Angular Momentum to a Distant Observer

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First observation of a binary black hole merger

Observation of Gravitational Waves from a Binary Black Hole Merger

B. P. Abbott *et al.*[*]
(LIGO Scientific Collaboration and Virgo Collaboration)
(Received 21 January 2016; published 11 February 2016)

On September 14, 2015 at 09:50:45 UTC the two detectors of the Laser Interferometer Gravitational-Wave Observatory simultaneously observed a transient gravitational-wave signal. The signal sweeps upwards in frequency from 35 to 250 Hz with a peak gravitational-wave strain of 1.0×10^{-21}. It matches the waveform predicted by general relativity for the inspiral and merger of a pair of black holes and the ringdown of the resulting single black hole. The signal was observed with a matched-filter signal-to-noise ratio of 24 and a false alarm rate estimated to be less than 1 event per 203 000 years, equivalent to a significance greater than 5.1σ. The source lies at a luminosity distance of 410^{+160}_{-180} Mpc corresponding to a redshift $z = 0.09^{+0.03}_{-0.04}$. In the source frame, the initial black hole masses are $36^{+5}_{-4} M_\odot$ and $29^{+4}_{-4} M_\odot$, and the final black hole mass is $62^{+4}_{-4} M_\odot$, with $3.0^{+0.5}_{-0.5} M_\odot c^2$ radiated in gravitational waves. All uncertainties define 90% credible intervals. These observations demonstrate the existence of binary stellar-mass black hole systems. This is the first direct detection of gravitational waves and the first observation of a binary black hole merger.

Measuring the angular momentum carried away by the gravitational waves is more challenging:

- Q: What is the angular momentum of the system, before and after?
- Open problem due to "supertranslation ambiguity" since the 1960's.
- The angular momentum recorded by two distant observers of the same system may not be the same.
- Joint work with Po-Ning Chen, Jordan Keller, Mu-Tao Wang, and Ye-Kai Wang.

Reference

1. (P.-N. Chen, M.-T. Wang, Y.-K. Wang, and S.-T. Yau) "BMS charges without supertranslation ambiguity", arXiv: 2107.05316
2. (P.-N. Chen, M.-T. Wang, Y.-K. Wang, and S.-T. Yau) "Supertranslation invariance of angular momentum", to appear in Adv. Theor. Math. Phys. arXiv: 2102.03235
3. (P.-N. Chen, J. Keller, M.-T. Wang, Y.-K. Wang, and S.-T. Yau) "Evolution of angular momentum and center of mass at null infinity", to appear in Comm. Math. Phys., arXiv: 2102.03221

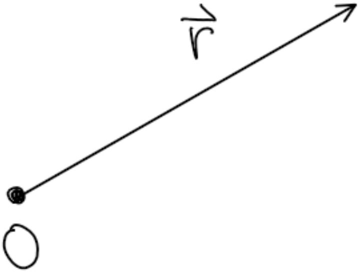
Outline

1. Angular momentum in classical mechanics and special relativity
2. Distant observer in the Bondi-Sachs formalism in general relativity
3. Supertranslation ambiguity
4. Chen-Wang-Yau angular momentum

Angular momentum

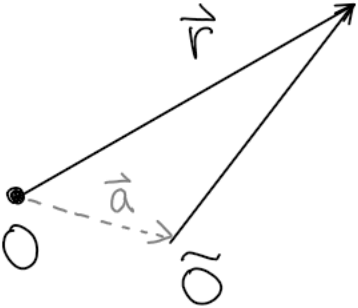
In classical mechanics, the angular momentum of a particle is defined as

$$J = m\mathbf{r} \times \mathbf{r}'.$$



If we translate the origin (or the coordinate system), the angular momentum gets shifted by linear momentum

$$J_{\tilde{O}} = m(\mathbf{r} - \mathbf{a}) \times \mathbf{r}' = J_O - \mathbf{a} \times \mathbf{p}.$$



The special relativity can be formulated in terms of the geometry of the Minkowski spacetime $\mathbb{R}^{3,1}$. The continuous symmetry of the $\mathbb{R}^{3,1}$ consists of

$$\frac{\partial}{\partial t}, \quad \frac{\partial}{\partial x^i}, \quad x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}, \quad t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t},$$

which generates the time translation, spatial translation, rotation, and Lorentz transformation respectively.

Noether's Theorem states that each continuous symmetry corresponds to a conserved quantity. For a particle γ in $\mathbb{R}^{3,1}$, we have

energy	$e = \langle \gamma', \frac{\partial}{\partial t} \rangle$
linear momentum	$p_i = \langle \gamma', \frac{\partial}{\partial x^i} \rangle$
angular momentum	$J_{ij} = \langle \gamma', x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \rangle$
center of mass	$C_i = \langle \gamma', t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t} \rangle$

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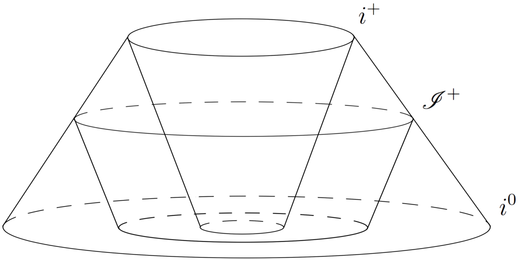
$$\begin{array}{ll} \text{energy} & e = \langle \gamma', \frac{\partial}{\partial t} \rangle \\ \text{linear momentum} & p_i = \langle \gamma', \frac{\partial}{\partial x^i} \rangle \\ \text{angular momentum} & J_{ij} = \langle \gamma', x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i} \rangle \\ \text{center of mass} & C_i = \langle \gamma', t \frac{\partial}{\partial x^i} + x^i \frac{\partial}{\partial t} \rangle \end{array}$$

If we translate the coordinate system $t \mapsto t + \alpha^0, x^i \mapsto x^i + \alpha^i$, the energy and linear momentum are the same but the angular momentum and center of mass get shifted

$$J'_{ij} = J_{ij} + \alpha_i p_j - \alpha_j p_i$$
$$C'_i = C_i + \alpha^0 p_i + \alpha^i e$$

Bondi-Sachs formalism

- An idealized distant observer is situated at future null infinity $\mathscr{I}^+$, where light rays approach along null geodesics.
- Descriptions of $\mathscr{I}^+$ of an isolated gravitating system include:
    - Bondi-Sachs coordinates (Bondi et al. 1962, Sachs 1962)
    - Penrose conformal compactification (Penrose 1965)
    - Christodoulou-Klainerman (1993)

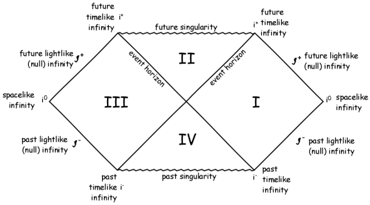


- Bondi-Sachs and Penrose formalisms are essentially equivalent; both of them imply the spacetimes have “peeling”–‘an idealized property that may not hold in general, as demonstrated in the work of Christodoulou-Klainerman (see also L. Bieri)
- We describe null infinity and distant observers using the Bondi-Sachs formalism. Our result can be extended to “polyhomogeneity” (Chrusciel-MacCallum-Singleton, 1995) $\mathscr{I}^+$ and the Christodoulou-Klainerman setting.

Recall the Schwarzschild spacetime in Eddington-Finkelstein coordinates:

$$-(1-\frac{2m}{r})du^2 - 2dudr + r^2\sigma_{AB}dx^A dx^B,$$

where $\sigma_{AB}dx^A dx^B = d\theta^2 + \sin^2\theta d\phi^2$.



- u is the "retarded time" and r is the luminosity distance from the black hole.
- $\mathscr{I}^+$ corresponds to $r = \infty$.
- $u = +\infty$ corresponds to future timelike infinity i^+ and $u = -\infty$ corresponds to spacelike infinity i^0.

- Bondi and his collaborators postulate a coordinate system (Bondi-Sachs) in which the metric tensor of the spacetime is given by

$$-UVdu^2 - 2Ududr + r^2 h_{AB}(dx^A + W^A du)(dx^B + W^B du).$$

- The spacetime is assumed asymptotically flat in the sense that $U, V \to 1$, $h_{AB} \to \sigma_{AB}$, $W^A \to 0$ as $r \to \infty$ and $\det h_{AB} = \det \sigma_{AB}$ (determinant condition) for r large.
- Moreover, outgoing radiation condition is imposed so that all metric coefficients U, V, h_{AB}, W^A can be expanded into power series of $\frac{1}{r}$. The assumption is relaxed to "polyhomogeneity" by Chrusciel-MacCallum-Singleton

- Bondi et al. found the the physics of gravitational field are encoded in the coefficients of power series expansion.
- Energy and linear momentum are well-understood in the Bondi-Sachs formalism. The expansion

$$V = 1 - \frac{2m(u,x)}{r} + O(r^{-2})$$

gives the *mass aspect* $m(u,x)$ on $\mathscr{I}^+$.
- The Bondi-Sachs energy-momentum (E, P^k), $k = 1,2,3$ associated with a u =const. slice is

$$E(u) = \int_{S^2} 2m(u,\cdot), \quad P^k(u) = \int_{S^2} 2m(u,\cdot)\tilde{X}^k,$$

where $\tilde{X}^1 = \sin\theta\cos\phi, \tilde{X}^2 = \sin\theta\sin\phi, \tilde{X}^3 = \cos\theta$ are the restriction of standard coordinate functions of $\mathbb{R}^3$ to S^2.

- Positivity of $E(u)$ (Schoen-Yau 1982, Horowitz-Perry 1982).
- Bondi energy loss formula:

$$\frac{d}{du}E(u) = -\frac{1}{4}\int_{S^2} |N_{AB}|^2 \leq 0,$$

where C_{AB} is the shear that appears in the expansion of h_{AB}:

$$h_{AB} = \sigma_{AB} + \frac{C_{AB}}{r} + O(r^2),$$

and $N_{AB} = \partial_u C_{AB}$ is called the *news*.
- Both C_{AB} and N_{AB} are traceless with respect to σ_{AB} because of the determinant condition $\det h_{AB} = \det \sigma_{AB}$.
- First theoretical study of gravitational radiation. In the linear case is due to Trautman and the nonlinear case is due to Bondi et al.

- The definition of angular momentum turns out to be more subtle largely because of the convergence of expansion above and the choice of the Bondi-Sachs coordinate.
- The corresponding Killing fields the rotation fields, which are of higher order near infinity compared to the translating Killing fields used in the definition of the energy and linear-momentum.
- As a result, a very precise comparison with the Minkowski spacetime is needed in order to find an appropriate definition of angular momentum.
- In the literature, there are many approaches to define angular momentum (Hamiltonian, spinor-twistor, Komar type etc.), leading to different definitions. [1]

[1] Newmann-Penrose 1962, Winicour-Tamburino 1966, Bramson 1975, Ashtekar-Hansen 1978, Penrose 1982, Ludvigsen-Vickers 1983, Dray-Streubel 1984, Moreschi 1986, Dougan-Mason 1991, Rizzi 1997, Chruściel-Jezierski-Kijowski 2002, Barnich-Troessaert 2011, Hawking-Perry-Strominger 2017, Klainerman-Szeftel 2019, etc.

All these definitions are of the form

$$\int_{S^2} Y^A(N_A + \cdots),$$

where Y^A is a rotation Killing field on S^2 and $N_A = \sigma_{AB}N^B$, called the *angular momentum aspect*[2], appears in the higher order expansion of W^A:

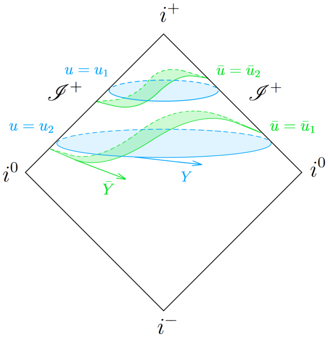
$$W^A = \frac{1}{2r^2}\nabla^D C^A_D + \frac{1}{r^3}\left(\frac{2}{3}N^A - \frac{1}{16}\nabla^A(C_{DE}C^{DE}) - \frac{1}{2}C_B{}^A\nabla^D C_D{}^B\right)$$
$$+ O(r^{-4}).$$

- All recover the Kerr angular momentum in a certain coordinate system.

[2] It corresponds to Ψ_1 in Newman-Penrose formalism

Supertranslation ambiguity

- There is no preferred coordinate system (in particular no preferred u coordinate) when radiation is present.
- What if we use a different Bondi-Sachs coordinate system?
- For any smooth function $f(x)$ on S^2, the change of coordinates $u = \bar{u} + f(x)$ is called supertranslation.
- Geometrically, supertranslation changes the foliation of $\mathscr{I}^+$.



- We decompose f into

$$f = \alpha_0 + \sum_{i=1}^{3} \alpha_i \tilde{X}^i + S$$

where $\alpha_0 + \sum_{i=1}^{3} \alpha_i \tilde{X}^i$ corresponds to (4-dimensional) ordinary translation and S is referred to as *pure supertranslation*.

- For a supertranslation $u = \bar{u} + f(x)$,

$$
\begin{aligned}
\bar{m}(\bar{u}, x) &= m(\bar{u} + f, x) + \frac{1}{2}(\nabla^B N_{BD})(\bar{u} + f, x)\nabla^D f \\
&\qquad + \frac{1}{4}(\partial_u N_{BD})(\bar{u} + f, x)\nabla^B f \nabla^D f + \frac{1}{4} N_{BD}(\bar{u} + f, x)\nabla^B \nabla^D f \\
\bar{C}_{AB}(\bar{u}, x) &= C_{AB}(\bar{u} + f(x), x) - 2\nabla_A \nabla_B f + \Delta f \sigma_{AB} \\
\bar{N}_{AB}(\bar{u}, x) &= N_{AB}(\bar{u} + f(x), x)
\end{aligned}
\tag{3.1}
$$

- Consequently, the total flux of energy

$$
\delta E = E(+\infty) - E(-\infty) = -\frac{1}{4}\int_{-\infty}^{+\infty}\int_{S^2} |N_{AB}|^2
$$

is supertranslation invariant and the energy radiated away is without ambiguity.

- On the other hand, under a supertranslation, the total flux of classical angular momentum (Dray-Streubel 1984)

$$\tilde{J} = \int_{S^2} Y^A (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB})$$

transforms according to

$$(\delta \tilde{J})_f - \delta \tilde{J} = -2 \int_{S^2} f \; Y^A \nabla_A (m(+) - m(-)),$$

where

$$m(\pm) = \lim_{u \to \pm\infty} m(u, x) \in C^\infty(S^2)$$

are the limits of the mass aspect at i^+ and i^0.

- In particular, if $Y^A \nabla_A (m(+) - m(-))$ has any $\ell \geq 2$ spherical harmonics component, the total flux of the classical angular momentum $\delta \tilde{J}$ can assume any value by supertranslations.[3]

[3] In fact, none of the definitions listed above were known to be supertranslation invariant .

- Penrose (Some unsolved problems in classical general relativity, Seminar on Differential Geometry 102, 1982):
  The very concept of angular momentum gets "shifted" by supertranslations and "it is hard to see in these circumstances how one can rigorously discuss such questions as the angular momentum carried away by gravitational radiation".
- There were efforts to eliminate supertranslation ambiguity by choosing special null foliations ("nice sections"[4] by Moreschi 1986, "preferred cuts"[5] by Rizzi 1997). However, both of them work only in special cases[6].

[4] with zero Bondi-Sachs momentum

[5] with zero electric part of the shear

[6] $|\dot{\sigma}|^2 < \sqrt{\frac{27}{4}}$ and $|\Xi|^2 < 16$, respectively.

Chen-Wang-Yau angular momentum

- Quasilocal angular momentum (Chen-M.-T. Wang-Y 2015) based on the theory of quasilocal mass (M.-T. Wang-Y 2009).
- Quasilocal formulation is essential in this work on angular momentum at null infinity. This is pointed out by Penrose in 1982 as he considered the definition of quasilocal mass and quasilocal angular momentum as his number one and number two unsolved problems in general relativity.
- Given a spacelike 2-surface Σ in a spacetime, extract the physical data $(\sigma, |H|, \alpha_H)$.
- For each time function $X^0 = \tau$, we solve the following equations

$$\sum_{i=1}^{3} \partial_a X^i \partial_b X^i = \sigma_{ab} + \partial_a X^0 \partial_b X^0 \tag{4.1}$$

- $X = (X^0, X^1, X^2, X^3)$ gives a configuration of Σ in $\mathbb{R}^{3,1}$. Extract the reference data $(\sigma, |H_0|, \alpha_{H_0})$ accordingly.
- From these two sets of data, we introduce ρ and j

$$
\begin{aligned}
\rho =& \frac{\sqrt{|H_0|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}} - \sqrt{|H|^2 + \frac{(\Delta\tau)^2}{1+|\nabla\tau|^2}}}{\sqrt{1+|\nabla\tau|^2}} \\
j =& \rho\nabla\tau - \nabla[\sinh^{-1}(\frac{\rho div(\nabla\tau)}{|H_0||H|})] - \alpha_{H_0} + \alpha_H
\end{aligned}
$$

- Integration of ρ gives rise to the energy of this embedding $E(X)$ which depends on the two sets of data.
- The Euler-Lagrange equation by minimizing $E(X)$ among all possible isometric embeddings X gives rise to the equation:

$$
\nabla^a j_a = 0. \tag{4.2}
$$

- For surfaces near the null infinity, the system of equations (4.1) and (4.2) has a unique solution, which can be calculated numerically.
- Transplant a rotation Killing field Y of $\mathbb{R}^{3,1}$ through this unique solution, we define the quasilocal angular momentum on Σ as

$$\int_{\Sigma} j(Y^T)$$

- Use the limit of quasilocal angular momentum to define total angular momentum at null infinity
- This can be carried out for very general spacetimes. In particular, we don't need any peeling structure of null infinity.

- Comparison of the classical angular momentum (Dray-Streubel 1984):

$$\tilde{J} = \int_{S^2} Y^A (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB}).$$

- CWY angular momentum:

$$J = \int_{S^2} Y^A (N_A - \frac{1}{4} C_A{}^D \nabla^B C_{DB} - c \nabla_A m),$$

where c is given by the decomposition of the shear tensor C_{AB} and has never occurred in any previous definition of angular momentum.

- The decomposition of C_{AB}:

$$C_{AB} = \nabla_A \nabla_B c - \frac{1}{2}\sigma_{AB}\Delta c + \frac{1}{2}(\epsilon_A{}^E \nabla_E \nabla_B \underline{c} + \epsilon_B{}^E \nabla_E \nabla_A \underline{c})$$

- c arises in the CWY angular momentum J through solving the the system of equations (4.1) and (4.2) to obtain the gravitational ground state.
- In particular, the additional term in the definition of J indeed comes from the reference term in the Hamiltonian theory.

Theorem (Chen-Keller-Wang-Wang-Y, 2021)

Suppose $N_{AB}(u,\cdot) = O(|u|^{-1-\epsilon})$ *as* $|u| \to \infty$. *Under a supertranslation* $f = \alpha_0 + \alpha_i \tilde{X}^i + S$, *the total flux of the CWY angular momentum*

$$\delta J(Y) = J(+\infty, Y) - J(-\infty, Y)$$

transforms according to

$$(\delta J)_f - \delta J = \alpha_i \epsilon^{ik}{}_j \delta P^j \text{ for } Y^A = \epsilon^{AB} \nabla_B \tilde{X}^k,$$

where δP^j *is the total flux of Bondi-Sachs linear momentum.*

In particular, δJ is invariant under pure supertranslation and the transformation law is the same as the special-relativistic angular momentum.

- The assumption on N_{AB} is needed even for the convergence of $\delta J(Y)$.

Concluding remarks

We obtain a complete set of ten conserved quantities (E, P^k, J^k, C^k) at null infinity (all as functions of the retarded time u and can be calculated numerically in a straight-forward way) that satisfy the following properties:

- (E, P^k, J^k, C^k) all vanish for any Bondi-Sachs coordinate system of the Minkowski spacetime where there is no gravity.
- In a Bondi-Sachs coordinate system of the Kerr spacetime, P^k and C^k vanish, and E and J^k recover the mass and angular momentum.
- On a general spacetime, the total fluxes of (E, P^k, J^k, C^k) are supertranslation invariant.
- (E, P^k, J^k, C^k) and their fluxes transform according to basic physical laws under ordinary translations.
- Since these are independent of the choice of the Bondi-Sachs coordinate, they can be computed in any Bondi-Sachs coordinate.
- Since the definition comes from quasilocal formulation, we should be able to calculate it at a finite distance.
- We would like to explore more detailed calculation of these conserved quantities for data coming from experiments.

Thank you !