New approaches to constrained dynamics and Hamilton-Jacobi procedures in general relativity

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References

See *Restoration of four-dimensional diffeomorphism covariance in canonical general relativity: An intrinsic Hamilton-Jacobi approach*, arXiv:1508.01277v6 for some background. This is corrected version of [Salisbury *et al.* , 2016]. Current work is in preparation. See *Observables and Hamilton-Jacobi approaches to general relativity. I. The Earlier History*, arXiv:2106.11894, for historical background.

Overview

1. CARTAN INVARIANT INTEGRAL - EXTENSION TO GAUGE FIELD THEORY

"Leçons sur les Invariants Intégraux ", Librairie Sceintifique A. Hermann et Fils (1922)

Demand that the integral I over a closed parameterized set $q^i(t(s);s)$, $v^j(t(s);s)$, $t(s)$ be independent of t where

$$I = \oint ds \left(\frac{\partial L}{\partial v^i} \left. \frac{\partial q^i}{\partial s} \right|_{t(s)} - \left(\frac{\partial L}{\partial v^i} v^i - L \right) \frac{dt}{ds} \right)$$

$$=: \oint \left(\frac{\partial L}{\partial v^i} dq^i - \left(\frac{\partial L}{\partial v^i} v^i - L \right) dt \right)$$

Free relativistic particle extension

$$L_p = \frac{1}{2N} v^2 - \frac{N}{2} m^2$$

$$I_p = \oint \left(N^{-1} v_\mu dq^\mu - \frac{1}{2} \left(N^{-1} v^2 + N m^2 \right) dt \right)$$

Require that this vanish under independent δ variations, integrating by parts over the closed interval to get

$$\delta I_p = \oint \left[\left(-N^{-2} \delta N v_\mu + N^{-1} \delta v_\mu \right) dq^\mu + \left(\frac{1}{2} N^{-2} \delta N v^2 - N^{-1} v_\mu \delta v^\mu \right.\right.$$

$$\left.\left. - \frac{1}{2} \delta N m^2 \right) dt - \left(\frac{1}{2} N^{-2} dN v^2 - N^{-1} v_\mu dv^\mu - \frac{1}{2} dN m^2 \right) \delta t \right] = 0$$

Free relativistic particle extension

Coefficient of δv^μ yields $v^\mu = \frac{dq^\mu}{dt} := \dot{q}^\mu$
Coefficient of δN yields $N = m^{-1} \left(-\dot{q}^2\right)^{1/2}$
Coefficient of δt is redundant, yielding $-m^2 \dot{N} = N^{-1} \dot{q} \cdot \ddot{q}$
Coefficient of δq^μ yields the equation of motion $\frac{d}{dt} \left(N^{-1} \dot{q}^\mu\right) = 0$

Free relativistic particle extension to phase space

We have a primary constraint $\pi = \frac{\partial L_p}{\partial \dot{N}} = 0$.
The required invariant integral becomes

$$I_p = \oint \left[p_\mu dq^\mu + \pi dN - \left(\frac{N}{2} \left(p^2 + m^2 \right) + \pi \dot{N} \right) dt \right]$$

resulting in

$$dI_p = \oint \left[\delta p_\mu \left(dq^\mu - N p^\mu dt \right) - \delta q^\mu dp_\mu + \delta N \left(-d\pi - \frac{1}{2}(p^2 + m^2) dt \right) \right.$$

$$+\delta\pi \left(dN - \dot{N} dt \right) + \delta t \left(\left(\frac{1}{2}(p^2 + m^2 \right) dN + N p^\mu \delta p_\mu + \delta\pi \dot{N} + \pi \delta \dot{N} \right)$$

$$\left. + \delta \dot{N} \pi dt \right] = 0$$

Free relativistic particle extension to phase space

The first and second terms result in $\frac{dq^\mu}{dt} = Np^\mu$ and $\frac{dp^\mu}{dt} = 0$. The third term yields the secondary constraint $p^2 + m^2 = 0$, and the forth $\dot{N} = \frac{dN}{dt}$. The remaining terms are redundant.

Extension to the ADM Lagrangian

Claim: The assumed invariant integral

$$I_{ADM} = \oint d^3x \left[p^{ab} dg_{ab} + P_\mu dN^\mu - \mathcal{H}_{ADM} dt - P_\mu \dot{N}^\mu dt \right]$$

where

$$\mathcal{H}_{ADM} = \frac{N}{\sqrt{{}^3g}} \left(p_{ab} p^{ab} - (p^a{}_a)^2 - N \sqrt{{}^3g}\, {}^3R - 2N^a p^b{}_{a|b} \right)$$

with primary constraints $P_\mu = 0$ yields the correct secondary constraints and the correct Einstein Hamiltonian equations.

These two systems are examples of a general new invariant integral approach to constrained Hamiltonian dynamcs, equivalent to the Rosenfeld-Bergmann-Dirac prodedure.

2. DERIVATION OF DIFFEOMORPHISM GENERATOR

Derivation of reparameterization generator for the free particle

Take ds in the Poincaré-Cartan increment to correspond to the change in solutions that results from an infinitesimal reparameterization $t' = t - ds\epsilon(t)$. Represent the change in the variables as $\bar{d}q^\mu = ds\dot{q}^\mu\epsilon$ and $\bar{d}N = ds(\dot{\epsilon}N + \epsilon\dot{N})$. (These are actually the Lie derivatives with respect to ϵ.)

Now use the identity which is the statement that the particle Lagrangian transforms as a scalar density under reparameterizations to deduce the existence of a vanishing Noether charge C_p:

$$\frac{dC_p}{dt} = \frac{d}{dt}\left(\frac{N\epsilon}{2}\left(p^2 + m^2\right) + \pi\left(\dot{N}\epsilon + N\dot{\epsilon}\right)\right) = 0$$

Non-projectability of the Noether charge

Must assume variable-dependent reparameterizations $\epsilon(t) = N^{-1}\xi(t)$ to get Legendre-projectable reparameterizations:

$$C_p = \frac{\xi}{2}\left(p^2 + m^2\right) + \pi\left(\dot{N}\epsilon + N\dot{\epsilon}\right)$$

$$= \frac{\xi}{2}\left(p^2 + m^2\right) + \pi\left(\dot{N}N^{-1}\xi - NN^{-2}\dot{N}\xi + \dot{\xi}\right)$$

$$= \frac{\xi}{2}\left(p^2 + m^2\right) + \pi\dot{\xi}$$

This is the phase space generator of the N-dependent reparameterizations.

New derivation of Pons-Salisbury-Shepley generator

Claim: The analogous substitution into the ADM vanishing Noether charge yields the Pons-Salisbury-Shepley generator $\mathcal{C}_{PSS}$ [Pons *et al.* , 1997] of diffeomorphism-induced general coordinate transformations $\epsilon^\mu(x) = \left(N^{-1}\xi^0(x), -N^{-1}N^a\xi^0(x) + \xi^a(x)\right)$:

$$\mathcal{C}_{PSS} = P_\mu\dot{\xi}^\mu + \mathcal{H}_\mu\xi^\mu + P_0(\xi^0 - N^a\xi^0_{,a} + N_{,a}\xi^a)$$

$$+P_a(N_{,b}e^{ab}\xi^0_{,b} - N\xi^0_{,b}e^{ab} + N_{,a}\xi^a + N^a_{,b}\xi^b - N^b\xi^a_{,b})$$

3. INTRINSIC COORDINATES AND DIFFEOMORPHISM INVARIANTS

Free particle evolution in proper time

Choose as the particle time evolution variable $T = -mq^0/p_0$. This is the proper time. Perform the canonical transformation in the Poincaré-Cartan increment

$$p_0 dq^0 = PdT + \frac{\partial G}{\partial q^0} dq^0 + \frac{\partial G}{\partial T} dT$$

The required generator is

$$G = -\frac{m}{2T}\left(q^0\right)^2 - mT + mT\ln(T)$$

with the resulting canonical conjugate

$$P = -m\ln\left(-\frac{mq^0}{p_0}\right) - \frac{p_0^2}{2m}$$

General covariance and gauge choices

System is still reparameterization covariant - with new generator

$$C_p = \xi \left[-P - m \ln(T) + \frac{1}{2} \left(p^a p_a + m^2 \right) \right] + \pi \dot{\xi}$$

Now make the proper time gauge choice by setting the evolution parameter equal to the intrinsic variable T, $t = T$.

Given particle solutions in any parameterization the gauge generator can be employed to transform to solutions satisfying the gauge condition. The remaining canonical variables are invariants under reparameterizations

Intrinsic spacetime coordinate variables

Choose appropriate spacetime scalars $X^\mu\left(g_{ab}, p^{cd}\right)$, constructed with the use of Weyl scalars, as intrinsic spacetime coordinates.

Perform canonical transformations in the Poincaré-Cartan increment so that the non-vanishing contribution becomes

$$dI_{ADM} = \int d^3x\, p^{ab} dg_{ab}$$

$$= \int d^3x \left(\pi_\mu dX^\mu + p^A dg_A + \frac{\delta G}{\delta g_{ab}} dg_{ab} + \frac{\delta G}{\delta \alpha_A} d\alpha_A + \frac{\delta G}{\delta X^\mu} dX^\mu\right)$$

Intrinsic spacetime coordinate gauge choice

Now make the gauge choices $x^{\mu} = X^{\mu}\left(g_{ab}, p^{cd}\right)$.

The diffeomorphism generator C_{PSS} could be employed to explicitly display the corresponding diffeomorphism invariants, as shown in [Pons *et al.* , 2009]

4. QUANTUM OUTLOOK

Quantum outlook

Claim: It is likely that every gravitational quantization procedure now undertaken either explicitly or implicitly invokes a choice of intrinsic coordinates

A scheme must be sought that takes into account the abundance of physically distinct gauge choices

References I

- Pons, Josep, Salisbury, Donald, & Shepley, Lawrence. 1997.
  Gauge transformations in the Lagrangian and Hamiltonian formalisms of generally covariant theories.
  Physical Review D, **55**, 658–668.

- Pons, Josep, Salisbury, Donald, & Sundermeyer, Kurt. 2009.
  Revisiting observables in generally covariant theories in light of gauge fixing methods.
  Physical Review D, **80**, 084015–1–084015–23.

- Salisbury, Donald, Renn, Jürgen, & Sundermeyer, Kurt. 2016.
  Restoration of four-dimensional diffeomorphism covariance in canonical general relativity: An intrinsic Hamilton-Jacobi approach.
  International Journal of Modern Physics A, **31**, 1650014–1–25.