Symmetric periodic trajectories of charged particles in Gutsunaev–Manko spacetime

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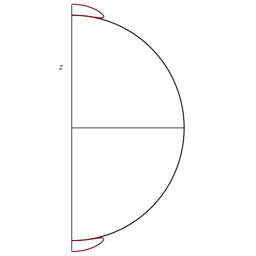
Gutsunaev–Manko solution[1]

Gutsunaev–Manko spacetime

- exact magnetostatic solution of Einstein–Maxwell equations
- axisymmetric
- magnetic field asymptotically dipole
- reduces to Schwarzschild spacetime in the absence of magnetic field
- singularities outside the horizon

Horizon and singularities



Gutsunaev–Manko solution details

The line element in prolate ellipsoidal coordinates

$$ds^2 = f dt^2 - \frac{k^2 e^{2\gamma}}{f}\left[\left(x^2 - y^2\right)\left(\frac{dx^2}{x^2-1} + \frac{dy^2}{1-y^2}\right) + \left(x^2-1\right)\left(1-y^2\right)d\varphi^2\right]$$

Metric functions

$$f = \frac{1}{2}\sqrt{\frac{x-1}{x+1}}\left(\frac{n_-}{d_-} + \frac{n_+}{d_+}\right)$$

$$e^{2\gamma} = \frac{x^2-1}{x^2-y^2} \cdot \frac{\left(n_- d_+ + n_+ d_-\right)^4}{\left(1+\alpha^2\right)^8\left(x^2-y^2\right)^8}$$

$$n_\pm = x^2 - y^2 \pm 2\alpha y\sqrt{x^2-1} + \alpha^2\left(x+1\right)^2$$

$$d_\pm = x^2 - y^2 \pm 2\alpha y\sqrt{x^2-1} + \alpha^2\left(x-1\right)^2$$

Gutsunaev–Manko solution details

Horizon and singularities

Horizon: $x = 1$ Singularities: $d_\pm = 0$

4-potential only nonzero component

$$A_\varphi = \frac{8k\alpha^3 \left(y^2-1\right)\left[2\left(1+\alpha^2\right)x^3+\left(1-3\alpha^2\right)x^2+y^2+\alpha^2\right]}{\left(1+\alpha^2\right)\left(n_-d_+ + n_+d_-\right)}$$

Mass and dipole moment

$$m = k\frac{1-3\alpha^2}{1+\alpha^2}, \qquad \mu = \frac{8m^2\alpha^3}{\left(1-3\alpha^2\right)^2}, \qquad 0 \le \alpha < 1/\sqrt{3}$$

Coordinates for illustrations

$$x = (r-m)/k, \qquad y = \cos\theta, \qquad \rho = \ln r/m \sin\theta, \qquad z = \ln r/m \cos\theta$$

Charged particle equations of motion

Lagrangian

$$\mathcal{L} = \frac{1}{2} g_{\mu\nu} u^{\mu} u^{\nu} + q A_{\mu} u^{\mu}$$

Conservation laws

$$E = \frac{\partial \mathcal{L}}{\partial u^t} = g_{tt} u^t, \qquad L = -\frac{\partial \mathcal{L}}{\partial u^{\varphi}} = -g_{\varphi\varphi} u^{\varphi} - q A_{\varphi}$$

$u_{\mu} u^{\mu} = 1$. Two-dimensional effective potential

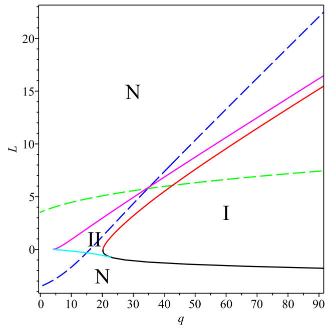
$$-g_{tt} \left(g_{xx} \left(\frac{dx}{ds} \right)^2 + g_{yy} \left(\frac{dy}{ds} \right)^2 \right) + g_{tt} \left(1 - \frac{(L - q A_{\varphi})^2}{g_{\varphi\varphi}} \right) = E^2$$

$$V_{eff}(x, y; \alpha, L, q) = g_{tt} \left(1 - \frac{(L - q A_{\varphi})^2}{g_{\varphi\varphi}} \right)$$

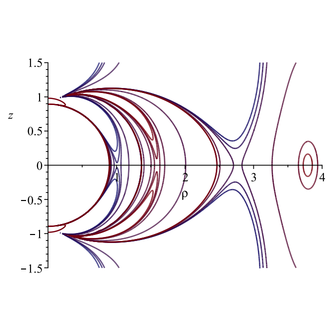
Effective potential

Number of stationary points and contour lines of effective potential

$$\alpha = \alpha_{\text{max}}/2 = 1/2\sqrt{3}$$



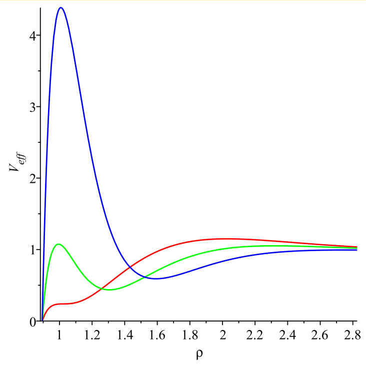
Areas of q and L with different numbers of stationary points. Dashed lines – zero hessian in the equatorial plane. N – no point above the plane, I – one point, II – two points



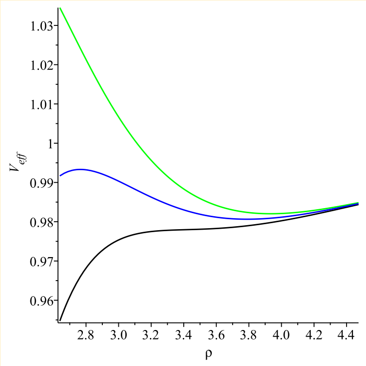
Contour lines for $L = 8, q = 90$

Effective potential in equatorial plane

$\alpha = 1/2\sqrt{3}, L = 8$



Magnetic well



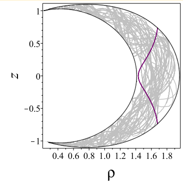
Gravitational well

red – $q = 42$, green – $q = 60$, blue – $q = 90$, **black** – $q = 120$

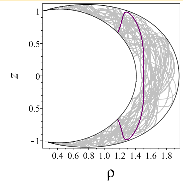
Poloidal projections of symmetric simple-periodic orbits

Markellos *et al* classification[2]

$\alpha = 1/2\sqrt{3}$, $L = 8$, $q = 90$, $E = 0.9$ ($E = 0.99$ for f1)



f0



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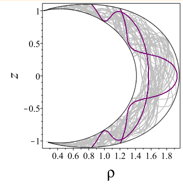
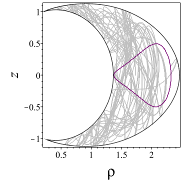
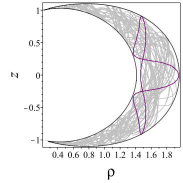


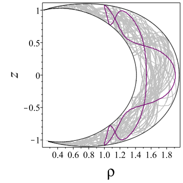
f4



f1



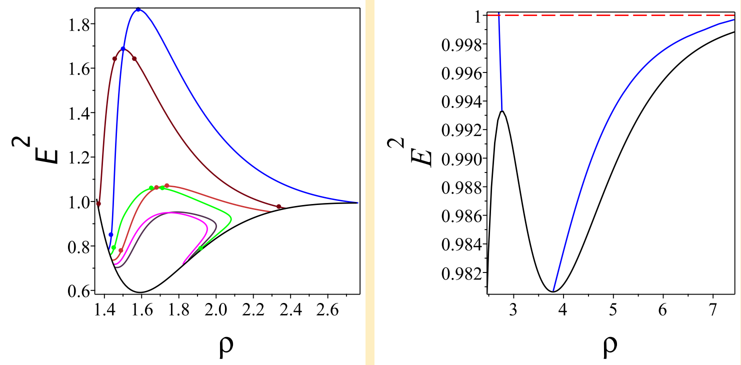
f3



f5

Families of orbits periodic in meridian plane

Energy vs starting point curves



black – V_{eff}
blue – f0, brown – f1,
orange – f2, red – f3,
violet – f4, magenta – f5

Stability[2]

Poincare map on equatorial plane

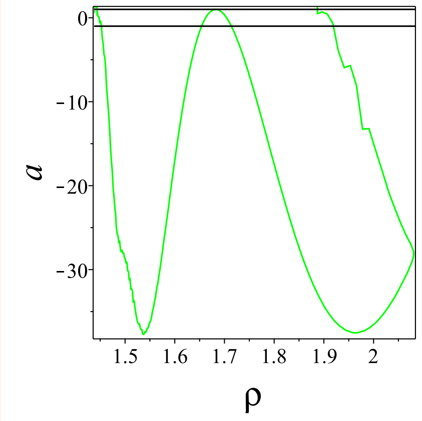
$$\begin{pmatrix} \Delta x_{int} \\ \Delta u^x_{int} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \Delta x_0 \\ \Delta u^x_0 \end{pmatrix}$$

Mirror symmetric orbits

$$a = d = \frac{\partial x_{int}}{\partial x_0}$$

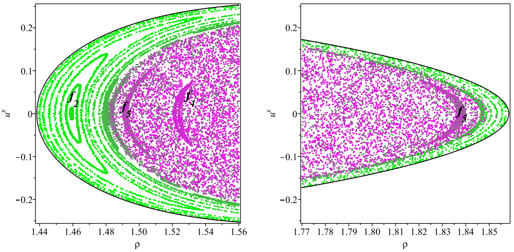
Stability condition

$$|a| < 1$$

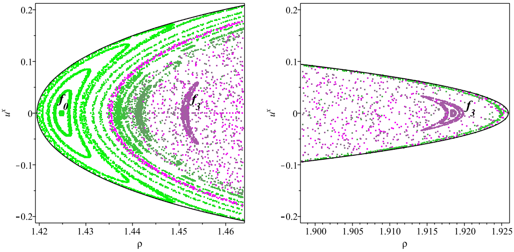


Stability parameter a for f3 family

Poincare surfaces for $\alpha = 1/2\sqrt{3}, L = 8, q = 90$

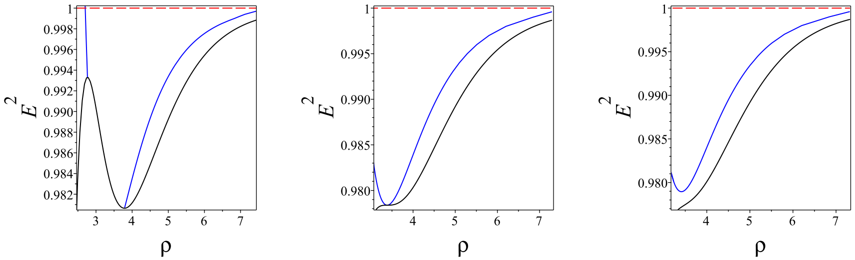


$E^2 = 0.7396$



$E^2 = 0.7868$

Behavior of $f0$ family with the increase of charge



$q = 90$ $\quad$ $q = 117.5$ $\quad$ $q = 125$

Conclusions

- The periodic orbits of charged particles in Gutsunaev–Manko spacetime are similar to the orbits in classical problem
- We are now in position to look how the families of orbits change with parameters variations
- The f0 (principal) family branches merge when the barrier between magnetic and gravitational wells disappears

Thank you for your attention!

[1] Gutsunaev Ts. I. and Manko V. S. *Gen. Rel. Grav.* **20**, 327 (1988)

[2] Markellos V. V., Klimopoulos S. and Halioulias A. A. *Celest. Mech.* **17**, 215 (1978)