# Dynamical systems approach to teleparallel cosmologies

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# Motivation

- Open questions in cosmology and gravity:
    - Accelerating phases in the history of the Universe - dark energy, inflation?
    - Relation between gravity, quantum theory and gauge theories?

$\Rightarrow$ $\Lambda$CDM model of general relativity challenged by theory & observations.

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  - $\Rightarrow$ $\Lambda$CDM model of general relativity challenged by theory & observations.
- Teleparallel gravity:
  - Based on a different (flat) connection - gravity is *not* mediated by curvature.
  - Gravity encoded in *torsion*: *metric* teleparallel gravity.
  - Gravity encoded in *nonmetricity*: *symmetric* teleparallel gravity.
  - Both *torsion & nonmetricity*: *general* teleparallel gravity.

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  - General functions of teleparallel scalar: $f(T)$, $f(Q)$, $f(G)$ gravity.
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  - Scalar fields coupled to teleparallel geometry.

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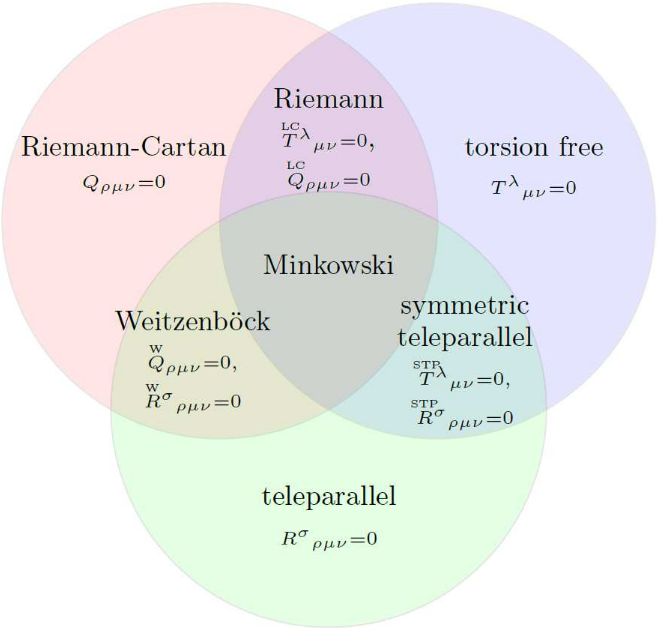
  - Scalar fields coupled to teleparallel geometry.
- Teleparallel cosmology:
  - Make use of cosmological symmetry in order to find most general geometry.
  - Modified Friedmann equations for each class of gravity theories.
  - Use method of dynamical systems to study cosmological dynamics.


Riemann-Cartan
$Q_{\rho\mu\nu}=0$
Riemann
$\overset{\text{LC}}{T}{}^{\lambda}{}_{\mu\nu}=0,$
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torsion free
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Minkowski
Weitzenböck
$\overset{\text{W}}{Q}{}_{\rho\mu\nu}=0,$
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symmetric teleparallel
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teleparallel
$R^{\sigma}{}_{\rho\mu\nu}=0$

# Cosmologically symmetric metric-affine geometry

1. Most general metric with cosmological symmetry:
   - Metric in space-time split:
   $$g_{\mu\nu} = -n_\mu n_\nu + h_{\mu\nu}\,. \tag{1}$$
   - Unit normal covector field:
   $$n_\mu \mathrm{d}x^\mu = -N\,\mathrm{d}t\,. \tag{2}$$
   - Spatial metric (gives projection onto spatial slices):
   $$h_{\mu\nu}\mathrm{d}x^\mu \otimes \mathrm{d}x^\nu = A^2\left[\frac{\mathrm{d}r \otimes \mathrm{d}r}{\chi^2} + r^2(\mathrm{d}\vartheta \otimes \mathrm{d}\vartheta + \sin^2\vartheta \mathrm{d}\varphi \otimes \mathrm{d}\varphi)\right]\,. \tag{3}$$

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   $\Rightarrow$ Metric depends on lapse $N(t)$ and scale factor $A(t)$.
2. Most general affine connection with cosmological symmetry:
   - Connection characterized by cosmologically symmetric torsion and nonmetricity:
   $$T^\mu{}_{\nu\rho} = \frac{2}{A}(\mathcal{T}_1 h^\mu_{[\nu} n_{\rho]} + \mathcal{T}_2 n_\sigma \varepsilon^{\sigma\mu}{}_{\nu\rho})\,, \quad Q_{\rho\mu\nu} = \frac{2}{A}(\mathcal{Q}_1 n_\rho n_\mu n_\nu + 2\mathcal{Q}_2 n_\rho h_{\mu\nu} + 2\mathcal{Q}_3 h_{\rho(\mu} n_{\nu)})\,. \tag{4}$$

   $\Rightarrow$ Connection depends on five free functions $\mathcal{T}_1(t), \mathcal{T}_2(t), \mathcal{Q}_1(t), \mathcal{Q}_2(t), \mathcal{Q}_3(t)$.

# Cosmologically symmetric teleparallel geometry

- Define conformal time derivative and conformal Hubble parameter:

$$f' = \frac{A}{N}\frac{\mathrm{d}f}{\mathrm{d}t}\,, \quad \mathcal{H} = \frac{A'}{A}\,. \tag{5}$$

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- $R^{\mu}{}_{\nu\rho\sigma} = 0$ if and only if:

$$(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2) + (\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2)' = 0\,, \tag{6a}$$
$$\mathcal{T}_2(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2) = 0\,, \tag{6b}$$
$$(\mathcal{Q}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) - (\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3)' = 0\,, \tag{6c}$$
$$\mathcal{T}_2(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) = 0\,, \tag{6d}$$
$$(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2)(\mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3) - \mathcal{T}_2^2 + u^2 = 0\,, \tag{6e}$$
$$\mathcal{T}_2' = 0\,. \tag{6f}$$

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$$\mathcal{T}_2' = 0\,. \tag{6f}$$

$\Rightarrow$ Different branches of solutions for $u = 0$ and $u \neq 0$.

# Solution branches

$$\begin{gathered}\mathcal{T}_2 = u \neq 0 \\ \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 = 0 \\ \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3 = 0\end{gathered}$$

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# Solution branches

$$Q_{\mu\nu\rho} = 0$$

$$\mathcal{T}_2 = u \neq 0 \qquad \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 = 0 \qquad \mathcal{H} - \mathcal{T}_1 + \mathcal{Q}_2 - \mathcal{Q}_3 = 0$$

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$$\swarrow \qquad \searrow$$

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$$\mathcal{T}_2 = 0 \qquad \mathcal{T}_1 - \mathcal{H} = iu \neq 0$$

# Solution branches

$$T^{\mu}{}_{\nu\rho} = 0$$

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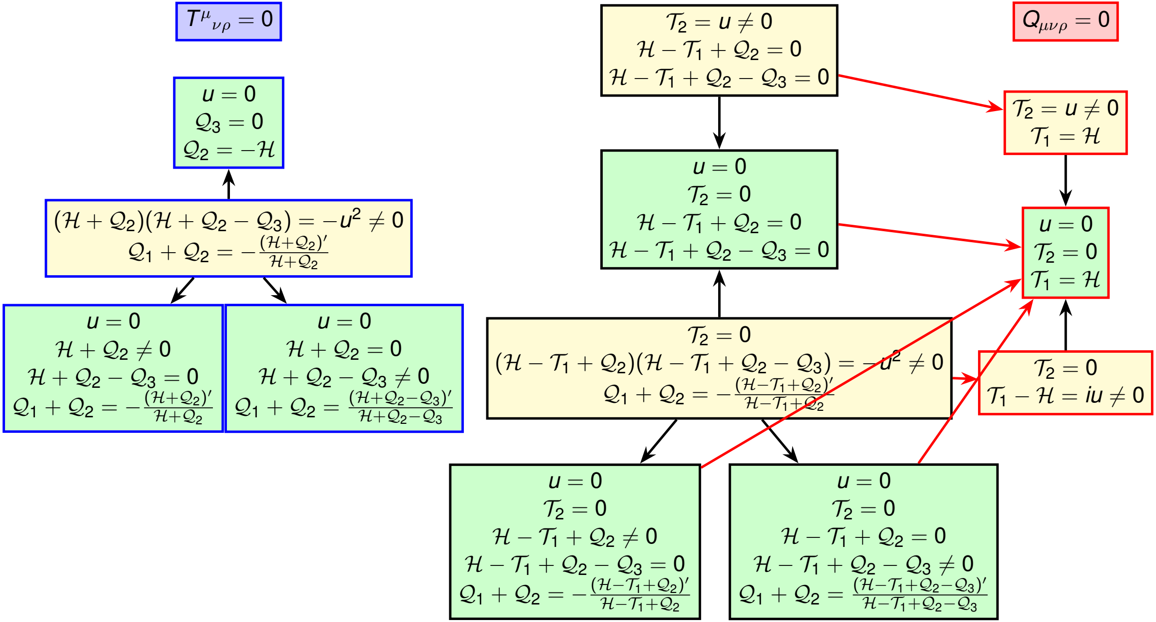

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# Aside: $f(K)$ and scalar-teleparallel theories

- General action of $f(K)$ gravity with $K \in \{T, Q, G\}$:

$$S = -\frac{1}{2\kappa^2} \int_M f(K) \sqrt{-g} \mathrm{d}^4 x \,. \tag{7}$$

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- Integrate out $\phi$ with $-2\kappa^2\mathcal{U}'(\psi) = (f')^{-1}(\psi)$:

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# Example 1: general spatially flat $f(T)$ cosmology

- Ansatz for spatially flat ($k = 0$) cosmology:

$$\theta^a{}_\mu = \operatorname{diag}(1, a(t), a(t), a(t)) \quad \Rightarrow \quad g_{\mu\nu}dx^\mu dx^\nu = -\mathrm{d}t^2 + a^2(t)\delta_{ij}\mathrm{d}x^i\mathrm{d}x^j\,. \tag{12}$$

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$$12H^2 f_T + f = 2\kappa^2\rho\,, \tag{14a}$$

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- Consider mixture of dust and radiation matter:

$$\rho = \rho_m + \rho_r \,, \quad p = p_m + p_r \,, \quad p_m = 0 \,, \quad p_r = \frac{1}{3}\rho_r \,. \tag{15}$$

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- Consider mixture of dust and radiation matter:

$$\rho = \rho_m + \rho_r\,, \quad p = p_m + p_r\,, \quad p_m = 0\,, \quad p_r = \frac{1}{3}\rho_r\,. \tag{15}$$

- Cosmological dynamics as a dynamical system [MH, Järv, Ualikhanova '17]:

$$W(H) = 12H^2 f_T + f\,,\ X = \frac{\rho_r}{\rho_r + \rho_m} \quad \Rightarrow \quad \dot{X} = HX(X - 1)\,,\ \dot{H} = -\frac{(X + 3)H}{(\ln W)_H}\,. \tag{16}$$

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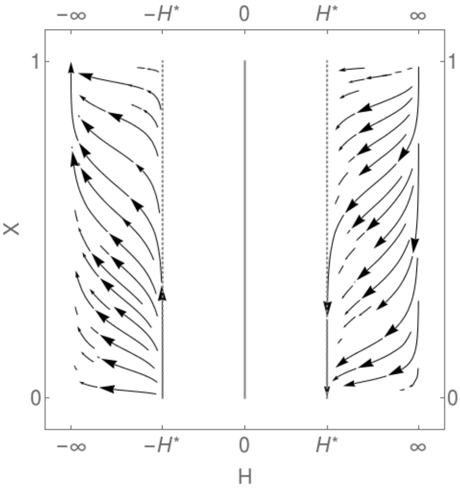
- For $\alpha > 0, \frac{1}{2} < n < 1$ or $\alpha < 0, n < \frac{1}{2}$:
    - Big bang at $H = \infty, X = 1$.
    - Transition from $\ddot{a} < 0$ to $\ddot{a} > 0$.
    - De Sitter attractor at $H = H^\star, X = 0$.
    - Phantom or non-phantom, no crossing.



# Example 2: newer general relativity of type 2 (vacuum)

- Parameters in cosmological field equations - non-degenerate system:

$$a_1 = 1 + 3\epsilon\,, \quad a_2 = 2\epsilon\,, \quad a_3 = \epsilon\,, \quad a_4 = 0\,. \tag{19}$$

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$\Rightarrow$ Dynamics obtained by solving for time derivatives $\dot{H}, \dot{K}, \dot{L}$.

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$$\dot{H} = \epsilon(3H + K)^2\,, \tag{20a}$$

$$\dot{K} = -\frac{1}{2}(3H + K)(9H + K) - 3\epsilon(3H + K)^2 - \frac{3H^2}{2\epsilon}\,. \tag{20b}$$

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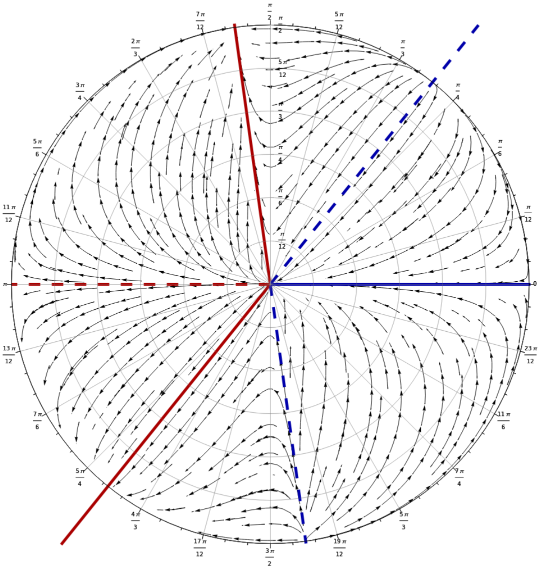
$$\dot{L} = \frac{27}{2}H^2 - 6H(4K + L) + 8K^2 + \frac{1}{2}L^2 + 3\epsilon(3H - 4K - L)^2 + \frac{3H^2}{2\epsilon}\,. \tag{22c}$$

# Example 2: phase diagram of branch 1, $\epsilon = -\frac{1}{6}$

- Use Poincaré disk model for $(H, K)$.



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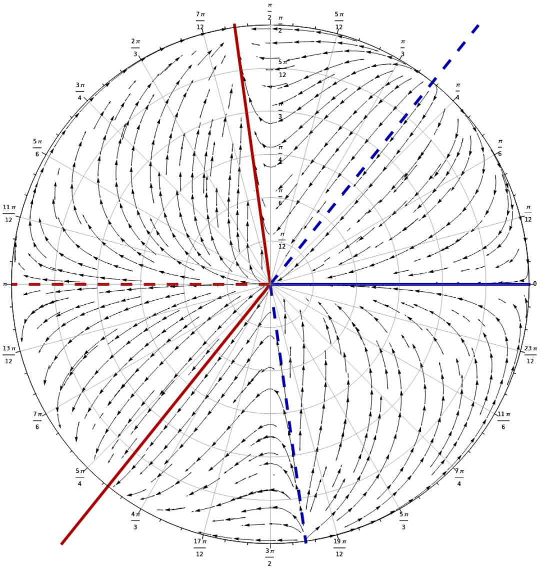

- Use Poincaré disk model for $(H, K)$.
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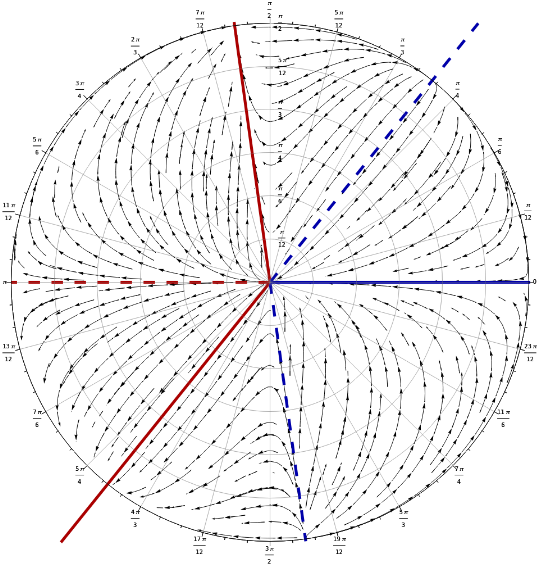

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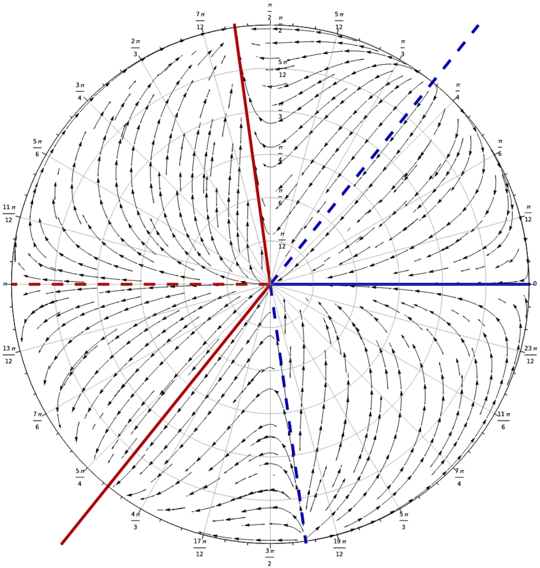

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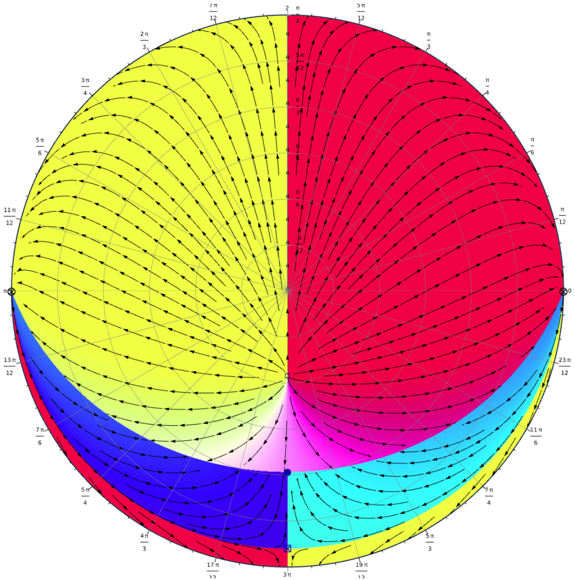

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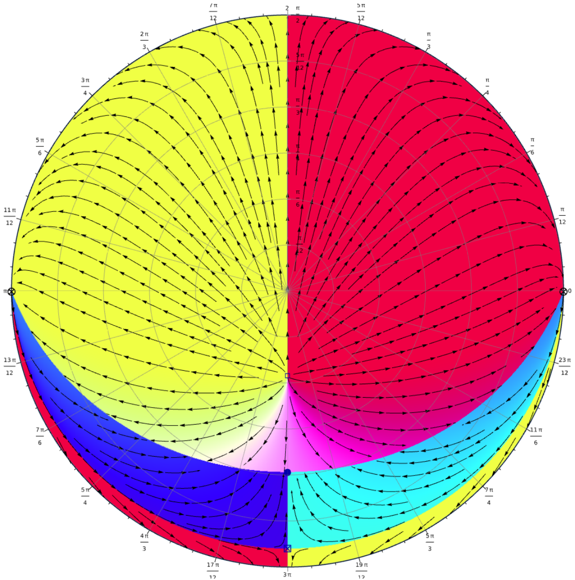

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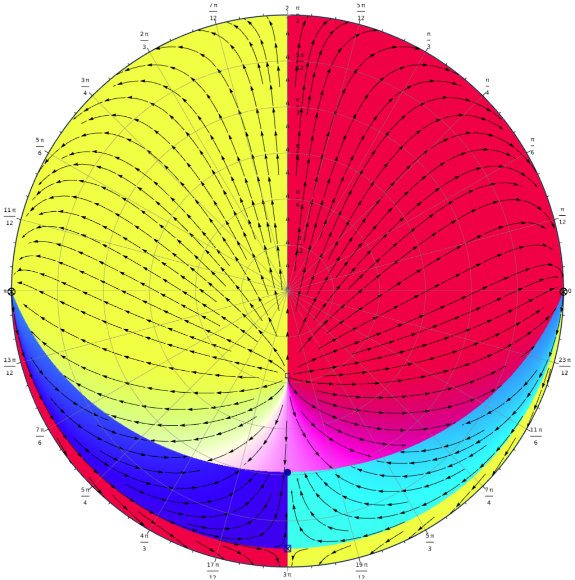

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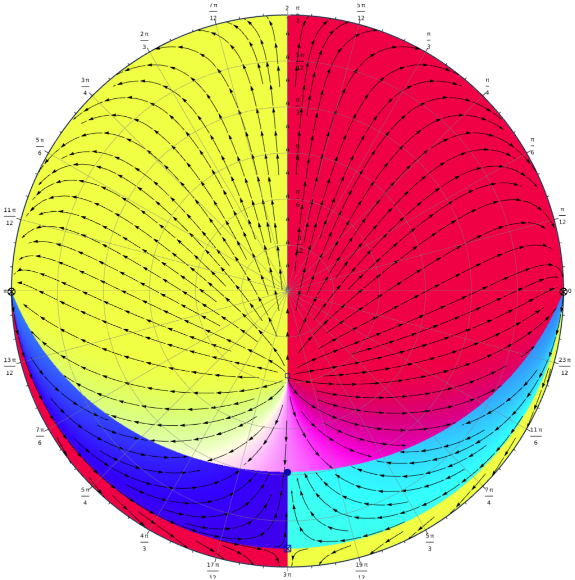

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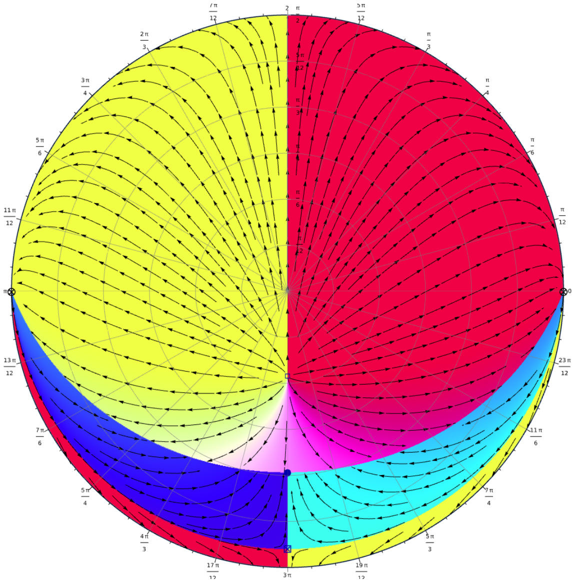

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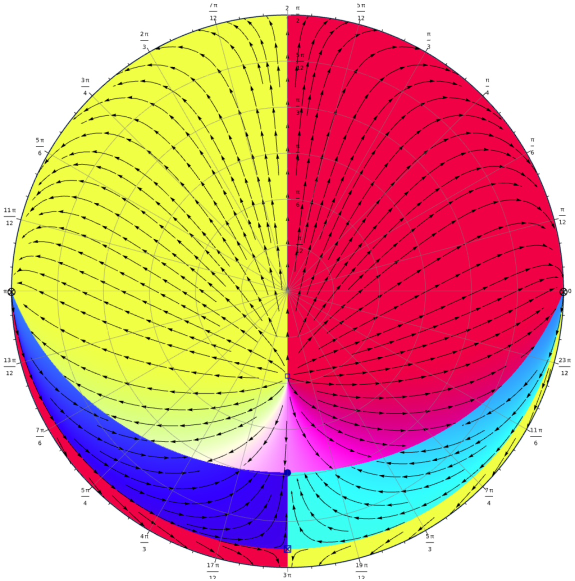

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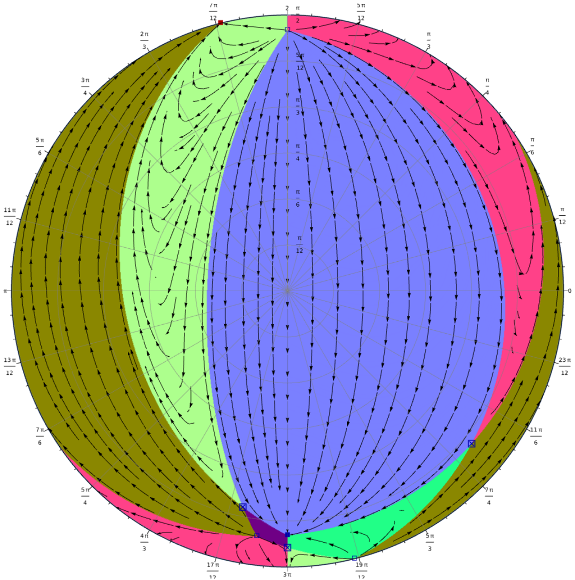

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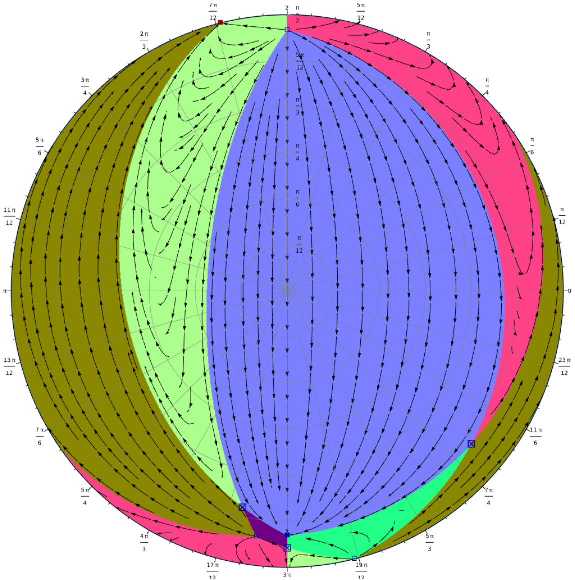

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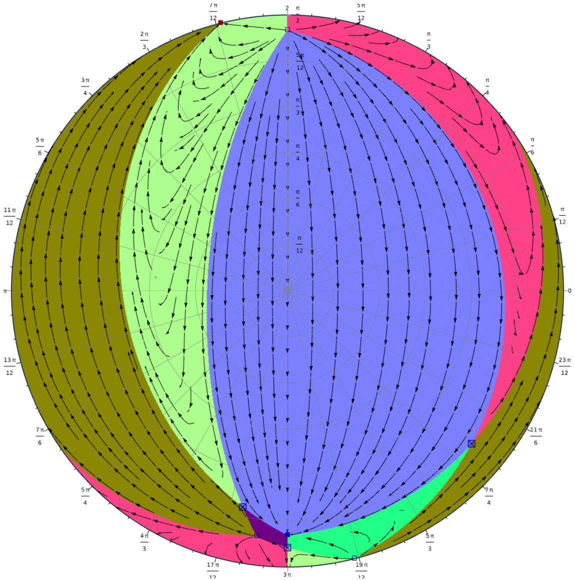

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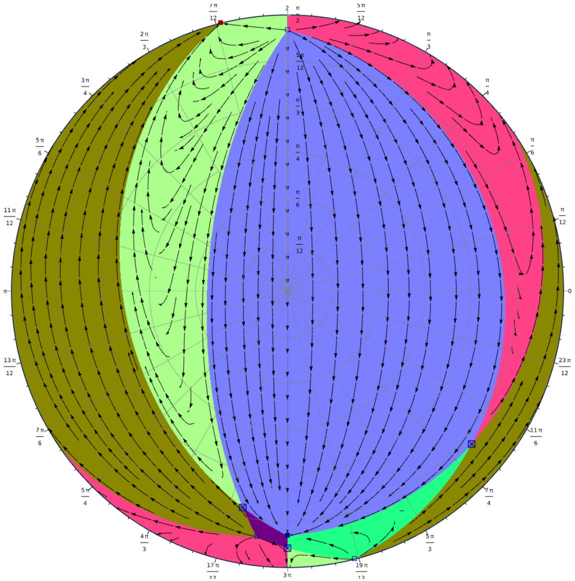

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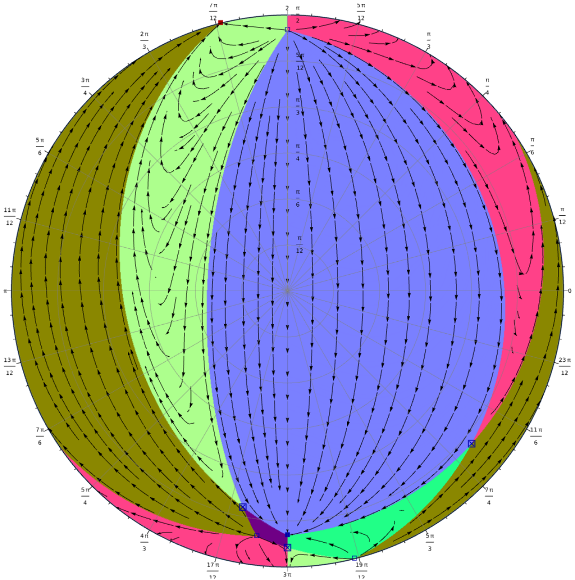

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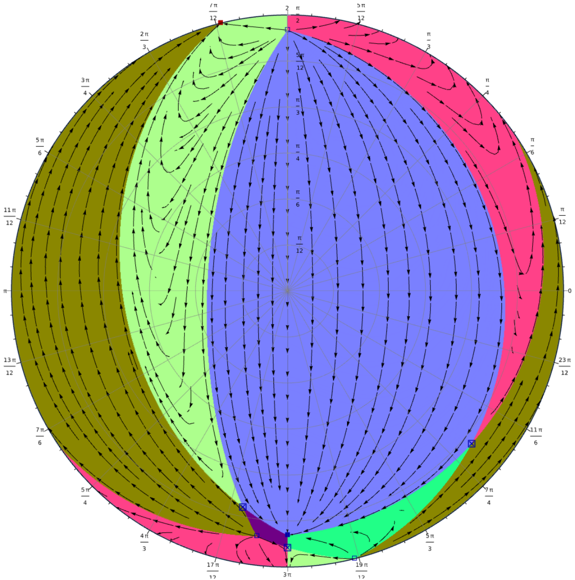

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- Hubble parameter: $\dot{H} = \epsilon(xH + yK + zL)^2$ for all three branches.
  - $\Rightarrow$ For $\epsilon > 0$: $\dot{H} \geq 0$, only bounces are possible, no turnarounds.
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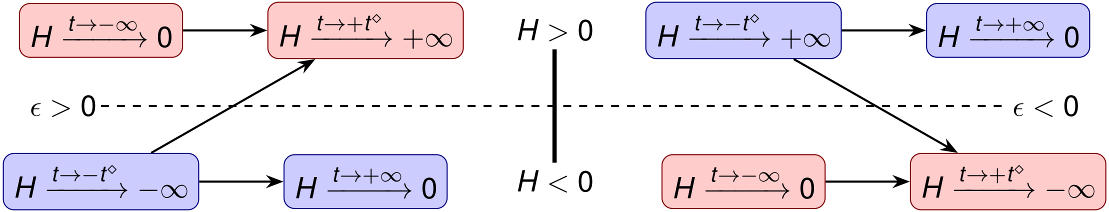

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- Teleparallel gravity:
  - Dynamical field is flat affine connection (vanishing curvature).
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  - Metric takes familiar Robertson-Walker form.
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$\Rightarrow$ Outlook: Complete classification of teleparallel cosmologies.

# References


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