The Role of General Relativity in the Structure of Matter

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ICRAnet-Isfahan Astronomy Meeting
Nov 3-5, 2021

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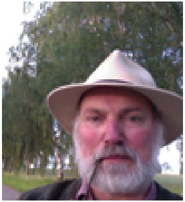

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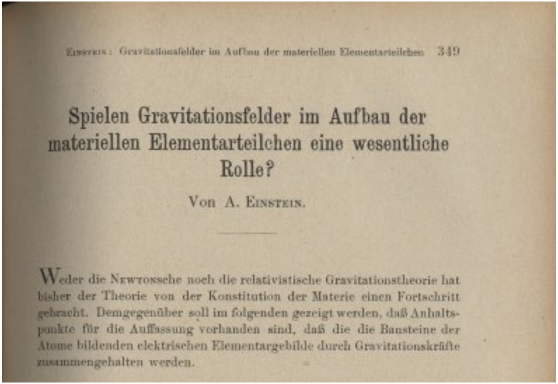
Figure : GRQM Team

Einstein's 1919 Question

- 

EINSTEIN: Gravitationsfelder im Aufbau der materiellen Elementarteilchen 349

Spielen Gravitationsfelder im Aufbau der materiellen Elementarteilchen eine wesentliche Rolle?

Von A. EINSTEIN.

Weder die NEWTONsche noch die relativistische Gravitationstheorie hat bisher der Theorie von der Konstitution der Materie einen Fortschritt gebracht. Demgegenüber soll im folgenden gezeigt werden, daß Anhaltspunkte für die Auffassung vorhanden sind, daß die die Bausteine der Atome bildenden elektrischen Elementargebilde durch Gravitationskräfte zusammengehalten werden.

- Does the gravitational field play an essential role in the structure of the elementary particles of matter?
- Our answer is YES!

Particles as Singularities of the Field

- **A toy model for joint evolution of fields and particles:** Let $x = q(t)$ be a timelike curve (to be determined), with $q(0) = \dot{q}(0) = 0$, that is the trajectory of the singularity of the solution to the following Cauchy problem for the wave equation:
- $$\begin{cases} u_{tt} - u_{xx} &= a\delta_0(x - q(t)), \qquad a \in \mathbb{R} \text{ (the "charge")} \\ u(0,x) &= -\frac{a}{2}|x| + v_0(x), \qquad v_0 \in C_c^\infty(\mathbb{R} \setminus \{0\}) \\ u_t(0,x) &= v_1(x), \qquad v_1 \in C_c^\infty(\mathbb{R} \setminus \{0\}) \end{cases}$$
- Equation of motion for the singularity, treated as a particle:
- **Classical relativistic mechanics** (Newton-Einstein):
  $$\begin{cases} \dot{q} &= \frac{p}{m\sqrt{1+p^2/m^2}} \qquad m = \text{"bare mass"} \\ \dot{p} &= F(t,q,\dot{q}), \qquad F = \text{force on the particle (to be found)} \end{cases}$$
- One can use momentum balance law to find F **(Kiessling, 2020)**
- $F(t,q,\dot{q}) = [u_t u_x]_{x=q(t)}\dot{q} + \frac{1}{2}[u_t^2 + u_x^2]_{x=q(t)}$
- Contains a "self-force" term: $F = -av_x + \frac{e^2}{2m^2}p\sqrt{m^2+p^2}$ where $v(t,x)$ satisfies $v_{tt} - v_{xx} = 0$, $v(0,x) = v_0(x)$, $v_t(0,x) = v_1(x)$.

Classical Law of Motion for Singularities

- Matter Particles = Singularities of Spacetime (Weyl, Einstein, ... 1920-1940's) Einstein-Infeld-Hoffmann, and its problems
- New approach **(Kiessling, Kiessling–Tahvildar-Zadeh)**:
- Field eqs. derivable from an Action Principle $\implies \exists$ Energy-Mom. Tensor $T^{\text{fie}}_{\mu\nu}$ satisfying $\nabla^\mu T^{\text{fie}}_{\mu\nu} = 0$ *away from singularities.*
- **If** singularities are "just right", $T_{\mu\nu}$ can be continued into their location, and we have $T^{\text{tot}}_{\mu\nu} = T^{\text{fie}}_{\mu\nu} + T^{\text{par}}_{\mu\nu}$
- **If** $\nabla^\mu T^{\text{tot}}_{\mu\nu} = 0$ still holds (in the sense of distributions), then according to momentum balance law:
- Force on the particle $= F_i(t,x)\delta(x - q(t)) = \nabla^\mu T^{\text{par}}_{\mu i} = -\nabla^\mu T^{\text{fie}}_{\mu i}$
- so that force can be computed by integrating both sides of above on a suitable nbhd of the trajectory and using divergence theorem:
- $F_i(t, q(t)) = -\int \nabla^\mu T^{\text{fie}}_{\mu i} d^3x = -\partial_t \int \Pi^{\text{fie}}_i d^3x.$
- This procedure succeeds provided the **field momentum density is integrable** in a neighborhood of the singularity.

Einstein Equations and Weak Bianchi Identities

- But, what would compel $T^{\text{tot}}_{\mu\nu}$ to still be divergence-free?
- Answer: GRAVITY would: If Einstein's Equations are satisfied

$$G_{\mu\nu} = \kappa T^{\text{tot}}_{\mu\nu}, \qquad G_{\mu\nu} := R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R$$

  then the **Weak** version of the *Twice Contracted Second Bianchi identity* $\nabla^\mu G_{\mu\nu} = 0$ implies that $\nabla^\mu T^{\text{tot}}_{\mu\nu} = 0$ holds in a weak sense.
- What are the conditions on spacetime metric g that guarantee Weak Second Bianchi holds in a neighborhood of spacetime singularities?
- For *static, spherically symmetric* spacetimes with a central timelike singularity, we have an answer, in terms of the *mass function*

$$m(r) := \frac{c^2 r}{2G} \left(1 - g(\nabla r, \nabla r)\right), \qquad r = \text{area-radius}.$$

- **(Burtscher, Kiessling, T.-Z. 2021)** Twice-contracted weak second Bianchi identity holds in a nbhd of $r = 0$ if $m(r) = m_0 + m_1 r + O(r^2)$ with either $m_0 < 0$ or $m_0 = 0$ and $m_1 < \frac{c^2}{2G}$.

Weak Twice-Contracted Second Bianchi Identity

- Second Bianchi Identity, satisfied by the Riemann curvature tensor of any sufficiently regular (at least C^3) metric:

$$R_{\alpha\beta[\lambda\mu;\nu]} = \nabla_\nu R_{\alpha\beta\lambda\mu} + \nabla_\lambda R_{\alpha\beta\mu\nu} + \nabla_\mu R_{\alpha\beta\nu\lambda} = 0.$$

- Once-contracted with the metric $g^{\alpha\lambda}$, one gets an identity satisfied by the Ricci curvature of the metric:

$$\nabla_\nu R_{\beta\mu} - \nabla_\lambda R^\lambda{}_{\beta\mu\nu} - \nabla_\mu R_{\beta\nu} = 0.$$

- Contracted again, with $g^{\beta\nu}$ we obtain:

$$\nabla_\lambda \left(R^\lambda{}_\mu - \frac{1}{2} R \delta^\lambda_\mu \right) = 0.$$

- Weak version of the above: multiply by $\nabla\psi$ and integrate by parts

$$\int_{\mathcal{M}} G^\mu{}_\nu \nabla_\mu \psi^\nu d\mu_g = 0, \qquad G^\mu{}_\nu := R^\mu{}_\nu - \frac{1}{2} R \delta^\mu_\nu.$$

for any compactly supported vector field $\psi \in \mathfrak{X}_c(\overline{\mathcal{M}})$.

Properties of $m(r)$

- $m(\infty) = m_{ADM}$ the total mass (energy content) of spacetime.
- $m(r) = \lim\limits_{\mathcal{S} \searrow S_r} m_{Hawking}(\mathcal{S})$.
- $\lim\limits_{r \to 0^+} m(r) = m_0$ is the **bare mass** of the singularity at $r = 0$.
- $m_0 > 0$ implies that the singularity is shielded by an event horizon.
- We want no horizons to be there so let's take $m_0 \leq 0$.
- Some examples:
  1. Schwarzschild blackhole: $m(r)$ is a positive constant M.
  2. Negative-mass Schwarzschild: $m(r)$ is a negative constant. (Naked singularity)
  3. Reissner-Weyl-Nordström: $m(r) = M - \frac{Q^2}{2r}$ Thus $m_0 = -\infty$. Can be either naked ($GM^2 < Q^2$) or black hole ($GM^2 \geq Q^2$).
  4. Prototype spacetime of a point charge: m is a C^2 function of r with $m(r) \sim \begin{cases} m_0 + m_1 r & r \to 0, \\ M - \frac{M_1}{r} & r \to \infty, \end{cases}$ for constants $m_0 < 0$, $m_1, M_1 > 0$.

Particle spacetimes

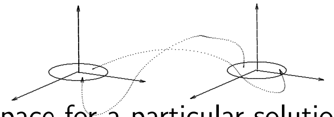
- Example of spacetimes where WSB is **not** satisfied: Reissner-Weyl-Nordstrom (RWN): $m_{\text{RWN}}(r) = M - \frac{Q^2}{2r}$.
- Example of spacetimes where WSB **is** satisfied: **Hoffmann's**. $m_{Hoff}(r) = M - \frac{1}{c^2}\int_r^\infty \zeta\left(\frac{Q^2}{2s^4}\right) s^2 ds$, with $\zeta(\mu) := \sqrt{1+2\mu} - 1$.
- More generally, WSB holds for *Electrovacuum* spacetimes for which the *reduced Hamiltonian* $\zeta(\mu)$ behaves like $\sqrt{\mu}$ as $\mu \to \infty$.
- For Maxwell's linear law of electromagnetic vacuum, where $\mathbf{E} = \mathbf{D}$ and $\mathbf{B} = \mathbf{H}$, we have $\zeta(\mu) = \mu$, so, no chance!
- Other electromagnetic vacuum laws: Born-Infeld (nonlinear) , Bopp-Lande-Thomas-Podolsky (linear, higher order).
- Existence of static, spherically symmetric solution to Einstein-Maxwell-BLTP **(Amorim, 2020)**
- WANTED: Two point-charge solution of Ein-Max-BLTP (no longer spherically symmetric, but still axisymmetric).

Quantum Law of Motion for Point Test-Particles

- A single electron placed in the electrostatic and gravitational field of a much more massive charge (nucleus).
- Nucleus modeled as the central singularity of a static spherically symmetric electrovacuum spacetime, and Dirac's equation for a test electron is studied on that background.
- Dirac equation on Hoffmann spacetime with $m_0 = 0$: Hamiltonian is essentially selfadjoint, continuous spectrum is $\mathbb{R} \setminus [-m, m]$, with ∞-many eigenvalues in the gap (**Balasubramanian, 2015**)
- Weak Bianchi is not the only factor: On **any** static spher. sym. electrovac spacetime with $m_0 < 0$, the Dirac Hamiltonian has uncountably many self-adjoint extensions. Any of these extensions has continuous spectrum $\mathbb{R} \setminus [-m, m]$, with ∞-many eigenvalues in the gap (**Kiessling, T.-Z., & Toprak, 2020**)
- Recall, for RWN, $m_0 = -\infty$.

Ring-like Particles

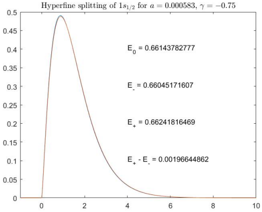
- Sommerfeld (1896) Generalization of Riemann surfaces to 3-D
- Two copies of $\mathbb{R}^3$ cross-glued at a disk (of radius a):

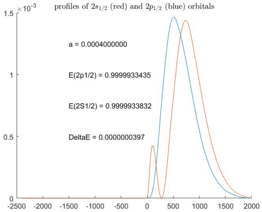


- Branched Riemann Space for a particular solution of Maxwell's equations, the Appell Field (Found by Appell in 1888).
- Add a time dimension to get a static, axisymmetric spacetime $\mathcal{M}_0$.
- Points on the ring are *conical singularities* for the metric.
- Appell field is such that the ring appears as positively charged in one sheet and negatively charged in the other sheet!
- Kerr-Newman (1963) = Stationary-Axisym. sol. of Ein.-Max.
- $g_{KN} = g_{KN}(M, a, Q, G) = \bar{g}_{KN}(GM, a, GQ^2)$
- zero-G limit of Kerr-Newmann = The above $\mathcal{M}_0$ + the Appell field!
- $\lim_{G\to 0} g_{KN} = \bar{g}_{KN}(0, a, 0)$ locally Minkowski but not globally!

Dirac Equation for the "Bothon"

- **Kiessling & T.-Z. (2014)**: Dirac Hamiltonian on zero-G Kerr-Newman (zGKN) spacetime is essentially self-adjoint and its spectrum is symmetric about zero.
- The continuous spectrum is $(-\infty, -m] \cup [m, \infty)$
- **Ling, Kiessling, & T.-Z. (2021)** Discrete spectrum of zGKN consists of an infinite family of eigenvalues, indexed by 3 integers, accumulating at the endpoints.

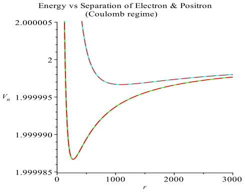




- Evidence of "Hyperfine splitting" and "Lamb shift"-like effects.

True Ground State of Positronium

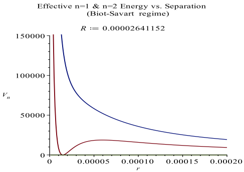
- **(Kiessling, 2018)** "Do particles and anti-particles really annihilate each other?" [arXiv:1807.06428]
- Positronium = bound state between an electron and a positron
- Binding energy = a few eV; life span = $0.125 \times 10^{-3} \mu s$. Decays to 2γ
- Effective potential $V_n(r) = 2\sqrt{1 + \frac{n^2}{r^2}} - \frac{\alpha_S}{r}$.



- The above ignores magnetic dipole-dipole interactions. If include them, then $V_n(r) = 2\sqrt{1 + \frac{n^2}{r^2}} - \frac{\alpha_S}{r} - \frac{\alpha_S^3}{8\pi^2 r^3}$ which is unbounded below. This is for point particles.

Tight bound state between ring-electron and ring-positron

- If replace point particles with ring-like particles, with ring radius R, then $V_n(r) = 2\sqrt{1+\frac{n^2}{r^2}} - \frac{\alpha_S}{\pi R\sqrt{1+r^2/4R^2}} K(\frac{1}{\sqrt{1+r^2/4R^2}}) - \frac{\alpha_S^3}{4\pi^3 R^3}\sqrt{1+r^2/4R^2}\left[\left(2-\frac{1}{1+r^2/4R^2}\right)K(\frac{1}{\sqrt{1+r^2/4R^2}}) - 2E(\frac{1}{\sqrt{1+r^2/4R^2}})\right]$
- Which has a very tight bound state with nearly zero energy!



- In this tight bound state e^- and e^+ are so close they appear to have annihilated each other. The bound state is neutral and of spin zero, so it would hardly interact with matter, except gravitationally in bulk.
- Kiessling has a proposal for an experiment to find evidence for them.

Conclusions

- General Relativity plays an essential role in the structure of matter:
- On spacetimes that are solutions of Einstein equations, if the weak version of second Bianchi identity holds, momentum balance laws can be used to derive equations of motion for matter particles, which are identified with the singularities of the fields on that spacetime.
- Inclusion of relativistic gravity has a profound effect on the spectral properties of Dirac Hamiltonians for hydrogen and hydrogenic ions.
- Non-trivial topology of spacetimes such as Kerr-Newman remain even after G is set to zero. The singularities of these multi-sheeted spacetimes can be identified with ring-like elementary particles.
- The Dirac Hamitonian for a single electron on these multi-sheeted spacetimes is well-behaved and captures all the essential features one would expect, in addition to breaking the degeneracies in the standard theory, which leads to hyperfine splitting and Lamb shift-like effects.
- Ring-like particles facilitate the existence of a true ground state of Positronium, which could be the origin of dark matter in the universe.

THANK YOU FOR LISTENING!