Black hole shadows as diagnostics of nonlinear electrodynamics

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Singularity Problem in GR and Regular Black Holes

The Problem: The Curvature Singularity

- General Relativity (GR) breaks down at the center of black holes ($r = 0$).
- Density, curvature, and gravitational field strength become **infinite**.
- This signals a fundamental failure of classical GR and cannot be resolved within the standard framework.

The Workaround: Regular Black Holes

- **Definition:** Spacetime solutions where the singularity is replaced by a smooth, de Sitter-like core.
- **Mechanism:** Couple GR with Nonlinear Electrodynamics (NED).
- **Effect:** The black hole's charge "smears" out the central mass, preventing collapse to a point.
- **Origin:** First proposed by J. Bardeen in 1968.

GR+NED

The action of GR+NED

$$S = \frac{1}{16\pi}\int d^4x\sqrt{-g}\,(R+\mathcal{L})\ .$$

The equations of motion:

$$\underbrace{G_{\mu\nu} = T_{\mu\nu}}_{\text{Einstein equations}}\ , \qquad \underbrace{\nabla_\nu\left(\mathcal{L}_F F^{\mu\nu}\right) = 0}_{\text{Maxwell equations}}$$

The energy-momentum tensor of the electromagnetic field

$$T_{\mu\nu} = 2\left(\mathcal{L}_F F^{\alpha}_{\ \mu} F_{\nu\alpha} - \frac{1}{4} g_{\mu\nu}\mathcal{L}\right).$$

The Unified Metric Framework

Spacetime metric: [Fan and Wang, PRD 2016]

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2(d\theta^2 + \sin^2\theta d\phi^2)$$

Lagrangian density:

$$\mathcal{L} = \frac{4\mu}{\alpha}\frac{(\alpha F)^{\frac{\nu+3}{4}}}{\left[1+(\alpha F)^{\frac{\nu}{4}}\right]^{1+\frac{\mu}{\nu}}}$$

Metric function: $f(r) = 1 - \frac{2Mr^{\mu-1}}{(r^\nu+q^\nu)^{\mu/\nu}}$

Maxwellian ($\nu = 1$): $f(r) = 1 - \frac{2Mr^{\mu-1}}{(r+q)^\mu}$

Bardeen ($\nu = 2$): $f(r) = 1 - \frac{2Mr^{\mu-1}}{(r^2+q^2)^{\mu/2}}$

Hayward ($\nu = 3$): $f(r) = 1 - \frac{2Mr^{\mu-1}}{(r^3+q^3)^{\mu/3}}$

Note:

$\mu \geq 3$ ensures that the spacetime is regular everywhere.

Weak Field Regime Behavior

Maxwellian ($\nu = 1$):

$$f(r) = 1 - \frac{2M}{r} + \frac{2\mu M q}{r^2} + O\left(\frac{1}{r^3}\right),$$
$$\mathcal{L} = 4\mu F + O(F^{5/4})$$

Bardeen ($\nu = 2$):

$$f(r) = 1 - \frac{2M}{r} + \frac{\mu M q^2}{r^3} + O\left(\frac{1}{r^4}\right),$$
$$\mathcal{L} = 4\sqrt[4]{\alpha}\mu F^{5/4} + O(F^{7/4})$$

Hayward ($\nu = 3$):

$$f(r) = 1 - \frac{2M}{r} + \frac{2\mu M q^3}{3r^4} + O\left(\frac{1}{r^5}\right),$$
$$\mathcal{L} = 4\sqrt{\alpha}\mu F^{3/2} + O(F^{5/4})$$

Validating the Lagrangian

The Weak-Field Requirement

Any valid NED theory must reduce to **Maxwell's electrodynamics** in the weak-field limit (far from the black hole).

Maxwellian: Correct weak-field limit

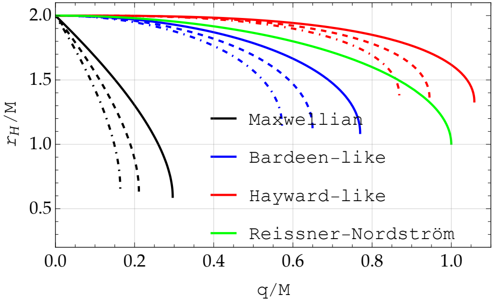
Bardeen: Pathological

Hayward: Pathological

- **The Pathology: Bardeen** and **Hayward** regular black holes fail this test.
  - Their Lagrangians do *not* reduce to Maxwell's theory.
  - Their metrics do *not* approach the Reissner-Nordström solution asymptotically.

Event Horizons: Black Holes vs. No-Horizon Objects

- Regular black holes possess two horizons (inner and outer).
- As charge (q) increases, the outer horizon shrinks.
- At the **extremal charge (q_{ext})**, the horizons merge.
- Beyond q_{ext}, the spacetime describes a **horizonless (no-horizon)** object.



solid curves: $\mu = 3$
dashed curves: $\mu = 4$
dotdashed curves: $\mu = 5$

Key Observation: Hayward-like black holes admit the largest charge range, while Maxwellian have the narrowest.

Photon Motion: Null Geodesics vs. Effective Geometry

- **In standard GR**: light follows the null geodesics of the background metric.
- **In NED:** This is **false**. In NED, light follows the null geodesics of an **effective metric** [M. Novello et al. PRD 2000.]

$$g^{\mu\nu}_{\text{eff}} = \mathcal{L}_F g^{\mu\nu} - \mathcal{L}_{FF} F^{\mu}_{\lambda} F^{\lambda\nu}$$

- The effective metric for magnetically charged black holes is written as:

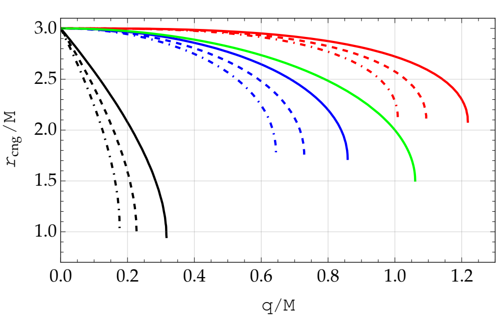
$$ds^2_{\text{eff}} = -\frac{f(r)}{\mathcal{L}_F} dt^2 + \frac{dr^2}{\mathcal{L}_F f(r)} + \frac{r^2}{\Phi} d\Omega^2$$

(where $\Phi = \mathcal{L}_F + 2F\mathcal{L}_{FF}$)

Consequence:

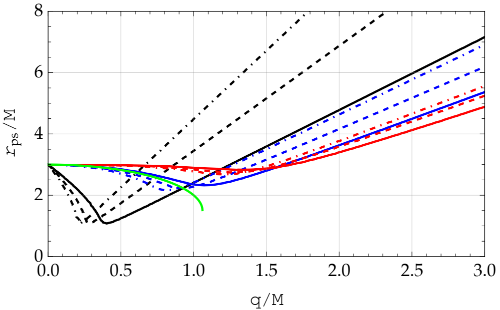
The geometry seen by photons is different from the geometry seen by massive particles.

A Surprising Result: CNG vs. Photonsphere



Circular Null Geodesics (CNG)

- Vanish completely at a critical charge value in the no-horizon regime.



Effective Photonsphere (PS)

- Never vanish. Persists for all charge values, even after the event horizon disappears.

Key Finding:

The true effective photonsphere is fundamentally more robust than standard null geodesics.

The Shadow Radius

- **Shadow** - the dark silhouette cast by the black hole against a bright background.
- The shadow radius (R_{sh}) is determined by the **radius of the effective photonsphere**.
- **Shadow radius:**

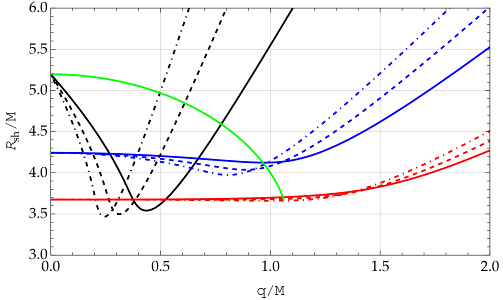
$$R_{\mathrm{sh}} = \sqrt{\frac{2r^2(2m' - rm'')}{(r-2m)(rm^{(3)} - m'')}}\Bigg|_{r=r_{\mathrm{ps}}}$$

- **Main takeaway:** A correct shadow requires a correct effective metric.

The Diagnostic "Smoking Gun": Shadows

The Physical Test

As the black hole's charge $q \to 0$, the spacetime must revert to a Schwarzschild black hole.
Schwarzschild Shadow radius: $R_{\text{Schw}}/M = 3\sqrt{3} \approx 5.196$.



Our Calculated Shadows at $q \to 0$:

- **Maxwellian / RN:** Pass.
  Matches Schwarzschild perfectly ($3\sqrt{3}$).
- **Bardeen-like:** Fail.
  Approaches $R_{\text{Schw}} \approx 4.2426M$.
- **Hayward-like:** Fail.
  Approaches $R_{\text{Schw}} \approx 3.6742M$.

Conclusion:

The Bardeen and Hayward shadows are fundamentally mathematically incorrect.

Evolution of the Shadow Radius

- **Black Hole Regime:** The shadow shrinks as the charge increases, making the black hole appear smaller.
- **No-Horizon Regime:** The shadow continues to shrink, reaches a minimum, then grows monotonically.
- **Minimum Values (Table 1):**

Model ($\mu = 3$)	Min. Shadow Radius (R_{sh}^{min}/M)
Maxwellian	3.5836
Bardeen	4.1420
Hayward	3.7617

Quasinormal Modes in High Energy Regime

- **QNMs:** The characteristic frequencies at which a perturbed black hole rings down (like a struck bell).
- **The Eikonal Limit (High ℓ):** In this high-frequency limit, QNM frequencies can be calculated using classical geodesic properties [V. Cardoso et al. PRD 2009].
- **Formula:**

$$\omega = \Omega_{\text{orbit}}\ell - i|\lambda_{\text{orbit}}|\left(n + \frac{1}{2}\right)$$

Which Orbits Govern Which Perturbations?

- **For Scalar and Gravitational Perturbations:** The QNMs are governed by the **Circular Null Geodesics (CNG)**:

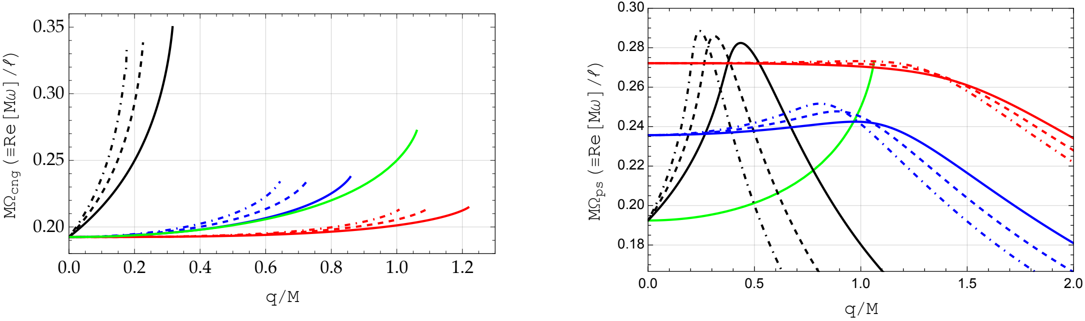
$$\omega = \Omega_{\mathrm{CNG}}\ell - i|\lambda_{\mathrm{CNG}}|\left(n + \frac{1}{2}\right)$$

- **For Electromagnetic Perturbations:** Because light does not follow the CNG in NED, the QNMs are governed by the **Effective Photonsphere**:

$$\omega = \Omega_{\mathrm{ps}}\ell - i|\lambda_{\mathrm{ps}}|\left(n + \frac{1}{2}\right)$$

For details, see B. Toshmatov et al., PRD 2018, 2019.

QNM Results Confirming the Pathology



- **CNG (Ω_{CNG}):** For all families, Ω_{CNG} correctly reduces to Schwarzschild at $q = 0$.
- **Photonsphere (Ω_{ps}):** This is where pathology appears:
  - **Maxwellian:** $\Omega_{\text{ps}}(q \to 0) = \Omega_{\text{Schw}}$ $\to$ Passes.
  - **Bardeen/Hayward:** $\Omega_{\text{ps}}(q \to 0) \neq \Omega_{\text{Schw}}$ $\to$ Fails.

Conclusion: The electromagnetic ringing of Bardeen/Hayward black holes is physically incorrect from the start.

The Shadow and Photonsphere Angular Velocity

- There is a mathematically beautiful inverse relationship between the shadow radius and the angular velocity of the photonsphere (Ω_{ps}):

$$R_{\mathrm{sh}} = \frac{1}{\Omega_{\mathrm{ps}}}$$

- **Implication:** A wrong shadow radius directly implies a wrong angular velocity of light at the photonsphere.

The Final Unified Picture

- We can rewrite the electromagnetic QNM frequency purely in terms of the shadow radius:

$$\omega = \frac{\ell}{R_{\rm sh}} - i|\lambda_{\rm ps}|\left(n + \frac{1}{2}\right)$$

- **Explanation:** The real part of the ringing frequency (the oscillation) is simply the multipole number divided by the shadow radius.
- **The Takeaway:** If you measure the shadow radius, you directly predict the electromagnetic ringing frequency. Since Bardeen/Hayward give the wrong shadow, they will also give the wrong ringing.

Recap: How Photons Behave Around Regular Black Holes

1. **Path:** Photons in NED follow an **effective metric**, not the background spacetime.
2. **Persistent Ring:** The true effective Photonsphere **never disappears**, even when the background null geodesics and event horizons vanish.
3. **The Diagnostic:** If the NED model is physically invalid (Bardeen/Hayward), the **shadow radius** and **electromagnetic QNMs** will fail the $q \to 0$ Schwarzschild limit.
4. **Unified Tool:** $R_{\rm sh}$ and $\Omega_{\rm ps}$ are inversely related, making shadow measurements direct probes of the black hole's ringing frequency.

Conclusions & Astrophysical Implications

- We have shown that shadows and eikonal QNMs are powerful tools to test the physical viability of NED.
- Only the Maxwellian family correctly recovers standard physics in the weak-field limit ($q \to 0$).
- **Key Takeaway:** The Bardeen and Hayward regular black hole models contain a hidden Lagrangian flaw that is exposed by the light that travels around them.

Grazie per l'attenzione!