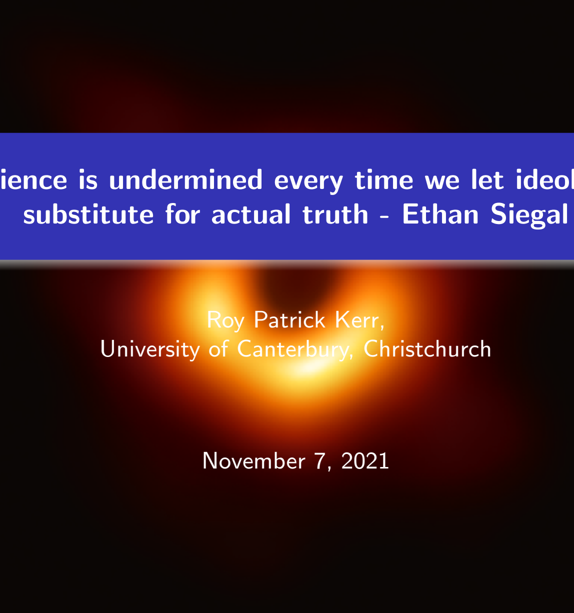
Science is undermined every time we let ideology substitute for actual truth - Ethan Siegal



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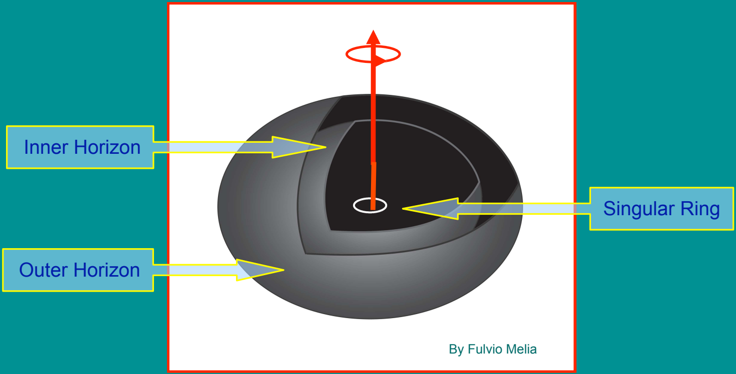
Event Horizons for Kerr Black Hole



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Boyer-Lindquist (Achilles Papapetrou):

$$ds^2 = -\frac{\Delta}{\rho^2}\left(dt - a\sin^2\theta\, d\phi\right)^2 + \frac{\sin^2\theta}{\rho^2}\left(\left(r^2 + a^2\right)\, d\phi - a\, dt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2\, d\theta^2$$

$$\text{where} \quad \Delta = r^2 - 2mr + a^2, \quad \rho^2 = r^2 + a^2\cos^2\theta$$

Boyer-Lindquist $\longrightarrow$ **Kerr-Schild** *Ingoing & Outgoing*:

$$dt \to dt \mp \frac{2mr}{\Delta}dr, \quad d\phi \to d\phi \mp \frac{a}{\Delta}dr.$$

Kerr-Schild *Ingoing & Outgoing*:

$$ds^2 = -dt^2 + dr^2 \mp 2a\sin^2\theta\; dr d\phi + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta\; d\phi^2 + \frac{2mr}{\rho^2}(\pm dt \mp a\sin^2\theta\; d\phi - dr)^2$$

Mathematical Representations: Kerr-Schild

Kerr-Schild *Ingoing & Outgoing*:

$$ds^2 = -dt^2 + dr^2 \mp 2a \sin^2\theta \; dr d\phi + \rho^2 d\theta^2 + (r^2 + a^2)\sin^2\theta \; d\phi^2 + \frac{2mr}{\rho^2}(\pm dt \mp a \sin^2\theta \; d\phi - dr)^2$$

Kerr-Schild (*Cartesian*) *Ingoing & Outgoing*:

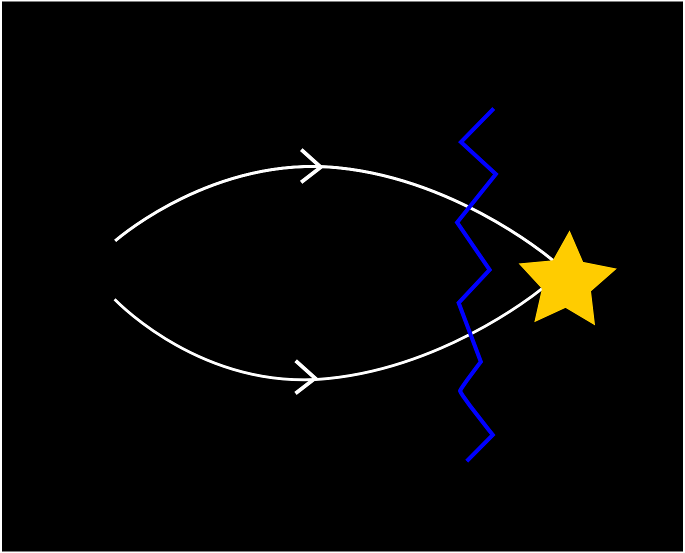
$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 + \frac{2mr^3}{r^4 + a^2 z^2}\,(k_\mu dx^\mu)^2$$

$$\text{where:} \quad k_\mu = \left(\pm 1, \frac{rx \pm ay}{r^2 + a^2}, \frac{ry \mp ax}{r^2 + a^2}, \frac{z}{r}\right), \quad |\underset{\sim}{k}|^2 = 0$$

$$\text{where:} \quad k^\mu = \left(\mp 1, \frac{rx \pm ay}{r^2 + a^2}, \frac{ry \mp ax}{r^2 + a^2}, \frac{z}{r}\right), \quad |\underset{\sim}{k}|^2 = 0$$

The surfaces of constant r are confocal ellipsoids of revolution,

$$\frac{x^2 + y^2}{r^2 + a^2} + \frac{z^2}{r^2} = 1.$$



Geodesics on the axis of the Kerr geometry

There are two light-like geodesics along the axis of rotation of Kerr in incoming K-S coordinates. These are the two principal null vectors (PNV). This is the one situation where it is easy to identify the other PNV. Since $x = 0, y = 0, z = r$, the equations for the two geodesics are

$$ds^2 = -dt^2 + dr^2 + \frac{2mr}{r^2 + a^2}(dr + dt)^2 = 0. \tag{1}$$

and so $\frac{dr}{dt} = -1$, the incoming geodesic, or

$$\frac{dr}{dt} = \frac{r^2 - 2mr + a^2}{r^2 + 2mr + a^2}$$

Since this is negative between the two horizons, both PNVs are pointing inwards in this region. The "fast" null geodesics continue straight through the inner horizon but the "slow" one trying to escape is compelled to move inwards also. Its affine length is $2\sqrt{m^2 - a^2}$, if r is used as affine parameter.

The simple, geometric proof that these geodesics are trapped between the horizons is that there can only be two double PNVs at a point.at each point of the horizons there are already two such vectors, the infalling one and the generator of the event horizon (a PNV!). There cannot be a third.

Geodesics on the axis of the Kerr geometry

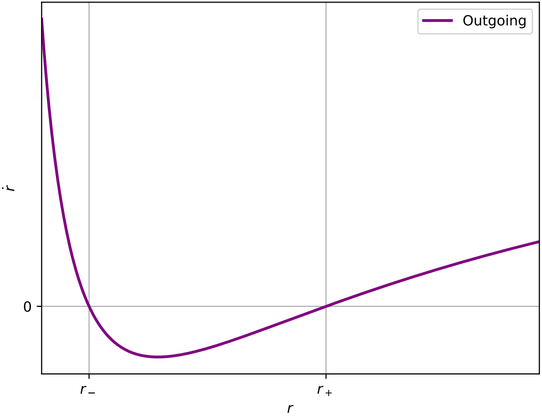


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Figure: $\dot{r}$ against r for ‘outgoing’ null geodesics on the axis of the Kerr geometry. r_- and r_+ represent the inner and outer event horizons respectively. The fixed points at these horizons mean that outgoing geodesics between the two horizons must stay between the two horizons.

The EIH Equations

The 1938 paper by Einstein, Infeld and Hoffman was one of the most important papers in the history of Relativity. The metric was expanded as a series using an indeterminate parameter λ. This may be taken as c, the velocity of light, but it is simpler to use $\lambda = 1$.

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \sum_{l} \underset{l}{h}^{\mu\nu}$$

The key assumption was that the bodies are far enough apart so that the motion is slow and therefore that for any metric function, $\underset{l}{F}$, of weight l, its time derivative $\underset{l}{F}_{,0}$ has weight $l+1$.

The expanded metric was substituted into the Einstein field equations and terms of the same order collected together,

$$\sqrt{-g}\,G^{\mu\nu} = \sum_{l}(\underset{l}{\Phi}^{\mu\nu} - \underset{l}{\Lambda}^{\mu\nu})$$

where $\underset{l}{\Phi}^{\mu\nu}$ is linear in second spatial derivatives of $\underset{l}{h}^{\mu\nu}$. From its form

$$\underset{l}{\mathfrak{C}}^{\mu} = \frac{1}{2\pi}\int \underset{l}{\Phi}^{\mu n} dS_n = 0$$

$$\underset{l}{\mathfrak{U}}_m = \varepsilon_{mst}\frac{1}{4\pi}\int x^s \underset{l}{\Phi}^{tn} dS_n = 0$$

General approach to Approximating 2-body problem

1. Set up a recursive system. Assume two bodies each with mass $M_a(t)$, spin vector $S_a^n(t)$ centred at $p_a^n(t)$, where $t = x^0$ is the fourth coordinate and $a\varepsilon\{1,2\}$.
2. Assume that metric can be written as a series in an indeterminate parameter, λ,

$$\mathfrak{g}^{\mu\nu} = \sqrt{-g}g^{\mu\nu} = \eta^{\mu\nu} + \sum_l \lambda^i \underset{l}{h}^{\mu\nu}$$

3. The expanded metric was substituted into the Einstein field equations and terms of the same order collected together,

$$\sqrt{-g}G^{\mu\nu} = \sum_l \lambda^i (\underset{l}{\Phi}^{\mu\nu} - \underset{l}{\Lambda}^{\mu\nu}) = 0,$$

where $\underset{l}{\Phi}^{\mu\nu}$ is linear in second derivatives of $\underset{l}{h}^{\mu\nu} = 0$.

4. The problem is that one cannot just equate the coefficients of λ^i to zero. This is what stymied Einstein and Infeld for 20 years.
5. . At the same time as one expands the metric in powers of λ one has to also expand the equations of motion:

$$M_{,0} = \sum_l \lambda^l \underset{l}{E}, \qquad Mx^i_{,00} = \sum_l \lambda^l \underset{l}{F}^i, \qquad S^i_{,0} = \sum_l \lambda^l \underset{l}{C}^i.$$

Coordinate Conditions

If the approximation equations are taken to be $\underset{l}{\Phi}^{\mu\nu} - \underset{l}{\Lambda}^{\mu\nu}$ for all l, then the following surface integrals must be zero for all orders, r,

$$\underset{r}{\mathfrak{C}}^{\mu} = \frac{1}{2\pi}\int \underset{r}{\Lambda}^{\mu n} dS_n = 0, \qquad \underset{r}{\mathfrak{U}}_m = \varepsilon_{mst}\frac{1}{4\pi}\int x^s \underset{r}{\Lambda}^{tn} dS_n = 0.$$

By choosing the coordinates so that

$$\underset{l}{h}^{\mu n}, n = 0,$$

the field equations became

$$\underset{l}{h}^{\mu\nu}, nn = 2\underset{l}{\Lambda}^{\mu\nu}).$$

The Bianchi Identities were used to prove that for any solution of the second equations $\underset{l}{h}^{\mu n}, n$ is a harmonic function, so the fundamental problem is whether we can satisfy the coordinate conditions by adding a harmonic function to $\underset{l}{h}^{\mu\nu}$.

All the higher poles can be eliminated but not the simple poles in the coordinate condition (corresponding to the fourEIH surface integral conditions) and the skew-symmetric part of the dipole terms. The former lead to the equations of energy and linear momentum, the latter to the equations for angular momentum.

Kerr-Schild Approximation: 2-body problem

1. Both the slow(EIH = Einstein, Infeld and Hoffman) and Minkowski approximation methods start with a poor approximation to spinning Black Holes. The main reason for this is that it was believed that one should use a consistent choice for the coordinate conditions. Since using the radiation gauge condition, $\mathfrak{g}^{\mu\nu}{}_{,\nu} = 0$, reduces the linear operator $\underset{l}{\Phi}^{\mu\nu}$ to a simple $\nabla^2\mathfrak{g}^{\mu\nu}$ (Einstein, Infeld Hoffman) or $\Box^2\mathfrak{g}^{\mu\nu}$ (Minkowski approximation) this was the obvious condition to use.
2. However, as I proved in 1958, one can change these conditions from step to step, choosing whatever is convenient at the time.
3. In particular, the best starting point for Black Holes is the outgoing Kerr-Schild form since this is already an exact linear solution of the Einstein equations.
4. Since we have to allow the particle parameters to vary if we are to solve for the higher order terms, we replace $M, S^i etc.$, by functions that are constant on the light cones centered at the individual Black Holes. This is similar to what is done in the Minkowski approximation scheme.
5. We are already fairly sure of what the equations of motion will be in the second approximation. Each Black Hole will move like a test-particle in the field of the other, giving the same results as Papapetrou [1951] and Kerr [1959].

Conclusions

The work by Jose Rodriguez, Jorge Rueda *et.al showed that using test body motion near a Black Hole gives templates that are very close to those derived by numerical relativity. These require far less effort to construct. Furthermore, it may even be that when combined with the EOB (equivalent one-body) approach they give better results, i.e. when applied to a Ligo observation the residual correlation between the two instruments is less.*

Kerr-Schild approximation: The big advantage of this is that Ligo is looking straight down the outgoing geodesics from the two Black Holes. The perturbation that Ligo is measuring comes from the flickering motion as the pair circle each other millions of light years away. The method is designed to catch this.

Singularities: Despite the nearly 60 years that has elapsed since the Kerr metric was discovered there is still no reason to believe that rotating Black Holes contain singularities inside. Raychaudhuri's equation has proved ineffective in proving their existence.