[1] A dynamical systems formulation for inhomogeneous LRS-II spacetimes

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[1] JCAP 06 (2024) 054

Going beyond ΛCDM

- Cosmological constant problem, H_0 problem, σ_8 problem etc inspires us to look for models beyond the standard model of cosmology, i.e. beyond ΛCDM. Such models describe a dynamical dark energy, as opposed to a "constant" dark energy given by the cosmological constant.

- These alternative models are constructed by adding one or more additional dynamical degrees of freedom to the standard model based on GR, either by adding additional matter components (canonical or non-canonical fields, non-ideal fluids or combinations thereof) or altering the theory of gravity altogether (scalar-tensor theories dominating this direction).

- There are clear problems associated with both approaches. For additional matter components, one may ask their explanation from the particle physics side. On the other hand, modified gravity theories are usually highly constrained by local gravity tests and many a time plagued by various instabilities.

An interesting way to go beyond

- An interesting alternative line of explanation is based on relaxing the assumption of homogeneity. Since a full non-perturbative treatment of generic inhomogeneous spacetime dynamics is very difficult, the toy model of spherically symmetric inhomogeneous spacetimes, namely the Lemaítre-Tolman-Bondi (LTB) model is usually considered.

- One class of works have studied LTB models with a purely formal notion of acceleration [1, 2, 3, 4]. Another class of works tried to explain the observed luminosity distance-redshift relation without any notion of an acceleration [5, 6, 7, 8]. Of course, such models also have their fair share of observational constraints.

- Two explicit theorems were proved by Mustapha, Hellaby and Ellis which make it clear that homogeneity cannot be proven without either a fully determinate theory of source evolution or availability of distance measures independent of the source evolution [9], which makes it worthwhile to study inhomogeneous cosmology.

Dynamical systems in cosmology

- Dynamical system analysis is an invaluable tool for studying the dynamics of homogeneous spacetimes, since the governing equations are ordinary differential equations. It has been used extensively for studying dark energy models based on FLRW. For inhomogeneous spacetimes, it cannot be applied in a straightforward manner because the governing equations are partial differential equations. Some attempts *are* there.
- The *orthonormal frame based approach* by Wainwright and Uggla [10, 11, 12, 13] is quite mathematical intricate: even the constraint equations are not algebraic but differential. Sussman's approach in terms of quasi-local variables [14, 15] has algebraic constraints, but the variables do not directly related to actual covariant quantities relevant to a comoving observer.
- I will talk about a new dynamical system formulation for inhomogeneous LRS-II spacetimes based on the semitetrad 1+1+2 covariant decomposition approach introduced by Clarkson [16, 17], which itself is built up on the semitetrad 1+3 covariant decomposition approach popularized by Ellis [18, 19]. In our approach the dynamical variables are directly related to actual covariant quantities (like Wainwright and Uggla) but the constraints are algebraic as well (like Sussman).

Covariant 1+3 and 1+1+2 splitting

- 1+3 covariant decomposition splits all the covariant quantities into a part along the timelike unit 4-vector u^μ ($u^\mu u_\mu = -1$) of the congruence of cosmological fundamental observers and one perpendicular to it. Metric decomposes as $g_{\mu\nu} = U_{\mu\nu} + h_{\mu\nu}$, where $U_{\mu\nu} = -u_\mu u_\nu$ is the projector along u^μ and $h_{\mu\nu}$ is the projector on the 3-surface orthogonal to u^μ ($h_{\mu\nu} u^\nu = 0$).
- The specific choice of the congruence (i.e. u^μ) is called a "frame" choice. 1+3 decomposition leaves us with scalars, 3-vectors, 3-tensors ("3-quantities" lie completely on the 3-space and orthogonal to u^μ).
- 1+1+2 covariant decomposition further splits all the "3-quantities" into a part that lies along a "preferred" spacelike unit 3-vector e^μ ($e^\mu e_\mu = 1$) that lie completely on the 3-surface and one perpendicular to e^μ (the "sheet"). $h_{\mu\nu}$ further decomposes as $h_{\mu\nu} = E_{\mu\nu} + N_{\mu\nu}$, where $E_{\mu\nu} = e_\mu e_\nu$ is the projector along e^μ and $N_{\mu\nu}$ is the projector on the 2-sheet orthogonal to both u^μ and e^μ ($N_{\mu\nu} u^\mu = 0 = N_{\mu\nu} e^\mu$).
- 1+1+2 decomposition leaves us with scalars, 2-vectors, 2-tensors ("2-quantities" lie completely on the 2-sheet and are orthogonal to both u^μ and e^μ).

LRS-II spacetime

- For LRS spacetimes, if the preferred 3-vector e^μ is chosen along the LRS direction (i.e. direction of the local rotational symmetry), 2-vectors and 2-tensors identically vanish, leaving us only with scalars. For a scalar ψ, covariant derivatives along u^μ, e^μ are $\dot{\psi} = u^\mu \nabla_\mu \psi = u^\mu \partial_\mu \psi$ and $\hat{\psi} = e^\mu D_\mu \psi = e^\mu h^\nu_\mu \nabla_\nu \psi = e^\mu \partial_\mu \psi$
- We consider the LRS-II class (which contains LTB as a subcase)

$$ds^2 = -\frac{1}{A^2(t,r)} dt^2 + B^2(t,r) dr^2 + C^2(t,r)[dy^2 + D_k^2(y) dz^2],$$

where t, r are affine parameters along u^μ and e^μ, $k = (-1, 0, 1)$ denotes open, flat and closed geometry of the 2-sheets respectively. The function $D_k(y)$ is

$$D_k(y) = \begin{cases} \sin y, & k = +1 \\ y, & k = 0 \\ \sinh y, & k = -1 \end{cases} .$$

LRS-II spacetime: continued...

The covariant scalars characterizing the LRS-II dynamics in presence of dust are

- θ; the expansion of the congruence with the timelike vector u^μ.
- ϕ; the expansion of the 2-sheet along the LRS direction.
- Σ; projection of the shear tensor along the LRS direction.
- $\mathcal{E}$; projection of the electric part of the Weyl curvature tensor along LRS direction.
- ${}^3R$; the 3-curvature of the 3-space orthogonal to u^μ.
- μ; fluid energy density measured locally by a cosmological fundamental observer.

For a perfect fluid the acceleration vector $a^\mu = u^\nu \nabla_\nu u^\mu$ for the timelike congruence vanishes, implying that the worldline of the fundamental observers are geodesics. Since the time coordinate t is the affine parameter along u^μ, we can identify the congruence as the congruence of comoving geodesics.

LRS-II covariant 1+1+2 equations

LRS-II covariant 1+1+2 equations in presence of dust are

Evolution equations:

$$\dot{\phi} = -\frac{1}{2}\phi\left(\frac{2}{3}\theta - \Sigma\right),$$
$$\dot{\theta} = -\frac{1}{3}\theta^2 - \frac{3}{2}\Sigma^2 - \frac{1}{2}\mu,$$
$$\dot{\mu} = -\theta\mu,$$
$$\dot{\Sigma} = -\frac{1}{2}\Sigma^2 - \frac{2}{3}\theta\Sigma - \mathcal{E},$$
$$\dot{\mathcal{E}} = -\frac{3}{2}\mathcal{E}\left(\frac{2}{3}\theta - \Sigma\right) - \frac{1}{2}\mu\Sigma.$$

Propagation equations: Spacelike constraints conserved in time.

$$\hat{\phi} = -\frac{1}{2}\phi^2 + \left(\frac{1}{3}\theta + \Sigma\right)\left(\frac{2}{3}\theta - \Sigma\right) - \frac{2}{3}\mu - \mathcal{E},$$
$$\hat{\Sigma} - \frac{2}{3}\hat{\theta} = -\frac{3}{2}\phi\Sigma,$$
$$\hat{\mathcal{E}} - \frac{1}{3}\hat{\mu} = -\frac{3}{2}\phi\mathcal{E}.$$

- **Friedmann constraint**: $\frac{\theta^2}{9} + \frac{{}^3R}{6} = \frac{\mu}{3} + \frac{\Sigma^2}{4}$.
- For any scalar ψ, $\dot{\hat{\psi}} - \hat{\dot{\psi}} = \left(\frac{1}{3}\theta + \Sigma\right)\hat{\psi}$.

LRS-II covariant 1+1+2 equations rewritten

The idea is to promote the covariant derivatives along the LRS direction, i.e. the $\hat{(..)}$ variables to seperate dynamical variables and calculate their evolution equation using the commutation relation. The propagation equations now act as purely algebraic constraints helping us to eliminate some $\hat{(..)}$ variables. The reduced system is

$$\begin{aligned}
\dot{\phi} &= -\frac{1}{2}\phi\left(\frac{2}{3}\theta - \Sigma\right), \\
\dot{\theta} &= -\frac{1}{3}\theta^2 - \frac{3}{2}\Sigma^2 - \frac{1}{2}\mu, \\
\dot{\mu} &= -\theta\mu, \\
\dot{\Sigma} &= -\frac{1}{2}\Sigma^2 - \frac{2}{3}\theta\Sigma - \mathcal{E}, \\
\dot{\mathcal{E}} &= -\frac{3}{2}\mathcal{E}\left(\frac{2}{3}\theta - \Sigma\right) - \frac{1}{2}\mu\Sigma, \\
\dot{\hat{\theta}} &= -\theta\hat{\theta} - 3\Sigma\hat{\theta} + \frac{9}{2}\Sigma^2\phi - \frac{1}{2}\hat{\mu}, \\
\dot{\hat{\mu}} &= -\left(\frac{4}{3}\theta + \Sigma\right)\hat{\mu} - \mu\hat{\theta}.
\end{aligned}$$

A dynamical system formulation for LRS-II

- Expansion normalized dynamical variables: $x_1 \equiv \frac{\phi}{\theta}$, $x_2 \equiv \frac{3\mu}{\theta^2}$, $x_3 \equiv \frac{3}{2}\frac{\Sigma}{\theta}$, $x_4 \equiv \frac{\mathcal{E}}{\theta^2}$, $y_1 \equiv \frac{\hat{\phi}}{\theta^2}$, $y_2 \equiv \frac{3\hat{\mu}}{\theta^3}$, $y_3 \equiv \frac{3}{2}\frac{\hat{\Sigma}}{\theta^2}$, $y_4 \equiv \frac{\hat{\mathcal{E}}}{\theta^3}$, $z \equiv \frac{\hat{\theta}}{\theta^2}$.
- Friedmann constraint: $x_2 + x_3^2 = 1 + \frac{3}{2}\frac{{}^3R}{\theta^2}$.
- Dynamical equations: $(d\tau = \theta dt,\ \theta > 0)$

$$
\begin{aligned}
\frac{dx_1}{\epsilon d\tau} &= \frac{1}{6}x_1(2x_3 + x_2 + 4x_3^2),\\
\frac{dx_2}{\epsilon d\tau} &= -\frac{1}{3}x_2(1 - x_2 - 4x_3^2),\\
\frac{dx_3}{\epsilon d\tau} &= -\frac{1}{6}x_3(2 + 2x_3 - x_2 - 4x_3^2) - \frac{3}{2}x_4,\\
\frac{dx_4}{\epsilon d\tau} &= -\frac{1}{3}x_4(1 - 3x_3 - x_2 - 4x_3^2) - \frac{1}{9}x_2x_3,\\
\frac{dy_2}{\epsilon d\tau} &= -\frac{1}{3}y_2 + \frac{1}{6}y_2(3x_2 - 4x_3) - x_2z + 2x_3^2y_2,\\
\frac{dz}{\epsilon d\tau} &= -\frac{1}{6}y_2 - \frac{1}{3}z + \frac{1}{3}x_2z - 2x_3z - 2x_1x_3^2 + \frac{4}{3}x_3^2z.
\end{aligned}
$$

Some important discussions and interpretation

- Consider a covariant quantity Q. Because of the inhomogeneity of the system, an observer has at his disposal two derivatives; $\dot{Q}$ and $\hat{Q}$. The $\dot{Q}$ gives the change in the value of Q at his frame, i.e. along his particular geodesic. $\hat{Q}$ can be interpreted as a "radial perturbation" respecting the local rotational symmetry. A nonvanishing $\hat{Q}$ stops the Q-distribution from being homogeneous. An important corollary: $Q = 0 \not\Rightarrow \hat{Q} = 0$.
- Physically, our approach goes into the frame of a comoving observer and follows the dynamics from his perspective. A fixed point $\{x_1^*, x_2^*, x_3^*, x_4^*, y_2^*, z^*\}$ is characterized by the vanishing of the following covariant scalar quantities:

  $$\{\phi - x_1^*\theta, 3\mu - x_2^*\theta^2, 3\Sigma - 2x_3^*\theta, \mathcal{E} - x_4^*\theta^2, 3\hat{\mu} - y_2^*\theta^3, \hat{\theta} - z^*\theta^2\} = \{0, 0, 0, 0, 0, 0\}.$$

  Since $Q = 0 \not\Rightarrow \hat{Q} = 0$, not all the comoving observers experience the same evolutionary phase (fixed point) simultaneously.
- If $\{y_1, y_2, y_3, y_4, z\} = \{0, 0, 0, 0, 0\}$, then $\{\hat{\phi}, \hat{\theta}, \hat{\Sigma}, \hat{\mathcal{E}}, \hat{\mu}\} = \{0, 0, 0, 0, 0\}$: necessarily spatial homogeneity (example: Kantowski-Sachs). However, converse is not true (example: $\hat{\phi} \neq 0$ for FLRW).

LTB phase space: key results

- Spherically symmetric Lemaítre-Tolman-Bondi (LTB) spacetime belongs to the LRS-II class. With LTB it is possible to explain late-time cosmological observations without invoking any dark energy component [7, 8, 20, 21, 22].
- LTB metric: $ds^2 = -dt^2 + \frac{R'^2(t,r)}{1-\mathcal{K}(r)}dr^2 + R^2(t,r)d\Omega^2$.
- We get get spatially flat matter dominated epoch as a saddle fixed point (intermediate cosmological epoch), as expected.
- Milne solutions form a line of future attractors. An expanding LTB cosmology will asymptotically approach Milne (DS if we considered ΛLTB). This conclusion is in agreement with those available in the literature [23, 24].
- Schawzschild interior solutions form a past attractor for expanding LTBs.
- Spatially flat solutions do *not* form an invariant submanifold because of nonvanishing shear.

LTB phase space: key results

- Vacuum solutions constitute an invariant submanifold ($\mathcal{S}_\mu$), as expected. However, a particular comoving observer being on $\mathcal{S}_\mu$ does not mean that a neighbouring comoving observer will also be on it. Even if a particular comoving observer experiences vacuum locally, a neighbouring observer can experience locally a non-vanishing energy density because of a generally non-vanishing radial perturbation $\hat{\mu}$.
- An important invariant submanifold is $\{x_3, x_4, y_2, z\} = \{0, 0, 0, 0\}$, which contains FLRW solutions ($\mathcal{S}_{\mathrm{FLRW}}$). This correspond to vanishing $\{\Sigma, \mathcal{E}\}$ and vanishing $\{\hat{\theta}, \hat{\mu}, \hat{\Sigma}, \hat{\mathcal{E}}\}$. This is actually a homogeneous situation where *all* the comoving observers experience the same $\{\theta, \mu\}$, i.e. same matter density and spacetime geometry.
- Stability of $\mathcal{S}_{\mathrm{FLRW}}$ determines the evolution of almost-FLRW LTB geometries. We found that an almost-FLRW expanding LTB will homogenize for $0 \leqslant \frac{\mu}{\theta^2} < \frac{4}{9} - \frac{\sqrt{10}}{9}$. Qualitatively, larger matter density hinders the homogenization of an almost-FLRW expanding LTB (assists in the homogenization of an almost-FLRW contracting LTB).

Outlook for future work

- Logical next step is inclusion of a cosmological constant term (ΛLTB model) and/or a non-vanishing pressure.

- Our approach even allows for including a non-perfect fluid; still describes the dynamics from the point of view of a comoving observer (whose worldlines may not be geodesics anymore).

- Formal comparison between our 1+1+2-based dynamical system formulation and the orthonormal frame approach of Uggla-wainwright and Sussman's quasi-local variables approach. Does there exist a one-to-one correspondence between fixed points and invariant submanifolds?

- Cosmic no-hair theorem (an inflating spacetime isotropizes asymptotically) in the presence of inhomogeneity.

- Particularly interesting: gravitational collapse and cosmic censorship conjecture.

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