \xi^0 \dot{S}_{(0)} \qquad (S = \varepsilon, n, \theta)$
  - Gauge of problem of cosmology:
    - $S_{(1)}$ is a solution $\Rightarrow$ $S'_{(1)}$ is also a solution
    - Solution contains the gauge mode $\xi^0 \dot{S}_{(0)}$ $\Rightarrow$ $S_{(1)}$ has no physical significance
- Conclusion: the perturbations $\varepsilon_{(1)}$, $n_{(1)}$ and $\theta_{(1)}$ to the three scalars of the universe have no physical significance

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Previous Attempts to Solve the Gauge Problem

- First occurrence of the problem: Lifshitz [1946]
- Vast literature on cosmological density perturbations
  - Physical quantities should be independent of the choice of a coordinate system, i.e., gauge-invariant.
  - No agreement on which gauge-invariant quantities are the true energy density perturbation and particle number density perturbation
  - None of the cosmological perturbation theories in the literature has a correct non-relativistic limit
    - Physical interpretation of the gauge-invariant variables?
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Real, Measurable Density Perturbations

What we know for sure

- $\varepsilon^{\text{phys}}_{(1)}$, $n^{\text{phys}}_{(1)}$ and $\theta^{\text{phys}}_{(1)}$ are gauge-invariant
- Physics of density perturbations is hidden in the general solution of the perturbation equations
  - $\varepsilon^{\text{phys}}_{(1)}$, $n^{\text{phys}}_{(1)}$ and $\theta^{\text{phys}}_{(1)}$ are linear combinations of solutions of the perturbation equations such that the gauge mode is eliminated
- Non-relativistic limit: local fluid flow is small with respect to the speed of light $\quad \boldsymbol{u}^{\text{phys}}_{(1)} \to \boldsymbol{0} \quad \Leftrightarrow \quad p \to 0$
  - The relativistic gauge transformation must become equal to the Newtonian gauge transformation, where time and space are decoupled
- Gauge problem of cosmology: find linear combinations for $\varepsilon^{\text{phys}}_{(1)}$, $n^{\text{phys}}_{(1)}$ and $\theta^{\text{phys}}_{(1)}$

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Solution of the Gauge Problem of Cosmology

Gauge-invariant Quantities

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- $\theta^{\text{phys}}_{(1)} = 0$: local density perturbations have no influence on the global expansion of the universe
- Are $\varepsilon^{\text{phys}}_{(1)}$, $n^{\text{phys}}_{(1)}$ and $\theta^{\text{phys}}_{(1)} = 0$ the real, physical perturbations?

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Decomposition Theorems

- General covariance ⇒ use synchronous coordinates
  - In Newtonian gravity all coordinates are synchronous.
  - Decomposition theorem for symmetric tensors of rank 2 of 3-spaces of constant time (York, jr. [1974], Stewart [1990])
    - Decomposition of the perturbed metric:

      $$h^i{}_j = h^i{}_{\parallel j} + (h^i{}_{\perp j} + h^i{}_{*j}) \qquad \boxed{h^i{}_{\parallel j} = \frac{2}{c^2}(\phi\delta^i{}_j + \zeta^{|i}{}_{|j})}$$

    - Decomposition of the perturbed Ricci tensor:
      $R^i_{(1)j} = R^i_{(1)\parallel j} + (R^i_{(1)\perp j} + R^i_{(1)*j})$
    - Decomposition of the spatial fluid velocity:
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Scalar Perturbations

- General solution of the linearized Einstein equations and conservation laws is a linear combination of
  - (scalar + vector + tensor) perturbations
- Scalar perturbations
  - Metric: $h^i{}_j \to h^i_{\parallel j}$
  - Curvature: $R^i_{(1)j} \to R^i_{(1)\parallel j}$ $\quad (R_{(1)} := R^k_{(1)k})$

$$R_{(1)} = g^{ij}_{(0)}\left(h^k{}_{k|i|j} - h^k{}_{i|k|j}\right) + \tfrac{1}{3} R_{(0)} h^k{}_k$$

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Evolution Equations for Scalar Perturbations

- Constraint equations

$$2H(\theta_{(1)} - \vartheta_{(1)}) = \tfrac{1}{2} R_{(1)} + \kappa\varepsilon_{(1)}$$

$$\dot{R}_{(1)} = -2HR_{(1)} + 2\kappa\varepsilon_{(0)}(1+w)\vartheta_{(1)} - \tfrac{2}{3} R_{(0)}(\theta_{(1)} - \vartheta_{(1)})$$

- Conservation laws $(\tilde{\nabla}^2 f := \tilde{g}^{ij} f_{|i|j})$

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- Evolution of $\varepsilon^{\text{phys}}_{(1)} := \varepsilon_{(1)} - \dfrac{\dot{\varepsilon}_{(0)}}{\dot{\theta}_{(0)}}\theta_{(1)}, \quad n^{\text{phys}}_{(1)} := n_{(1)} - \dfrac{\dot{n}_{(0)}}{\dot{\theta}_{(0)}}\theta_{(1)}$

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Solution of the Gauge Problem of Cosmology

Newtonian Limit

- Flat FLRW universe: $R_{(1)} = \dfrac{4}{c^2}\phi^{|k}{}_{|k} = -\dfrac{4}{c^2}\dfrac{\nabla^2\phi}{a^2}$
- Non-relativistic limit: $\boldsymbol{u}^{\text{phys}}_{(1)} \to \boldsymbol{0} \quad \Leftrightarrow \quad p \to 0$:
  - Momentum constraint equation: $\nabla^2\dot{\phi} = 0$
  - Energy density constraint equation: $\nabla^2\phi(\boldsymbol{x}) = \dfrac{4\pi G}{c^2}\varepsilon^{\text{phys}}_{(1)}(\boldsymbol{x})$
  - $n^{\text{phys}}_{(1)} := n_{(1)} - \dfrac{\dot{n}_{(0)}}{\dot{\theta}_{(0)}}\theta_{(1)} \quad \Rightarrow \quad \varepsilon^{\text{phys}}_{(1)}(\boldsymbol{x}) = n^{\text{phys}}_{(1)}(\boldsymbol{x})mc^2$
  - Relativistic gauge transformation reduces to the Newtonian gauge transformation:
    $$x^0 \to x'^0 = x^0 - C \qquad x^i \to x'^i = x^i - \chi^i(\boldsymbol{x})$$
  - Conservation laws have only gauge modes as solution
- Conclusion: $\varepsilon^{\text{phys}}_{(1)}$, $n^{\text{phys}}_{(1)}$ and $\theta^{\text{phys}}_{(1)} = 0$ are indeed the real, physical perturbations, so that the gauge problem of cosmology is finally solved

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Solution of the Gauge Problem of Cosmology

Newtonian Limit

- Flat FLRW universe: $R_{(1)} = \dfrac{4}{c^2}\phi^{|k}{}_{|k} = -\dfrac{4}{c^2}\dfrac{\nabla^2\phi}{a^2}$
- Non-relativistic limit: $\boldsymbol{u}^{\text{phys}}_{(1)} \to \mathbf{0} \quad \Leftrightarrow \quad p \to 0$:
  - Momentum constraint equation: $\nabla^2\dot{\phi} = 0$
  - Energy density constraint equation: $\nabla^2\phi(\boldsymbol{x}) = \dfrac{4\pi G}{c^2}\varepsilon^{\text{phys}}_{(1)}(\boldsymbol{x})$
  - $n^{\text{phys}}_{(1)} := n_{(1)} - \dfrac{\dot{n}_{(0)}}{\dot{\theta}_{(0)}}\theta_{(1)} \quad \Rightarrow \quad \varepsilon^{\text{phys}}_{(1)}(\boldsymbol{x}) = n^{\text{phys}}_{(1)}(\boldsymbol{x})mc^2$
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Evolution Equations for δ_ε and δ_n

Open, Flat and Closed FLRW Universes

- Evolution of the energy density fluctuation $\delta_\varepsilon := \varepsilon^{\text{phys}}_{(1)}/\varepsilon_{(0)}$

$$\ddot{\delta}_\varepsilon + b_1\dot{\delta}_\varepsilon + b_2\delta_\varepsilon = b_3\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right]$$

- Evolution of the entropy perturbation ($\delta_n := n^{\text{phys}}_{(1)}/n_{(0)}$)

$$\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right] = \frac{3Hn_{(0)}p_n}{\varepsilon_{(0)}(1+w)}\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right]$$

- Coefficients b_1, b_2 and b_3 are determined by the background equations and the equation of state $p = p(n,\varepsilon)$
- Adiabatic perturbations: no heat exchange, only gravity

$$p = p(\varepsilon) \quad \Rightarrow \quad p_n := \left(\frac{\partial p}{\partial n}\right)_\varepsilon = 0 \quad \Rightarrow \quad b_3 = 0$$

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Diabatic Perturbations

- Equation of state: $p = p(n, \varepsilon)$
- Thermodynamics: $\mathrm{d}E = \text{đ}Q - p\mathrm{d}V + \mu\mathrm{d}N \quad (\text{đ}Q = T\mathrm{d}S)$
  - Background conservation laws:
    - Entropy of the background universe is constant: $\dot{s}_{(0)} = 0$
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$$T_{(0)} s_{(1)}^{\mathrm{phys}} = -\frac{\varepsilon_{(0)}(1+w)}{n_{(0)}}\left[\delta_n - \frac{\delta_\varepsilon}{1+w}\right]$$

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Outline

Radiation Dominated Universe

- Equations of state

$$\varepsilon = a_{\mathrm{B}} T_{\gamma}^4, \quad p = \tfrac{1}{3} a_{\mathrm{B}} T_{\gamma}^4 \quad \Rightarrow \quad p = \tfrac{1}{3}\varepsilon$$

- General relativistic evolution equations

$$\ddot{\delta}_{\varepsilon} - H\dot{\delta}_{\varepsilon} - \left[\frac{1}{3}\frac{\nabla^2}{a^2} - \tfrac{2}{3}\kappa\varepsilon_{(0)}\right]\delta_{\varepsilon} = 0$$

$$\delta_n - \tfrac{3}{4}\delta_{\varepsilon} = 0$$

  - Equations hold true for all kinds of particles that interact through gravity
  - From the entropy equation it follows that fluctuations in CDM are coupled to fluctuations in the energy density

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General Solution

$$\delta_\varepsilon(\tau, \boldsymbol{q}) = \left[A_1(\boldsymbol{q}) \sin\left(\mu_\mathrm{r}\sqrt{\tau}\right) + A_2(\boldsymbol{q}) \cos\left(\mu_\mathrm{r}\sqrt{\tau}\right)\right]\sqrt{\tau}, \quad \tau := \frac{t}{t_0}$$

$$\mu_\mathrm{r} := \frac{2\pi}{\lambda_0}\frac{1}{H(t_0)}\frac{v_\mathrm{s}}{c}, \quad \lambda_0 := \lambda a(t_0), \quad \frac{v_\mathrm{s}}{c} = \frac{1}{\sqrt{3}}, \quad |\boldsymbol{q}| = \frac{2\pi}{\lambda}$$

- Finite scale perturbations oscillate with a growing amplitude proportional to $t^{1/2}$
- Large-scale perturbations $\delta_\varepsilon = c_1 t + c_2 t^{1/2}$
- Conclusion:
  - CDM and energy density fluctuations are coupled by gravitation $\Rightarrow$ no seeds from CDM for structure formation
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Universe after Decoupling of Matter and Radiation

Equations of State

- Equations of state for a monatomic perfect fluid:

$$\varepsilon(n,T) = nmc^2 + \tfrac{3}{2} n k_{\mathrm{B}} T, \quad p(n,T) = n k_{\mathrm{B}} T$$

$$\beta^2 := \frac{\dot{p}_{(0)}}{\dot{\varepsilon}_{(0)}} \approx \frac{v_{\mathrm{s}}^2}{c^2} = \frac{5}{3}\frac{k_{\mathrm{B}} T_{(0)}}{mc^2}, \quad w := \frac{p_{(0)}}{\varepsilon_{(0)}} \approx \frac{k_{\mathrm{B}} T_{(0)}}{mc^2} \lesssim 2.7\times 10^{-10}$$

- Background energy conservation law: $T_{(0)} \propto a^{-2}$
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  - background universe: $w \ll 1 \quad \Rightarrow \quad p_{(0)} = 0$
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  - background universe: $w \ll 1 \quad \Rightarrow \quad p_{(0)} = 0$
  - perturbed universe: $p_{(1)}^{\mathrm{phys}} = 0 \quad \Rightarrow$ Newtonian limit

    Therefore $p_{(1)}^{\mathrm{phys}} \neq 0$:

    - $p_{(1)}^{\mathrm{phys}} = 0 \quad \Leftrightarrow \quad \boldsymbol{u}_{(1)}^{\mathrm{phys}} = \boldsymbol{0} \quad \Rightarrow$ no structure formation
    - no local pressure perturbations $\Leftrightarrow$ no local fluid flow

Universe after Decoupling of Matter and Radiation

Equations of State

- Equations of state for a monatomic perfect fluid:

$$\varepsilon(n,T) = nmc^2 + \tfrac{3}{2}nk_{\mathrm{B}}T, \quad p(n,T) = nk_{\mathrm{B}}T$$

$$\beta^2 := \frac{\dot{p}_{(0)}}{\dot{\varepsilon}_{(0)}} \approx \frac{v_{\mathrm{s}}^2}{c^2} = \frac{5}{3}\frac{k_{\mathrm{B}}T_{(0)}}{mc^2}, \quad w := \frac{p_{(0)}}{\varepsilon_{(0)}} \approx \frac{k_{\mathrm{B}}T_{(0)}}{mc^2} \lesssim 2.7\times 10^{-10}$$

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Universe after Decoupling of Matter and Radiation

Evolution Equations

- General relativistic evolution equations

$$\ddot{\delta}_\varepsilon + 3H\dot{\delta}_\varepsilon - \left[\beta^2\frac{\nabla^2}{a^2} + \tfrac{5}{6}\kappa\varepsilon_{(0)}\right]\delta_\varepsilon = -\frac{2}{3}\frac{\nabla^2}{a^2}(\delta_n - \delta_\varepsilon)$$

$$\frac{1}{c}\frac{\mathrm{d}}{\mathrm{d}t}(\delta_n - \delta_\varepsilon) = -2H(\delta_n - \delta_\varepsilon)$$

- Solution of the entropy equation $(H := \dot{a}/a)$

$$\delta_n - \delta_\varepsilon \propto a^{-2}$$

- Kinetic energy density fluctuation $(T_{(0)} \propto a^{-2})$

$$\delta_{\text{kin}}(t, \boldsymbol{x}) \approx \delta_\varepsilon(t, \boldsymbol{x}) - \delta_n(t, \boldsymbol{x}) \approx \frac{3}{2}\frac{k_{\text{B}}T_{(0)}(t)}{mc^2}\delta_T(t_{\text{dec}}, \boldsymbol{x}) \propto a^{-2}$$

- The entropy perturbation $s_{(1)}^{\text{phys}} \approx \frac{3}{2}k_{\text{B}}\delta_T(t_{\text{dec}}, \boldsymbol{x})$ is a constant and also a free parameter

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Universe after Decoupling of Matter and Radiation

Evolution Equation in dimensionless time

- Evolution equation in dimensionless time $\tau := t/t_{\rm dec}$

$$\delta_\varepsilon'' + \frac{2}{\tau}\delta_\varepsilon' + \left[\frac{4}{9}\frac{\mu_{\rm m}^2}{\tau^{8/3}} - \frac{10}{9\tau^2}\right]\delta_\varepsilon = -\frac{4}{15}\frac{\mu_{\rm m}^2}{\tau^{8/3}}\delta_T(t_{\rm dec}, \boldsymbol{q}), \quad \tau \geq 1$$

$$\mu_{\rm m} = \frac{2\pi}{\lambda_{\rm dec}}\frac{1}{cH(t_{\rm p})}\frac{1}{\left[z(t_{\rm dec}) + 1\right]}\sqrt{\frac{5}{3}\frac{k_{\rm B}T_{(0)\gamma}(t_{\rm p})}{m}} \approx \frac{16.48}{\lambda_{\rm dec}\ [\mathrm{pc}]}$$

- Entropy perturbation $s_{(1)}^{\rm phys} \approx \frac{3}{2}k_{\rm B}\delta_T(t_{\rm dec}, \boldsymbol{q})$ random variable
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Initial Values just after decoupling

Initial values at $t = t_{\text{dec}}$

- From observation (Planck Collaboration [2020]):
  - $|\delta_\varepsilon(t_{\text{dec}}, \boldsymbol{q})| \lesssim 10^{-5}$
- Initially no growth:
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  - $\delta_T(t_{\text{dec}}, \boldsymbol{q}) > 0 \quad \Rightarrow \quad$ large voids
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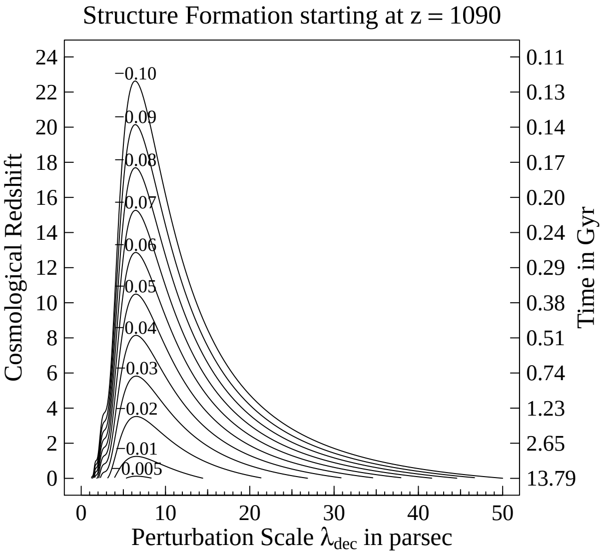
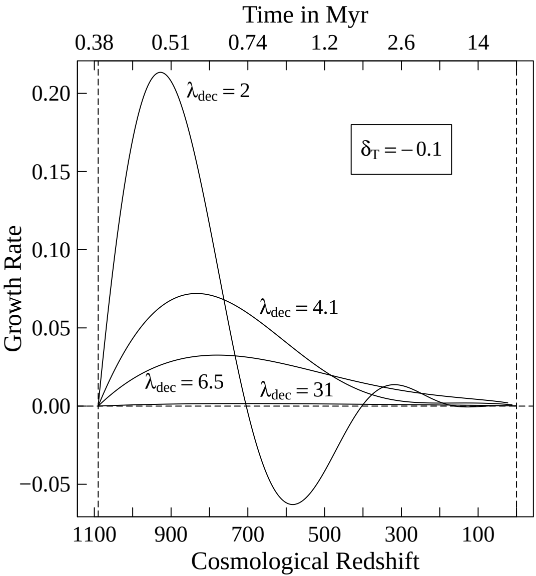

Structure Formation starting at z = 1090
Cosmological Redshift
Time in Gyr
Perturbation Scale λ_{dec} in parsec
0.10
0.09
0.08
0.07
0.06
0.05
0.04
0.03
0.02
0.01
0.005
24
22
20
18
16
14
12
10
8
6
4
2
0
0.11
0.13
0.14
0.17
0.20
0.24
0.29
0.38
0.51
0.74
1.23
2.65
13.79
0
10
20
30
40
50


Time in Myr
0.38
0.51
0.74
1.2
2.6
14
Growth Rate
0.20
0.15
0.10
0.05
0.00
−0.05
$\lambda_{\mathrm{dec}} = 2$
$\delta_{\mathrm{T}} = -0.1$
$\lambda_{\mathrm{dec}} = 4.1$
$\lambda_{\mathrm{dec}} = 6.5$
$\lambda_{\mathrm{dec}} = 31$
1100
900
700
500
300
100
Cosmological Redshift

Outline

Results

Theory

- Perturbation theory for open, flat and closed FLRW universes filled with a perfect fluid with an equation of state for the pressure $p = p(n, \varepsilon)$
  - Newtonian limit in an expanding universe
    - Gauge problem of cosmology is solved
  - Second-order evolution equation for the true, physical density fluctuation $\delta_\varepsilon := \varepsilon_{(1)}^{\text{phys}}/\varepsilon_{(0)}$
  - First-order evolution equation for entropy perturbations proportional to $\left[\delta_n - \dfrac{\delta_\varepsilon}{1+w}\right]$
  - Local density perturbations do not affect the global expansion of the universe

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Outline

Results

Structure Formation in a flat FLRW universe

- Radiation dominated universe:
  - δ_ε oscillates with an amplitude $\propto t^{1/2}$
  - $\delta_n = \frac{3}{4}\delta_\varepsilon$ also for CDM (baryons: Thomson scattering)
  - Structure formation can start only after decoupling
- Universe after decoupling:
  - Structure formation depends on both the initial value of the entropy perturbation and the initial scale λ_{dec} of a density perturbation
    - Only one assumption: $s_{(1)}^{\text{phys}} \approx \frac{3}{2}k_{\text{B}}\delta_T(t_{\text{dec}})$ is random
  - Relativistic Jeans scale: 6.5 pc $\approx$ 21 ly
  - Relativistic Jeans mass: $4.3 \times 10^3\ M_\odot$
  - First stars $z \approx$ 24–12 (110–300 Myr after the Big Bang)
- I have shown that the General Relativistic perturbation theory for a flat FLRW universe explains structure formation in the early universe, without the need of Cold Dark Matter

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  - $\delta_n = \frac{3}{4}\delta_\varepsilon$ also for CDM (baryons: Thomson scattering)
  - Structure formation can start only after decoupling
- Universe after decoupling:
  - Structure formation depends on both the initial value of the entropy perturbation and the initial scale λ_{dec} of a density perturbation
    - Only one assumption: $s^{\mathrm{phys}}_{(1)} \approx \frac{3}{2} k_{\mathrm{B}} \delta_T(t_{\mathrm{dec}})$ is random
  - Relativistic Jeans scale: 6.5 pc $\approx$ 21 ly
  - Relativistic Jeans mass: $4.3 \times 10^3\, M_\odot$
  - First stars $z \approx$ 24–12 (110–300 Myr after the Big Bang)
- I have shown that the General Relativistic perturbation theory for a flat FLRW universe explains structure formation in the early universe, without the need of Cold Dark Matter

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Further Reading, Software and Contact

P. G. Miedema.
General Relativistic Evolution Equations for Density Perturbations in Closed, Flat and Open FLRW Universes
ArXiv e-eprint, September 2014
https://arxiv.org/abs/1410.0211

P. G. Miedema.
Structure Formation in the Early Universe
ArXiv e-print, January 2016
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