The Universality of the diagonal model, or the Abelianization of the Gauss constraint in Loop Quantum Cosmology

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Two points program

- Introduction to nondiagonal Bianchi models
  - Minisuperspace and Ashtekar variables
  - Flux quantization procedure

[Montani, MB ‘23]
[gr-qc] 2302.03638

- Abelianization of the Gauss constraint
  - Gauge freedom and canonical transformation
  - Revised Gauss Constraint and Quantum-level implications

[Montani, MB ‘23]
[gr-qc] 2306.10934

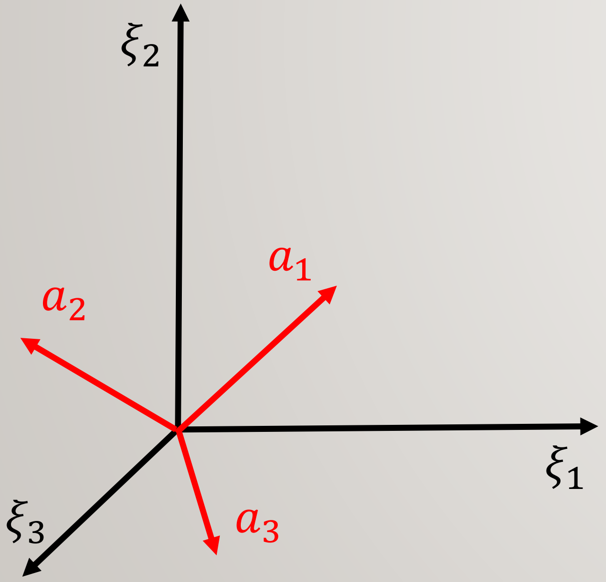
Nondiagonal Bianchi models
Minisuperspace

[Landau, Lifshits ‘74]
[Belinski ‘14]
[Montani, MB ‘23]

Globally hyperbolic spacetime $\mathcal{M} = \mathbb{R} \times \Sigma$

Homogeneous space Σ prescription $q_{ij}(t,x) = \eta_{IJ}(t)\omega_i^I(x)\omega_j^J(x)$

Nondiagonal metric decomposition $\eta_{IJ} = \Gamma_{AB} R_I^A R_J^B$



Maurer-Cartan equation

$$d\omega^I + \frac{1}{2} f^I_{JK}\omega^J\omega^K = 0$$

Lie algebra generators

$$[\xi_I, \xi_J] = f^K_{IJ}\xi_K$$

Metric configuration variables $\{a_1, a_2, a_3, \theta, \psi, \phi\}$

Nondiagonal Bianchi models

Ashtekar variables

Lagrangian

$$L_{ADM} = N|\det(\omega^I_i)|\sqrt{\det(\Gamma_{AB})}\Big[\bar{R} + \frac{1}{4N^2}(\Gamma^{AC}\Gamma^{BD}\dot{\Gamma}_{AB}\dot{\Gamma}_{CD} + 2\Gamma^{AB}\Gamma_{CD}(R\dot{\Lambda})^D_A(R\dot{\Lambda})^C_B + 2(R\dot{\Lambda})^B_C(R\dot{\Lambda})^C_B$$

$$+ 2N^A N^B\big(f^I_{AJ}f^J_{BI} + \eta^{IJ}\eta_{KL}f^K_{AI}f^L_{BJ}\big) + 4N^K\eta^{IJ}\dot{\eta}_{JL}f^L_{KI} - \Gamma^{IJ}\dot{\Gamma}_{IJ}\Gamma^{KL}\dot{\Gamma}_{KL})\Big]$$

Ashtekar connection

$$A^a_i = \left[\frac{1}{2}\epsilon^{abc}\frac{a_c}{a_b}\Lambda^J_b R^c_K f^K_{IJ} - \frac{1}{4}\epsilon^{abc}\frac{1}{a_b a_c}\eta_{IJ}\Lambda^K_b\Lambda^L_c f^J_{LK} + \frac{\gamma}{2N}a_{(a)}R^a_L\big(\eta^{LJ}\dot{\eta}_{JI} + N^A\eta^{LK}\eta_{IJ}f^J_{AK} + N^A f^L_{AI}\big)\right]\omega^I_i$$

Electric field

$$E^i_a = |\det(\omega^I_i)|\,\mathrm{sgn}\big(a_{(a)}\big)\,|a_b a_c|\Lambda^I_a\xi^i_I$$

Nondiagonal Bianchi I model

Ashtekar variables

Lagrangian

$$L_{ADM} = N|\det(\omega_i^I)|\sqrt{\det(\Gamma_{AB})}\Big[\cancelto{0}{\bar{R}} + \frac{1}{4N^2}\big(\Gamma^{AC}\Gamma^{BD}\dot{\Gamma}_{AB}\dot{\Gamma}_{CD} + 2\Gamma^{AB}\Gamma_{CD}(R\dot{\Lambda})_A^D(R\dot{\Lambda})_B^C + 2(R\dot{\Lambda})_C^B(R\dot{\Lambda})_B^C$$

$$+ \cancelto{0}{2N^AN^B\left(f_{AJ}^I f_{BI}^J + \eta^{IJ}\eta_{KL}f_{AI}^K f_{BJ}^L\right)} + \cancelto{0}{4N^K\eta^{IJ}\dot{\eta}_{JL}f_{KI}^L} - \Gamma^{IJ}\dot{\Gamma}_{IJ}\Gamma^{KL}\dot{\Gamma}_{KL}\big)\Big]$$

Ashtekar connection

$$A_i^a = \left[\cancelto{0}{\frac{1}{2}\epsilon^{abc}\frac{a_c}{a_b}\Lambda_b^J R_K^c f_{IJ}^K} - \cancelto{0}{\frac{1}{4}\epsilon^{abc}\frac{1}{a_b a_c}\eta_{IJ}\Lambda_b^K\Lambda_c^L f_{LK}^J} + \frac{\gamma}{2N}a_{(a)}R_L^a\left(\eta^{LJ}\dot{\eta}_{JI} + \cancelto{0}{N^A\eta^{LK}\eta_{IJ}f_{AK}^J} + \cancelto{0}{N^A f_{AI}^L}\right)\right]\omega_i^I$$

Electric field

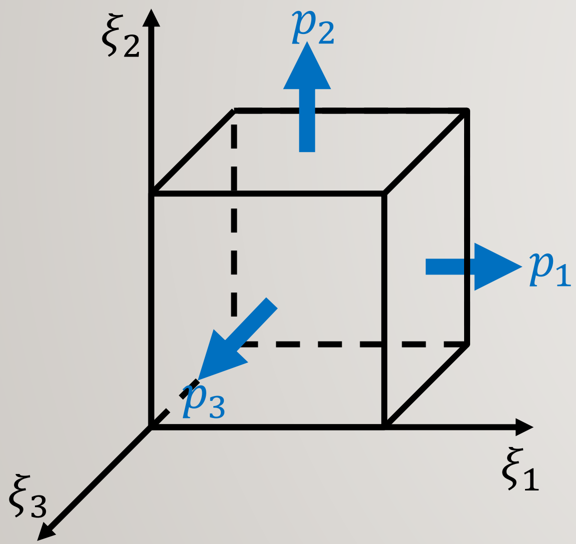
$$E_a^i = |\det(\omega_i^I)|\,\mathrm{sgn}\left(a_{(a)}\right)|a_b a_c|\Lambda_a^I\xi_I^i$$

Nondiagonal Bianchi I model

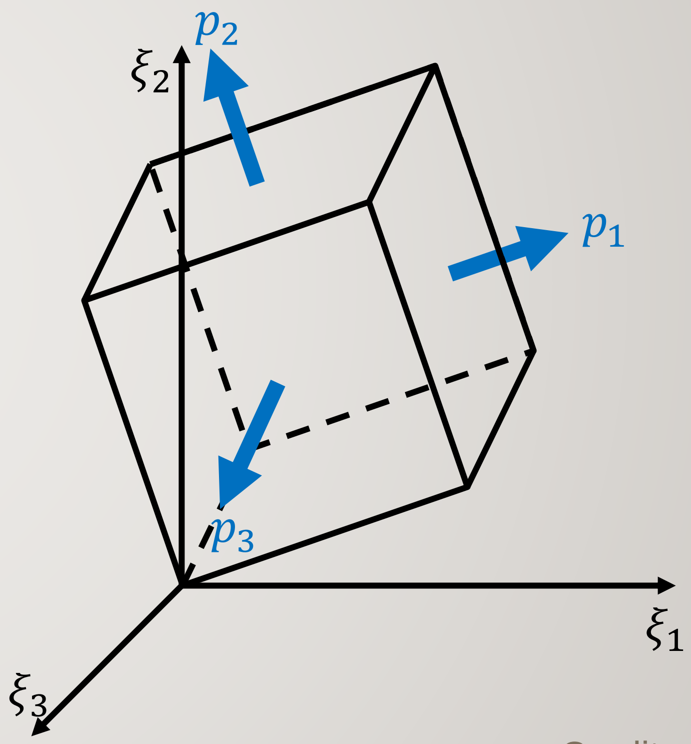
Flux quantization

Quantization in the flux polarization as in [Ashtekar, Wilson-Ewing '09]

The fluxes computed on the faces of the fiducial cell



Fiducial volume
$V_0 = L_1 L_2 L_3$



Credits: Beatrice Gorga

The fluxes have the same form of the diagonal case

Nondiagonal Bianchi I model

Hilbert space

$$\begin{cases} p_1 = a_2 a_3 \\ p_2 = a_1 a_3 \\ p_3 = a_1 a_2 \end{cases}$$

Basis states of the Hilbert space: $|p_1, p_2, p_3, \theta, \psi, \phi\rangle$

$$\mathcal{H} = \bigoplus_{a \in SO(3)} \mathcal{H}_a^{diag}$$

Kinematical geometric operators

Volume operator $Vol = V_0 \sqrt{p_1 p_2 p_3}$
Area operator $Ar(\sigma_i) = |p_i| L_j L_k \epsilon_{ijk}$

The kinematic depends only on the “diagonal” fluxes

Holonomy along the i-th edge

$$h_{l_i} = \exp(\lambda_i \phi_i^a \tau_a)$$

$$\phi_i^a = \begin{pmatrix} L_1 c_1 & L_2 w_3 & L_3 w_2 \\ L_1 w_3 & L_2 c_2 & L_3 w_1 \\ L_1 w_2 & L_2 w_1 & L_3 c_3 \end{pmatrix}$$

w_i functions of fluxes and angles, linear in angles velocities

Nondiagonal Bianchi I model

Dynamics

$$H = -\frac{1}{8\pi\gamma G}(p_1p_2c_1c_2 + p_2p_3c_2c_3 + p_1p_3c_1c_3)$$

$$+\left(\frac{p_1}{p_2} - \frac{p_2}{p_1}\right)^{-2}\pi_\theta^2$$

$$+\left(\frac{p_2}{p_3} - \frac{p_3}{p_2}\right)^{-2}\left(-\cos\theta\cot\psi\,\pi_\theta - \sin\theta\,\pi_\psi + \cos\theta\csc\psi\,\pi_\phi\right)^2$$

$$+\left(\frac{p_1}{p_3} - \frac{p_3}{p_1}\right)^{-2}\left(-\sin\theta\cot\psi\,\pi_\theta + \cos\theta\,\pi_\psi + \sin\theta\csc\psi\,\pi_\phi\right)^2$$

Free diagonal part

Quadratic in angles momenta

No holonomies

Abelianization of the Gauss constraint

Gauge freedom

$$G_a|_{\wp_{Mt}} = 0 \qquad \wp_{Mt} = \{a_1, a_2, a_3, \theta, \psi, \phi, \pi_1, \pi_2, \pi_3, \pi_\theta, \pi_\psi, \pi_\phi\}$$

M. Bojowald's suggestion in [Bojowald '00, '13]

$$\left.\begin{aligned} A_i^a(t,x) &= \phi_I^a(t)\omega_i^I(x) \\ E_a^i(t,x) &= |\det(\omega(x))| p_a^I(t)\xi_I^i(x) \end{aligned}\right\} \quad G_a = \epsilon_{abc}\phi_I^b p_c^I$$

Mismatch in the number of degrees of freedom!

Recover the gauge freedom adding a rotation

$$\wp_{\overline{Mt}} = \{a_1, a_2, a_3, \theta, \psi, \phi, \alpha, \beta, \gamma, \pi_1, \pi_2, \pi_3, \pi_\theta, \pi_\psi, \pi_\phi, \pi_\alpha, \pi_\beta, \pi_\gamma\}$$

Three abelian constraints $\begin{cases} \pi_\alpha = 0 \\ \pi_\beta = 0 \\ \pi_\gamma = 0 \end{cases}$

Abelianization of the Gauss constraint

Canonical transformation

Lie condition $\phi_I^a dp_a^I - \pi_n dq_n = 0$ provides, perturbative in configurational variables, a linear dependence between Gauss constraint and gauge momenta

Ansatz

Gauss constraint is linear in the gauge momenta $G_a = L_{ag}\pi_g$

System of 9 independent equations $\epsilon_{abc} = L_{ag}(O^t)_d^c \frac{\partial O_b^d}{\partial q_g}$

Admits a unique solution!

$$L_{ag} = \begin{pmatrix} -\csc\beta\cos\gamma & \sin\gamma & \cot\beta\cos\gamma \\ \csc\beta\sin\gamma & \cos\gamma & -\cot\beta\sin\gamma \\ 0 & 0 & 1 \end{pmatrix}$$

Abelianization of the Gauss constraint

The Abelian contraints

$$G_a = \begin{pmatrix} -\csc\beta\cos\gamma\,\pi_\alpha & \sin\gamma\,\pi_\beta & \cot\beta\cos\gamma\,\pi_\gamma \\ \csc\beta\sin\gamma\,\pi_\alpha & \cos\gamma\,\pi_\beta & -\cot\beta\sin\gamma\,\pi_\gamma \\ 0 & 0 & \pi_\gamma \end{pmatrix}$$

From a SU(2) symmetry, three U(1) appear!

The Gauss constraint is recast into three abelian constraints, namely the gauge momenta [Loran '02]

This feature holds at the quantum level $\hat{G}_a|\Psi\rangle = 0 \iff \hat{\pi}_g|\Psi\rangle = 0$

The wavefunction factorizes $\Psi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3, \alpha, \beta, \gamma) = \varphi(\alpha, \beta, \gamma)\Phi(p_1, p_2, p_3, \theta_1, \theta_2, \theta_3)$

$$\hat{\pi}_g|\Psi\rangle = 0 \;\Rightarrow\; \varphi = const$$

The Hilbert space previously defined is the gauge-invariant one

Abelianization of the Gauss constraint

The relation with SU(2)

$\{G_a, G_b\}_{\wp_{\overline{Mt}}} = \epsilon_{abc} G_c$ The Gauss constraint has the usual commutation relation on $\wp_{\overline{Mt}}$

Considering the following parametrization of $SO(3)$ $R = \exp \alpha_i j_i$ j_i generators of $\mathfrak{so}(3)$

$$G = \left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \frac{1}{2}\begin{pmatrix} 0 & \alpha_3 & -\alpha_2 \\ -\alpha_3 & 0 & \alpha_1 \\ \alpha_2 & -\alpha_1 & 0 \end{pmatrix} + \mathcal{O}(\alpha^2)\right]\begin{pmatrix} \pi_1 \\ \pi_2 \\ \pi_3 \end{pmatrix}$$

$\in \mathfrak{so}(3) \cong \mathfrak{su}(2)$

Conclusions

- The diagonal quantization in LQC can be extended to the nondiagonal case, in which the kinematic and the geometric operators conserve their simple expression
- The Abelianization of the quantum theory is a feature of the minisuperspace. The three U(1) symmetries arise from decomposing the Gauss constraint in three abelian ones
- The loop quantization of the diagonal case is quite general within the minisuperspace approach. The introduction of nondiagonal terms and the gauge freedom yield to the same kinematical picture

Thank you for your attention

The Universality of the diagonal model in Loop Quantum Cosmology

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