



# The dynamical tidal Love numbers

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# Outline

- The static and dynamic tidal Love numbers.
- Dynamical tidal Love numbers for black holes
- Dynamical tidal Love numbers for reflective compact object.
- Tidal Love number and asymptotic properties of spacetime.

## References

- Nair, **SC** and Sarkar, **PRD 107**, 124041 (2023) + Work in Progress.
- **SC**, Maggio, Pani and Silvestrini, arXiv:2310.06023 + Work in Progress.
- Bhatt, **SC** and Bose, **PRD 108**, 084013 (2023) + arXiv 2406.09543.



# Tides are Everywhere

- Tides are ubiquitous and captures the true nature of gravity, as it depends on the Riemann.
- Deformation produced by the tides depends on the detail of the constituent of the object being deformed.
- What about black holes? Can they be deformed?
- In the context of GW, the effects due to tidal deformation appears at 5 pN, thus unless the deformation is large, detection can be challenging.
- What happens if we consider objects as compact as BHs, but with reflective surface? Can they be deformed?
- Does deformation depends on asymptotic properties of spacetime? Can a Schwarzschild-de Sitter BH be deformed?



# Newtonian Deformation

- Newtonian gravity is solely governed by the potential.
- The tidal field arises from double derivative of the potential.

$$\mathcal{E}_{ij} = \frac{\partial^2 \Phi_{\text{ext}}}{\partial x^i \partial x^j}.$$

- The total potential will consist of potential due to the external tidal field and the potential of the deformed object. **[Thorne, Phys. Rev. D 58, 124031 (1998)]**

$$\frac{(1 - g_{tt})}{2} = -\frac{M}{r} - \frac{3Q_{ij}}{2r^3} \left( n^i n^j - \frac{1}{3} \delta^{ij} \right) + O\left(\frac{1}{r^3}\right) + \frac{1}{2} \mathcal{E}_{ij} x^i x^j + O(r^3),$$

- Quadrupole moment of the deformed object arises due to tidal field.

$$Q_{ij} = -\lambda \mathcal{E}_{ij}.$$

$$k_2 = \frac{3}{2} G \lambda R^{-5}.$$



# Static Love number of BHs

- BH perturbation theory becomes directly applicable.

$$g_{\alpha\beta} = g_{\alpha\beta}^{(0)} + h_{\alpha\beta},$$

$$h_{\alpha\beta} = \text{diag} [e^{-\nu(r)} H_0(r), e^{\lambda(r)} H_2(r), r^2 K(r), r^2 \sin^2 \theta K(r)] Y_{2m}(\theta, \varphi).$$

- Among the three unknown functions, perturbed Einstein's equations demand  $H_2 = H_0 \equiv H$ , and  $K$  can be expressed in terms of  $H$ .

$$(x^2 - 1) H'' + 2xH' - \left(6 + \frac{4}{x^2 - 1}\right) H = 0.$$

[Hinderer, arXiv: 0711.2420]

- Exact solution exist, in terms of associated Legendre polynomials.

$$H = c_1 \left(\frac{r}{M}\right)^2 \left(1 - \frac{2M}{r}\right) \left[ -\frac{M(M-r)(2M^2 + 6Mr - 3r^2)}{r^2(2M-r)^2} + \frac{3}{2} \log \left(\frac{r}{r-2M}\right) \right] + 3c_2 \left(\frac{r}{M}\right)^2 \left(1 - \frac{2M}{r}\right).$$

[near horizon]

[infinity]

$$H = \frac{8}{5} \left(\frac{M}{r}\right)^3 c_1 + O\left(\left(\frac{M}{r}\right)^4\right) + 3 \left(\frac{r}{M}\right)^2 c_2 + O\left(\left(\frac{r}{M}\right)\right),$$



# Love number of compact objects

- Generically, compact objects are described by two parameters

$$r_0 = r_+ (1 + \epsilon),$$

$$\mathcal{R}(\omega) = \left[ \frac{1 - \frac{i}{\bar{\omega}} \left( \frac{1}{X} \frac{dX}{dr_*} \right)}{1 + \frac{i}{\bar{\omega}} \left( \frac{1}{X} \frac{dX}{dr_*} \right)} \right]_{r_*^0}$$

- In the static case, the reflectivity makes sense for Dirichlet and Neumann conditions on the surface.

[Cardoso [+](#), arXiv: 1701.01116]

$$k_2^{\text{polar}} = \frac{8}{5(7 + 3 \ln \epsilon)}$$

- Thus, departure from BH leads to non-zero static Love number with a Logarithmic behaviour.



# Questions to Address

- **Q1:** How to remove gauge ambiguities? Can we provide a formalism based on scalars?
- **Q2:** What happens in a dynamical situation?
- The GW observations of tidal Love number happens in the in-spiral phase, which is inherently dynamical.
- **Q3:** Do BHs have vanishing dynamical tidal Love numbers?
- **Q4:** Does Logarithmic behaviour for compact objects retained in dynamical context as well?



# Tidal Love number from scalar

- For the Love number to be gauge invariant, one solves the Teukolsky equation for the Weyl scalars and impose boundary conditions near the horizon.

$$\lim_{c \rightarrow \infty} c^2 \psi_0 = \sum_{\ell m} \alpha_{\ell m}(t) r^{\ell-2} \left[ 1 + 2k_{\ell m} \left( \frac{R}{r} \right)^{2\ell+1} \right] {}_2Y_{\ell m}(\theta, \varphi),$$

$$F_{\ell m}(\omega) = 2k_{\ell m} + i\omega\tau_0\nu_{\ell m} + \mathcal{O}(\omega^2).$$

- The real part determines the tidal Love numbers, while imaginary part determines the dissipation.

[Chia, arXiv:2010.07300]

[Le Tiec +, arXiv:2010.15795]





# Dynamical Love number for BHs

- Starting point is the radial Teukolsky equation in the ingoing null coordinate.
- In small frequency and near horizon limit, it can be solved exactly.

$$R(z) = (z + 1)^{2-N_3} \left[ c_1 z^2 {}_2F_1(3 + l - N_2, 2 - l - N_1; 3 + 2iP_+; -z) + c_2 z^{-2iP_+} {}_2F_1(l - 2iP_+ - N_2 + 1, -l - 2iP_+ - N_1; -1 - 2iP_+; -z) \right]$$

[Bhatt, SC and Bose, arXiv: 2406.09543]

- For BH, imposing purely ingoing boundary condition near the horizon, the second solution identically vanishes. Also,  $N_1$  and  $N_2$  are both  $\mathcal{O}(M\omega)$ .

$$P_{\pm} = \frac{am - 2r_{\pm}M\omega}{r_+ - r_-}$$

$$N_3 = \frac{12\omega(r_+ - M)^2}{am + 2i(r_+ - M)}$$



# Dynamical Love for Kerr BH

- The hypergeometric function can be further expanded in the intermediate region.

$$\Psi_4^{\text{intermediate}} \propto z^{l-2+N_1-N_3} \left\{ \frac{\Gamma(3+2iP_+) \Gamma(2l+N_1-N_2+1)}{\Gamma(l+2iP_+ + N_1 + 1) \Gamma(3+l-N_2)} \right\} \\ \times \left[ 1 + z^{-2l-1+N_2-N_1} \left\{ \frac{\Gamma(-2l-N_1+N_2-1) \Gamma(l+2iP_+ + N_1 + 1) \Gamma(3+l-N_2)}{\Gamma(2-l-N_1) \Gamma(-l+2iP_+ + N_2) \Gamma(2l+N_1-N_2+1)} \right\} \right]$$

- Assuming  $l$  to be a complex number, the Gamma functions can be simplified. For zero frequency, the response function for Kerr BH becomes

$$F_{lm}^{\text{static}} = - \left( \frac{iam}{r_+ - r_-} \right) \frac{(l-2)! (l+2)!}{(2l+1)! (2l)!} \prod_{j=1}^l \left[ j^2 + \left( \frac{2am}{r_+ - r_-} \right)^2 \right]$$

[Bhatt, SC and Bose, arXiv: 2406.09543]



# Love of Arbitrary Rotating BH

- For an arbitrary rotating BH, the response function with all linear-in-frequency term become: **[Bhatt, SC and Bose, arXiv: 2406.09543]**

$$F_{lm} = \frac{\Gamma(-2l - N_1 + N_2 - 1) \Gamma(l + 2iP_+ + N_1 + 1) \Gamma(3 + l - N_2)}{\Gamma(2 - l - N_1) \Gamma(-l + 2iP_+ + N_2) \Gamma(2l + N_1 - N_2 + 1)}$$

- The Gamma functions can be simplified using reflection formula, as well as doing a power series expansion, leading to di-gamma functions.
- The above response function turns out to have a real part. The associated tidal Love numbers become

$$k_{lm} = (am)^2 k_{lm}^{(0)} + am\omega k_{lm}^{(1)} + \mathcal{O}(M^2\omega^2)$$

# Love for extremal black hole

- The above approach is not directly applicable for extremal black hole. In which case, the response function becomes: **[Bhatt, SC and Bose, arXiv: 2406.09543]**

$$F_{lm} = -\{2i(m - 2M\omega)\}^{2l+1-2N} \frac{\Gamma(-2l + 2N) \Gamma(3 + l - 2iM\omega - N)}{\Gamma(2 - l - 2iM\omega + N) \Gamma(2l + 2 - 2N)}$$

- In the static case, the above response function reduces to,

$$F_{lm}^{\text{static}} = (-1)^{-1-l} im(2im)^{2l} \frac{\Gamma(3 + l)\Gamma(-1 + l)}{\Gamma(2l + 1)\Gamma(2l + 2)}$$

- While the dynamical response function has a real part:

$$k_{lm} = (2m)^{2l-2N-1} \{m - (2l + 1)2M\omega\} \left( \frac{\Gamma(-1 + l)\Gamma(3 + l)}{\Gamma(1 + 2l)\Gamma(2l + 2)} \right) \\ \times \left[ -\frac{2l(l + 1)(2l + 1)}{(l^2 + l + 4)} \{1 + N(2\psi(1 + 2l) + 2\psi(2 + 2l) - \psi(-1 + l) - \psi(3 + l))\} \right. \\ \left. + 4Mm\omega \left\{ \psi(-1 + l) - \psi(3 + l) - \frac{\pi m(l^2 + l + 4)}{l(l + 1)(2l + 1)} \right\} \right].$$



# Compact Objects can be deformed

- Compact Objects having non-zero reflectivity, in general, have non-zero Love number. [SC, Maggio, Silvestrini and Pani, arXiv: 2310.06023]

$$k_2 = \text{Re} \left[ \left( \frac{iM\omega}{30} \right) (1 + 16M^2\omega^2) (1 + 4M^2\omega^2) \times \left\{ \frac{1 - \frac{\mathcal{B}}{\mathcal{A}}\Gamma_1}{1 + \frac{\mathcal{B}}{\mathcal{A}}\Gamma_1} \right\} \right],$$

- Reflectivity is defined in terms of the Detweiler function, which can be related to the Teukolsky function.
- It turns out that the Love number is non-zero, if and only if

$$\mathcal{R}(\omega) = 1 + iM\omega\mathcal{R}_1$$

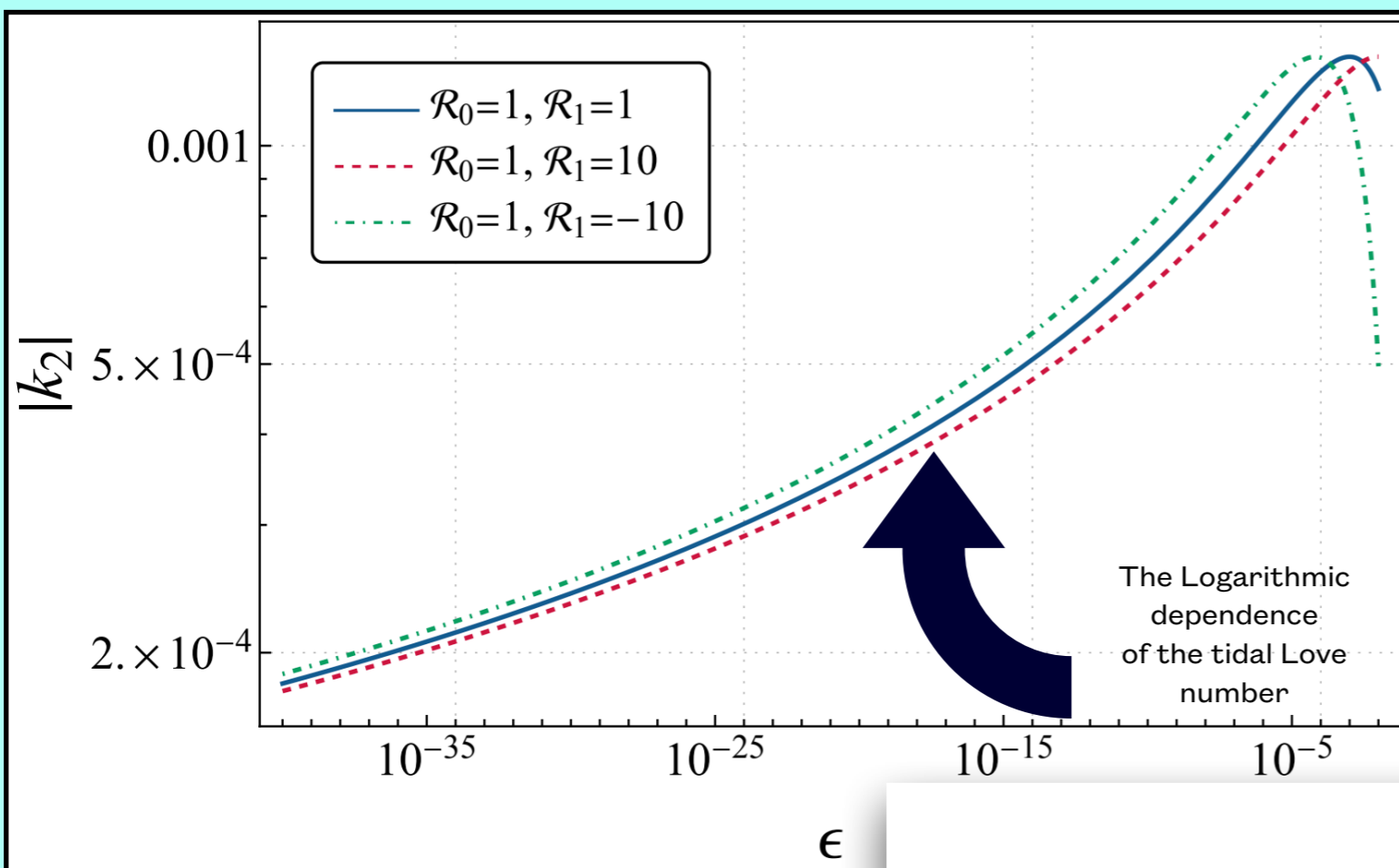


# Dynamical Love for Compact Object

- In general, the tidal Love number of compact object vanishes, except when the zero-frequency reflectivity is unity.

$$k_2 = \frac{2}{15} \text{Re} \left[ \frac{1}{-2\mathcal{R}_1 + \{7 + 16i\pi + 8(\epsilon + \ln \epsilon)\}} \right]$$

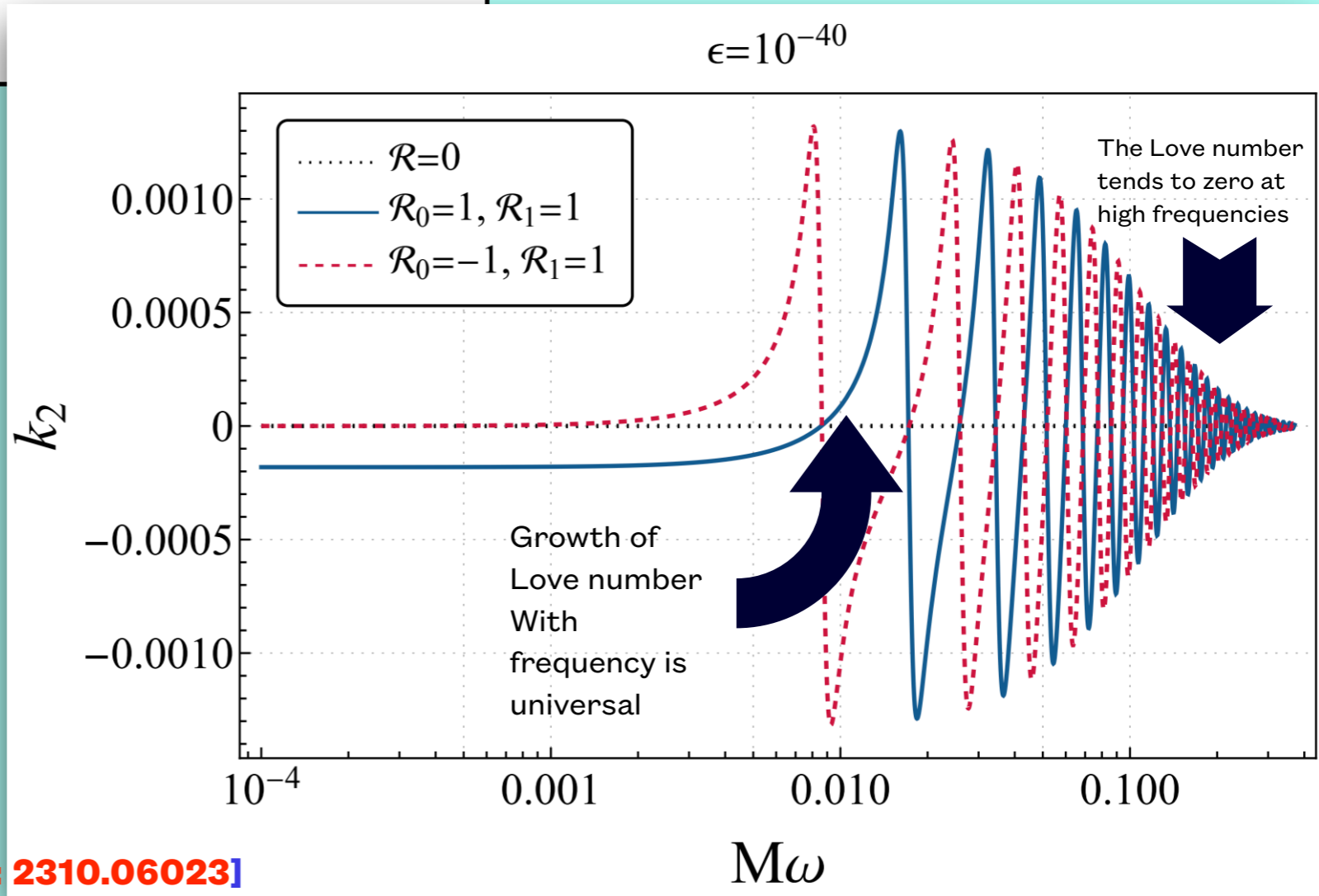
- The Logarithmic behaviour is retained.  
[SC, Maggio, Silvestrini and Pani, arXiv: 2310.06023]
- Does not match with the static result in the zero-frequency limit — the solution of the Teukolsky equation connected with the outgoing mode at the horizon diverges.
- Static limit must be taken separately.



← The frequency dependent Part of the reflectivity takes over

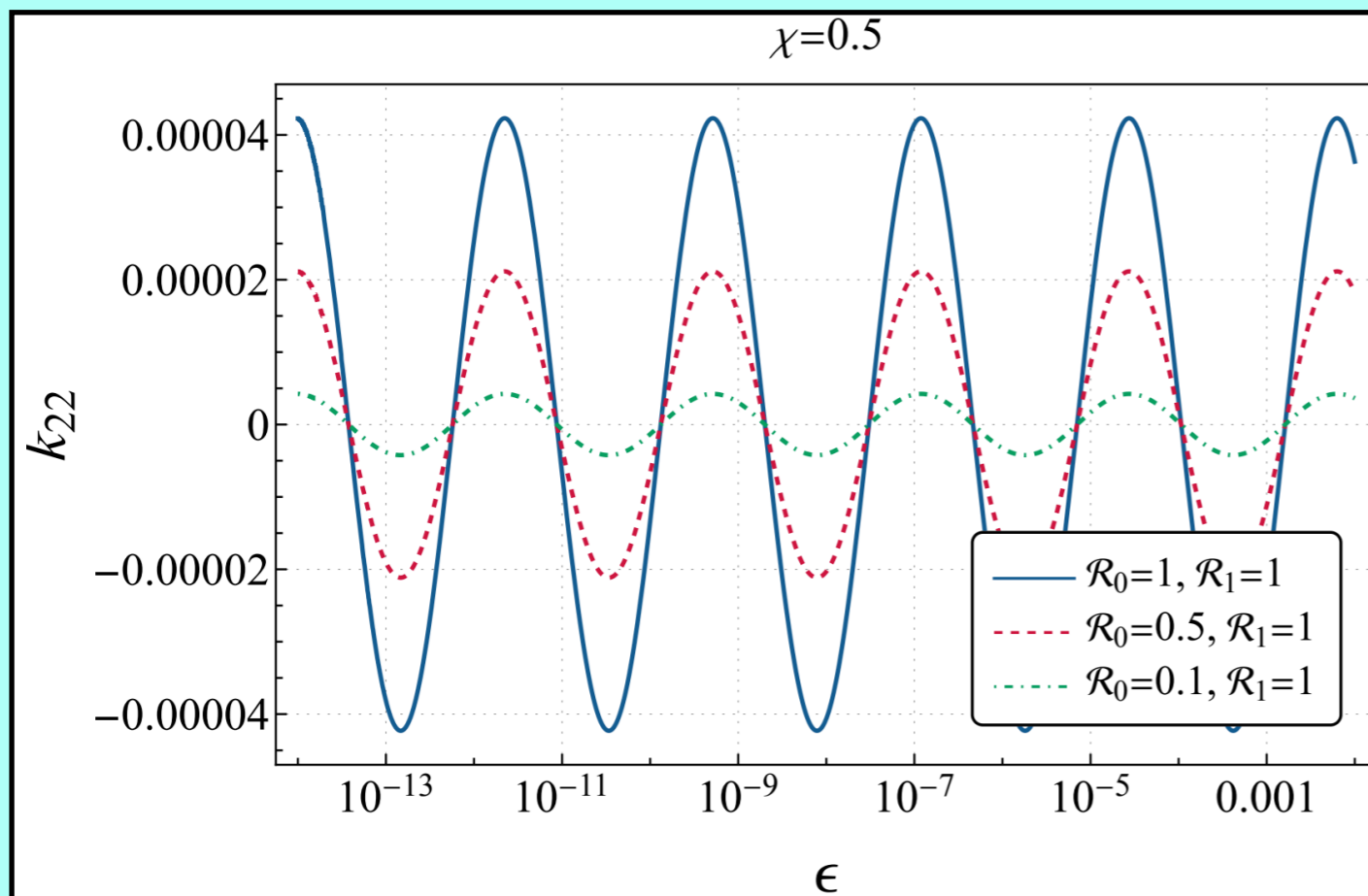
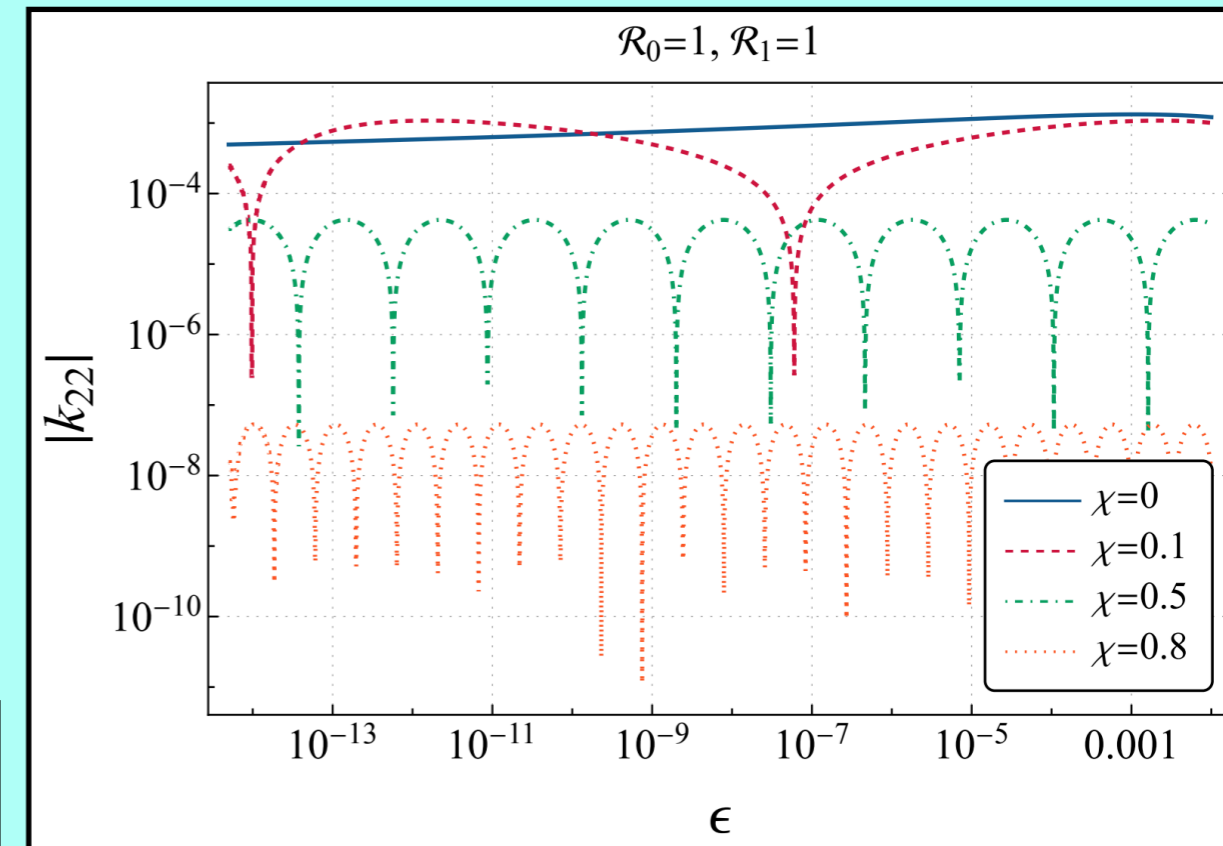
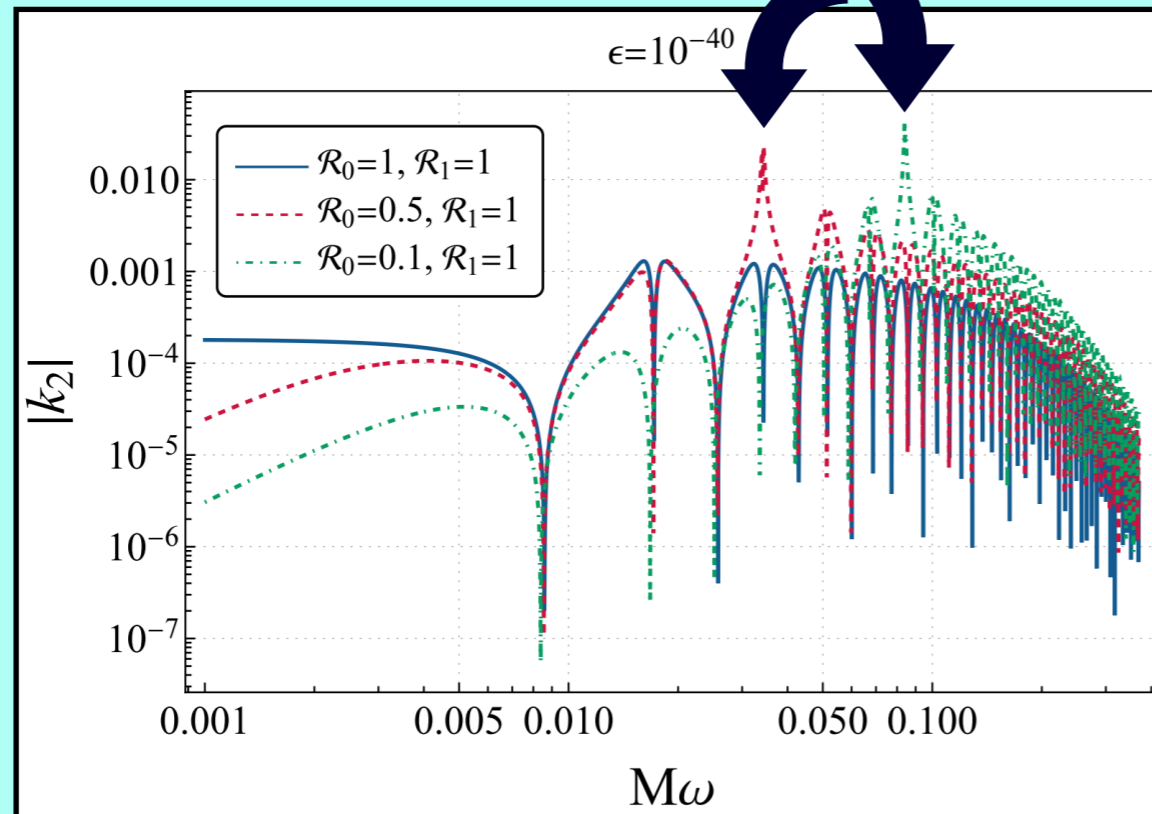
- The Logarithmic behaviour is specific to unit reflectivity.
- Love number heavily depends on the phase of the reflectivity.

- The tidal Love number grows with frequency.
- At higher frequencies the Love number tends to zero → the compact object effectively behaves as a BH.



[SC, Maggio, Silvestrini and Pani, arXiv: 2310.06023]

# Resonances at QNM frequencies



- There are resonances in the Love number at QNM frequencies.
- Increasing rotation and decreasing reflectivity decreases the Love number drastically.

[SC, Maggio, Silvestrini and Pani, arXiv: 2310.06023]





# Near-Horizon and Love Number

- Schwarzschild dS has two horizons.

$$f(R) = 1 - \frac{2M}{R} - H^2 R^2$$

$$HR_c = \left(1 - HM - \frac{3}{2}H^2 M^2 + \mathcal{O}(H^3 M^3)\right)$$

$$R_h = M \left(2 + 8H^2 M^2 + \mathcal{O}(H^3 M^3)\right)$$

- Keeping only leading order terms of  $z = (R - R_h)/R_h$ , the perturbation equation can be solved.
- Subsequent expansion revealed a non-zero Love number, dependent on the dimensionless combination  $H^2 M^2$ .

$$k_\ell^{(2)} = -\frac{24\ell(\ell+1)}{2\ell+1} H^2 M^2 \text{Im}(\mathcal{F}_\ell^{\text{Sch}})$$

$$\text{near } R^{(2)}(y) = (1+y)^{2iM\omega} y^{-2iM\omega} {}_2F_1[-\tilde{\ell}, \tilde{\ell}+1, 1-4iM\omega, -y]$$

$$\tilde{\ell} = \ell + \frac{24\ell(\ell+1)}{2\ell+1} H^2 M^2 .$$

[Nair, SC and Sarkar, arXiv: 2401.06467]



# Questions Again

- **Q1:** How to remove gauge ambiguities?
- **Ans1:** One needs to use Weyl scalar, which by definition is gauge invariant.
- **Q2:** What happens in a dynamical situation?
- **Ans2:** The Weyl scalar should be expanded and response function is to be determined, whose real part give the dynamical tidal Love numbers.
- **Q3:** Do BHs have vanishing dynamical tidal Love numbers?
- **Ans3:** Yes, for non-rotating BHs in asymptotically flat spacetimes. Not for frequency dependent part of rotating BH and for BHs in asymptotically dS spacetimes.
- **Q4:** Is the Logarithmic behaviour retained for compact objects?
- **Ans4:** Yes, the Logarithmic behaviour is retained, but only for specific reflectivity and the static limit must be taken carefully.



# Conclusion

- **Non-rotating BHs have zero tidal Love number.**
- **Rotating BHs have non-zero tidal Love number.**
- **Tidal Love number is gauge dependent.**
- **BHs can have non-zero tidal Love number at quadratic order in the frequency.**
- **Static limit of tidal Love number must be taken carefully.**
- **Tidal Love number depends on asymptotics.**
- **Asymptotically de Sitter BH has non-zero tidal Love number.**



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# Thank You