Geometrically thick tori around compact objects with a quadrupole moment

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Content

Effective potential and thick accretion tori in static spacetimes

From the normalisation condition $u_\mu u^\mu \overset{!}{=} -1$, we can describe the time component by using the *specific angular momentum* $l = \frac{g_{tt}}{g_{\varphi\varphi}}$:

$$u_t^{-2} = -g^{tt} - l^2 g^{\varphi\varphi}$$

Euler equation for a perfect fluid in circular motion:

$$\partial_\mu \ln |u_t| - \left(\frac{\Omega}{1-\Omega l}\right) = -\frac{1}{\rho h}\partial_\mu p$$

For a barotropic fluid, surfaces of constant l and Ω coincide. If $\mathrm{d}l \neq 0$, then $\Omega = \Omega(l)$ $\rightarrow$ *relativistic von Zeipel theorem*. In that case, integrate Euler equation to find

$$\mathcal{W} - \mathcal{W}_{in} := -\int_0^p \frac{\mathrm{d}p'}{\rho h} = \ln|u_t| - \ln|(u_t)_{in}| - \int_{l_{in}}^{l} \frac{\Omega}{1-\Omega l'}\mathrm{d}l' \ ,$$

For constant angular momentum and choosing $\mathcal{W}_{\mathrm{in}} = -\ln|(u_t)_{in}|$, this equation reduces to

$$\boxed{\mathcal{W}(l, r, \vartheta) = -\frac{1}{2}\ln\Big(-g^{tt}(r,\vartheta) - l^2 g^{\varphi\varphi}(r,\vartheta)\Big)}$$

$\Rightarrow$ 'Polish Doughnuts'

Circular lightlike and timelike geodesics I

Fluid element at the centre moves along circular timelike geodesic
→ study properties of circular geodesics in the considered spacetimes!
Consider geodesics in the equatorial plane → Lagrangian:

$$\mathcal{L} = \frac{1}{2}\left(g_{tt}\dot{t}^2 + g_{rr}\dot{r}^2 + g_{\varphi\varphi}\dot{\varphi}^2\right)$$

Constants of motion:

$$E = g_{tt}\dot{t}, \qquad L = g_{\varphi\varphi}\dot{\varphi}, \qquad \varepsilon = -g_{tt}\dot{t}^2 - g_{rr}\dot{r}^2 - g_{\varphi\varphi}\dot{\varphi}^2 \ ,$$

where $\varepsilon = 0$ for lightlike and $\varepsilon = 1$ for timelike geodesics.
Defining the effective potential $\mathcal{V}$:

$$-g_{tt}g_{rr}\dot{r}^2 + \mathcal{V} = E^2 \ \text{ with } \ \mathcal{V} = -g_{tt}(r)\left(\frac{L^2}{g_{\varphi\varphi}(r)} + \varepsilon\right)$$

Circular motion ($\dot{r} = 0$ and $\ddot{r} = 0$) is equivalent to

$$\mathcal{V}(\varepsilon, L, r) = E^2, \quad \frac{\partial \mathcal{V}(\varepsilon, L, r)}{\partial r} = 0$$

Circular lightlike and timelike geodesics II

For lightlike geodesics $\varepsilon = 0$, the two conditions are equivalent to

$$\frac{L^2}{E^2} = -\frac{g_{\varphi\varphi}}{g_{tt}}, \quad g_{tt} g'_{\varphi\varphi} = g_{\varphi\varphi} g'_{tt}$$

$\rightarrow$ position of the photon circle
For timelike geodesics $\varepsilon = 1$, the two conditions lead to the "Keplerian" constants of motion:

$$L_K^2 = \frac{g_{\varphi\varphi}^2 g'_{tt}}{g_{tt} g'_{\varphi\varphi} - g_{\varphi\varphi} g'_{tt}}, \quad E_K^2 = -\frac{g_{tt}^2 g'_{\varphi\varphi}}{g_{tt} g'_{\varphi\varphi} - g_{\varphi\varphi} g'_{tt}}$$

$\rightarrow$ Keplerian specific angular momentum (KSAM) and Keplerian angular velocity:

$$l_K^2 = \left(\frac{L_K}{E_K}\right)^2 = -\frac{\partial_r g^{tt}}{\partial_r g^{\varphi\varphi}}, \quad \Omega_K^2 = \left(\frac{g_{tt} L_K}{g_{\varphi\varphi} E_K}\right)^2 = -\frac{\partial_r g_{tt}}{\partial_r g_{\varphi\varphi}}$$

Marginally stable circular orbit (last stable circular orbit):

$$l'_K(r_{\mathrm{ms}}) \overset{!}{=} 0$$

Marginally bound circular orbit:

$$\mathcal{V}(1, L_K(r_{\mathrm{mb}}), r_{\mathrm{mb}}) \overset{!}{=} 1 \quad \Leftrightarrow \quad E_K^2(r_{\mathrm{mb}}) \overset{!}{=} 1$$

The q-metric

Simplest exact static exterior solution of the vacuum field equations with non-vanishing quadrupole moment, its metric:

$$ds^2 = -\left(1-\frac{2M}{r}\right)^{1+q} dt^2 + \left(1-\frac{2M}{r}\right)^{-q}\left[\left(1+\frac{M^2\sin^2\vartheta}{r^2-2Mr}\right)^{-q(2+q)}\left(\frac{dr^2}{1-\frac{2M}{r}}+r^2 d\vartheta^2\right)+r^2\sin^2\vartheta d\varphi^2\right],$$

where q: quadrupole parameter, M: mass parameter.
Mass multipole moments (Geroch-Hansen):

$$M_0 = (1+q)M, \qquad M_2 = -\frac{M^3}{3}q(q+1)(q+2)$$

For later comparison, express the q-metric in terms of M_0 and M_2 (restrict discussion to $q > -1$ and $M > 0$):

$$M = M_0\sqrt{3\frac{M_2}{M_0^3}+1}, \qquad q = \frac{1}{\sqrt{3\frac{M_2}{M_0^3}+1}} - 1$$

Marginally bound and marginally stable circular orbits

For lightlike geodesics, there is exactly one solution for photon circles:

$$r_c = (3+2q)M = \left(2 + \sqrt{1 + 3\frac{M_2}{M_0^3}}\right)$$

Marginally bound circular orbit: only numerically, up to one
Marginally stable circular orbit:

$$r_{\mathrm{ms}}^{\pm} = M\left(4 + 3q \pm \sqrt{5q^2 + 10q + 4}\right)$$

$\rightarrow$ splits the family of q-metrics intro three classes:

Class I : $\infty > q > -1/2$ or $-1/3 < M_2/M_0^3 < 1$

- Schwarzschild-like

Class II : $-1/2 > q \gtrsim -0.553$ or $1 < M_2/M_0^3 < 4/3$

- two marginally stable, but *no* photon circle anymore

Class III : $-0.553 \gtrsim q > -1$ or $4/3 < M_2/M_0^3 < \infty$

- all orbits $> 2M$ are stable

Depiction of orbits

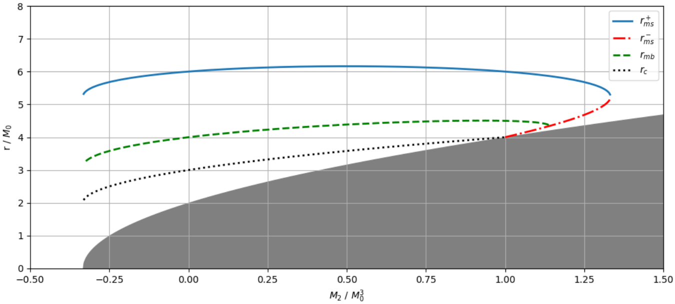
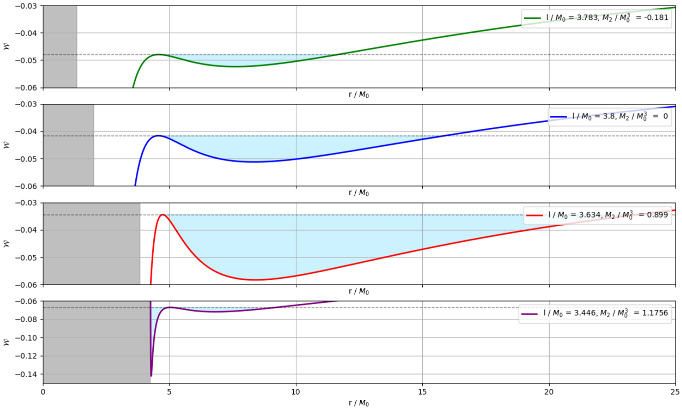


Abbildung: Circular orbits in the q-metric, depending on the quadrupole moment.

Effective potential in the equatorial plane

$$\mathcal{W}(r, l, \vartheta) = \frac{1}{2} \ln \left[\frac{r^2 \sin^2 \vartheta}{\left(1 - \frac{2M}{r}\right)^{-(1+q)} r^2 \sin^2 \vartheta - l^2 \left(1 - \frac{2M}{r}\right)^q} \right]$$



Effective potential in Class II spacetimes: the connection of double tori and fish

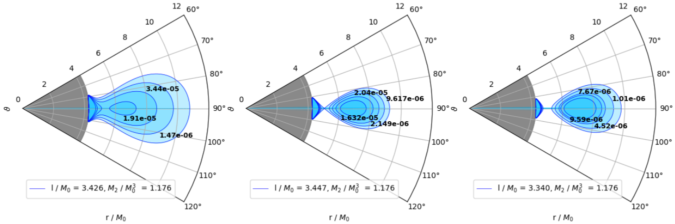


Abbildung: Polar dependency of double tori for Class II spacetimes, depending on the quadrupole moment, forming fish-like structures.

The Erez-Rosen spacetime

First found solution of Einstein's vacuum field equations identified as describing the gravitational field around a central object with a quadrupole moment

$$ds^2 = -fdt^2 + \frac{\sigma^2}{f}\left[e^{2\gamma}(x^2 - y^2)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + (x^2-1)(1-y^2)d\varphi^2\right]$$

with

$$\begin{aligned} f =& \frac{x-1}{x+1}e^{-2qP_2Q_2} \\ \gamma =& \frac{1}{2}(1+q)^2 \ln\frac{x^2-1}{x^2-y^2} + 2q\left(1-P_2\right)Q_1 + q^2\left(1-P_2\right)\cdot\Big[\left(1+P_2\right)\left(Q_1^2 - Q_2^2\right) \\ & + \frac{1}{2}\left(x^2-1\right)\left(2Q_2^2 - 3xQ_1Q_2 + 3Q_0Q_2 - Q_2'\right)\Big] \end{aligned}$$

with $Q = Q(x)$ and $P = P(y)$, and q: quadrupole parameter.
Transformation to Schwarzschild-like coordinates via $x = r/M - 1$ and $y = \cos\vartheta$.
Multipole moments (Geroch-Hansen):

$$M_0 = M, \quad M_2 = \frac{2}{15}q^3M^3$$

Marginally bound and marginally stable circular orbits

For lightlike geodesics, there are up to 2 solutions for photon circles, determined by:

$$r - 3M - qr(r - 2M)\partial_r Q_2(r/M - 1) \stackrel{!}{=} 0$$

Marginally bound circular orbit: only numerically, up to one
Marginally stable circular orbit: also, only numerically, up to two
Orbit properties divide Erez-Rosen spacetimes into 3:

Class I : $-\infty < q < 1$ or $-\infty < M_2/M_0^3 \lesssim 0.13$

- Schwarzschild-like

Class IIa : $1 < q \lesssim 2.25$ or $0.13 \lesssim M_2/M_0^3 \lesssim 1.52$

- *two* photon circles!

Class IIb : $2.25 \lesssim q < 4.8$ or $1.52 \lesssim M_2/M_0^3 \lesssim 25.8$

- two marginally stable, but no photon circle anymore - see Class II q-metric

Class III : $4.8 \lesssim q < \infty$ or $25.8 \lesssim M_2/M_0^3 < \infty$

- all orbits $> 2M$ are stable

Depiction of orbits

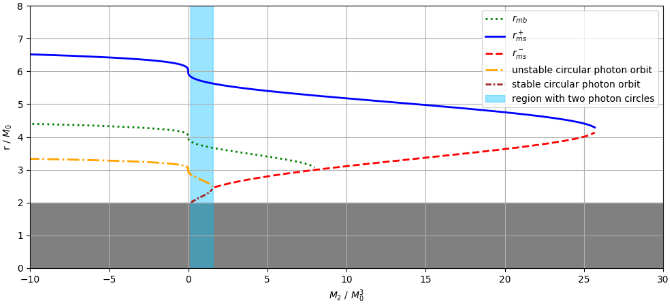
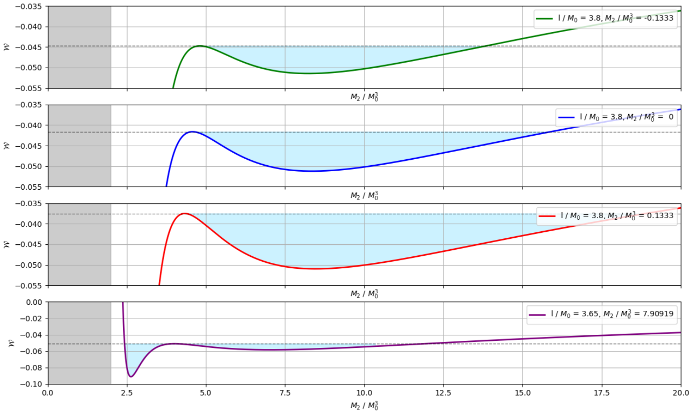


Abbildung: Circular orbits in Erez-Rosen spacetime, depending on the quadrupole moment.

Effective potential in the equatorial plane

$$\mathcal{W}(r,l,\vartheta) = \frac{1}{2}\ln\left[\frac{r^2(r-2M)e^{-2qP_2(\cos\vartheta)Q_2(r/M-1)}}{r^3\sin^2\vartheta - l^2(r-2)e^{-4qP_2(\cos\vartheta)Q_2(r/M-1)}}\right]$$



Effective potential in Class II spacetimes

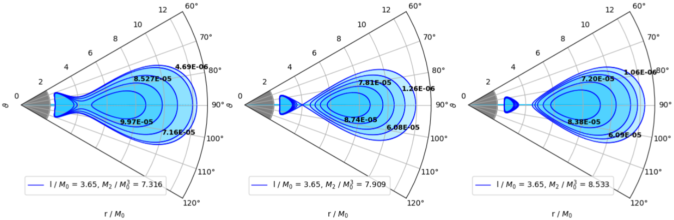


Abbildung: Polar dependency of the effective potential for class II spacetimes. The black numbers represent the density at the position of the equipotential surfaces.

Conclusion

- both q-metric and Erez-Rosen spacetime can be distinguished into 3 classes
- In class I, tori are qualitatively similar to the tori in Schwarzschild spacetime
- In class II, there are qualitative differences
  - double tori, fish-like structures
  - two centres
  - no accretion
- In class III, tori cannot have a cusp, thus no accretion