Revisiting compaction functions for PBH formation

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This talk is based on
TH, C.M. Yoo & Y. Koga PRD108 043515 (2023) [arXiv:2304.13284].

PBH formation in inflationary cosmology

- Cosmological perturbations can directly collapse to black holes, which are called primordial black holes (PBHs).
  - Fluctuations generated in inflation get stretched to super-horizon scale.
  - After inflation, they are described by long-wavelength solutions.
  - Once they enter the horizon, the long-wavelength scheme breaks down.
  - Only a nonlinearly large amplitude can collapse to a PBH in RD.

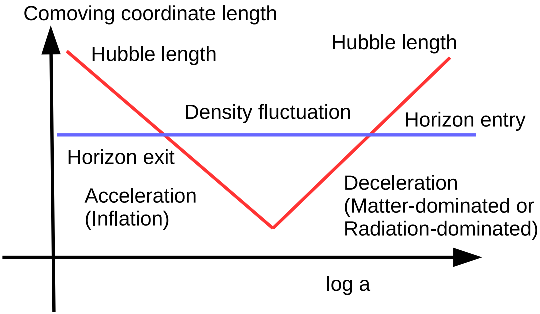


Figure: The time evolution of the Hubble length and the fluctuation scale

Cosmological conformal 3+1 decomposition

- Metric

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2(t)\tilde{\gamma}_{ij}(dx^i + \beta^i dt)(dx^j + \beta^j dt),$$

where $\tilde{\gamma} = \eta$ with η_{ij} being the flat 3D metric.

- $a(t)$: the scale factor of the flat FLRW solution
- Flat FLRW: $\alpha = 1$, $\beta^i = 0$, $\psi = 1$ and $\tilde{\gamma}_{ij} = \eta_{ij}$

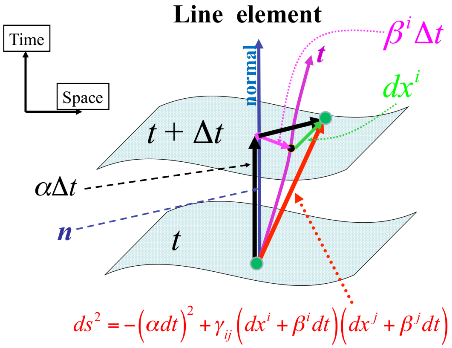
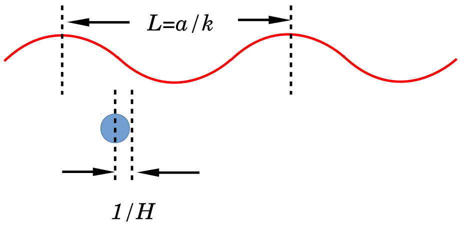


Figure: Slicing and threading with $\gamma_{ij} = \psi^4 a^2(t)\tilde{\gamma}_{ij}$

Long-wavelength limit

- A smoothing length is $L = a/k$, below which it is described by the FLRW, while the Hubble length is $\ell_H := H^{-1}$, where $H = \dot{a}/a$.
- Expansion parameter $\epsilon \ll 1$

$$\epsilon := \frac{\ell_H}{L} = \frac{k}{aH} \quad \text{with} \quad \frac{\partial_i \ln \psi}{aH} = O(\epsilon)$$



- In the decelerated expansion, the limit $\epsilon \to 0$ realises as $t \to 0$.

Long-wavelength solutions

- Gradient expansion. The Einstein eqs imply the following:
  - Metric: $\psi = \Psi(\mathrm{x})(1+\xi)$, $\alpha = 1+\chi$ and $\tilde{\gamma}_{ij} = \eta_{ij} + h_{ij}$.

    $$\Psi(\mathrm{x}) = O(\epsilon^0), \xi = O(\epsilon^2), \beta^i = O(\epsilon), \chi = O(\epsilon^2), h_{ij} = O(\epsilon^2)$$

  - Matter: $T^{\mu\nu} = (\rho + p)u^\mu u^\nu + p g^{\mu\nu}$, $\rho = \rho_b(1+\delta)$, $v^i = u^i/u^t$

    $$\delta = O(\epsilon^2), \quad v^i = O(\epsilon)$$

  - Extrinsic curvature:
    $K_{ij} = A_{ij} + (K/3)\gamma_{ij}$, $K = -3H(1+\kappa)$, $\tilde{A}_{ij} = \psi^{-4}a^{-2}A_{ij}$

    $$\kappa = O(\epsilon^2), \ \tilde{A}_{ij} = O(\epsilon^2)$$

Shibata & Sasaki (1999), Lyth, Malik & Sasaki (2005)

Shibata-Sasaki compaction function

- Shibata and Sasaki (1999) used the conformally flat coordinates

$$ds^2 = -\alpha^2 dt^2 + \psi^4 a^2(t)[(dr + \beta r dt)^2 + r^2 d\Omega^2],$$

  in spherical symmetry in the CMC slice, which is compatible with the LWL scheme.

- They calculated the Misner-Sharp (Kodama) mass excess and the compaction function as follows:

$$\begin{aligned} \delta M_{\rm SS} &:= 4\pi a^3 \rho_0 \int_0^r x^2 dx \delta_{\rm CMC} \psi^6 \left(1 + \frac{2x}{\psi}\frac{\partial \psi}{\partial x}\right) \\ \mathcal{C}_{\rm SS} &:= \frac{\delta M_{\rm SS}(t,r)}{R(t,r)} = \frac{\delta M_{\rm SS}(t,r)}{r\psi^2(t,r)a}, \end{aligned}$$

  where $R(t,r) = r\psi^2(t,r)a(t)$ is the areal radius

- $\mathcal{C}_{\rm SS}$ in the LWL solution becomes time-independent in the limit $\epsilon \to 0$ or $t \to 0$.

$\mathcal{C}_{\rm SS}$ gives a PBH formation threshold

- $\delta_{\rm CMC}(t,r)$ and $\mathcal{C}_{\rm SS}(r)$

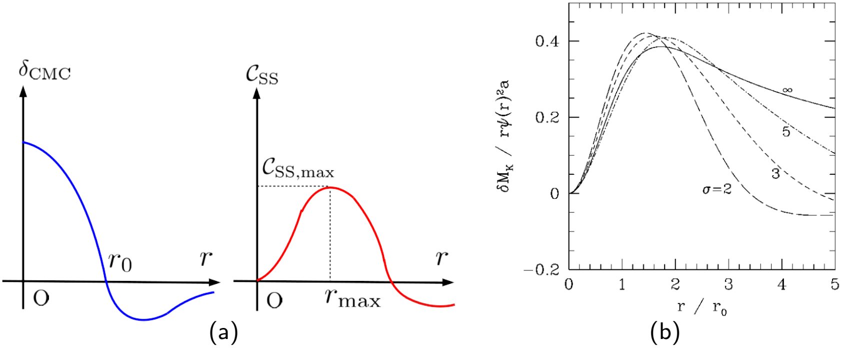


Figure: (a) $\delta_{\rm CMC}(t,r)$, $\mathcal{C}_{\rm SS}(r)$, (b) $\mathcal{C}_{\rm SS}$ for the critical cases

- Empirically, the maximum of $\mathcal{C}_{\rm SS}(r)$ gives a robust threshold $\simeq 0.4$ for PBH formation for radiation $\Gamma = 4/3$, where $p = (\Gamma - 1)\rho$.

Shibata & Sasaki (1999), Harada, Yoo, Nakama & Koga (2015), c.f. Escriva et al. (2019)

$\mathcal{C}_{\rm SS}$ in terms of Ψ

- LWL soln in the CMC slice in spherical symmetry

$$\begin{aligned}\delta_{\rm CMC} &\approx f\left(\frac{1}{aH}\right)^2, \; u_{{\rm CMC}r} \approx \frac{2}{3\Gamma(3\Gamma+2)H}\delta_{{\rm CMC},r}, \\ \Psi &= \Psi(r), \;\; f = f(r) = -\frac{4}{3}\frac{1}{r^2\Psi^5}\left(r^2\Psi'\right)'\end{aligned}$$

- $\mathcal{C}_{\rm SS}$ in the LWL soln can be rewritten as

$$\mathcal{C}_{\rm SS} \approx \frac{1}{2}\left[1-\left(1+2\frac{d\ln\Psi}{d\ln r}\right)^2\right]$$

This does not contain Ψ''.

TH, Yoo, Nakama & Koga (2015)

Mass and mass excess

- In reality, the Misner-Sharp mass also contains the contribution from the velocity perturbation.

$$
\begin{aligned}
M &= 4\pi \int_0^r x^2 dx a^3 \alpha \psi^6 T^t{}_\mu K^\mu \\
&= 4\pi a^3 \int_0^r dx (\psi^2 x)^2 \left\{ -[(\rho + p)u^t u_t + p](\psi^2 x)' \right. \\
&\quad \left. + (\rho + p) u^t u_r \frac{x}{a} (\psi^2 a)_{,t} \right\},
\end{aligned}
$$

where $K^\mu := -\epsilon^{AB}\partial_B R \left(\partial/\partial x^A\right)^\mu$.

- Mass excess
  - The mass excess from the flat FLRW spacetime is naturally defined as

$$
\delta M(t,r) = M(t,r) - M_{\mathrm{FF}}(t, \psi^2(t,r)r),
$$

i.e., the difference between masses enclosed by two spheres of the same areal radius.

$\mathcal{C}_{\rm SS} \neq \mathcal{C}_{\rm CMC}$

- Mass excess

$$\begin{aligned}\delta M_{\rm CMC} &\approx 4\pi a^3\rho_b \int_0^r dx(\Psi^2 x)^2 \left[\delta_{\rm CMC}(\Psi^2 x)' + \frac{2}{3(3\Gamma+2)}\delta'_{\rm CMC}(\Psi^2 x)\right] \\ &= 4\pi a^3 \rho_b \left[\frac{3\Gamma}{3\Gamma+2}\int_0^r dx(\Psi^2 x)^2(\Psi^2 x)'\delta_{\rm CMC} + \frac{2}{3(3\Gamma+2)}\delta_{\rm CMC}(t,r)(\Psi^2(r)r)^3\right],\end{aligned}$$

where integration by parts is implemented. This reduces to

$$\begin{aligned}\mathcal{C}_{\rm CMC} &:= \frac{\delta M_{\rm CMC}}{R} \\ &\approx \frac{3\Gamma}{3\Gamma+2}\mathcal{C}_{\rm SS}(r) + \frac{1}{3\Gamma+2}\delta_{\rm CMC}(t,r)(aH)^2(\Psi^2(r)r)^2.\end{aligned}$$

$\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm com}$

- In the comoving slice, which is compatible with the LWL solns,

$$\begin{aligned} \delta_{\rm com} &\approx \frac{3\Gamma}{3\Gamma+2} f \left(\frac{1}{aH}\right)^2, \quad u_{{\rm com}j} = 0, \\ \delta M_{\rm com}(t,r) &\approx \frac{3\Gamma}{3\Gamma+2} \delta M_{\rm SS}(t,r) \end{aligned}$$

- The legitimate compaction function in the comoving slice is thus directly related to $\mathcal{C}_{\rm SS}$ as

$$\mathcal{C}_{\rm com}(r) := \frac{\delta M_{\rm com}}{R} \approx \frac{3\Gamma}{3\Gamma+2} \mathcal{C}_{\rm SS}(r).$$

Summary

- Despite the initial intention, $\mathcal{C}_{\rm SS} \neq \delta M/R$ in the CMC slice but $\mathcal{C}_{\rm SS}$ happens to $\mathcal{C}_{\rm com}$ up to a constant factor.

- Both $\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm com}$ give a robust threshold for PBH formation $\mathcal{C}_{\rm SS} \simeq 0.4$ or $\mathcal{C}_{\rm com} \simeq 2/3 \times 0.4$ for radiation.

$\mathcal{C}_{\rm SS}$ and $\mathcal{C}_{\rm CMC}$

- $\mathcal{C}_{\rm SS}$ does not contain Ψ'' or δ. This is why $\mathcal{C}_{\rm SS}$ is empirically robust.
- Gedanken experiment:
  - Assume a spiky shell of trivial mass $\propto r_1^2\Delta^{1/2}$. Then, the perturbation will not collapse to a BH but disperse due to strong pressure gradient force.

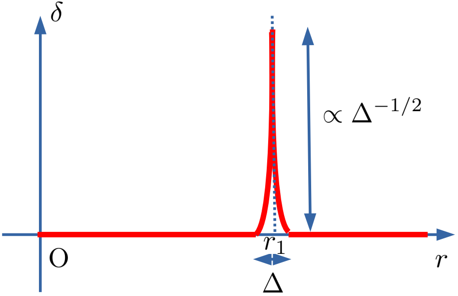


Figure: Spiky spherical shell

  - In this case, $\mathcal{C}_{\rm CMC}(r)$ has a large maximum of $O(\Delta^{-1/2})$ at r_1, whereas both $\mathcal{C}_{\rm SS}(r)$ and $\Psi(r)$ are kept small.