A geodesically complete ring wormhole

XVII Marcel Grossmann Meeting

Juan Carlos Del Águila Rodríguez

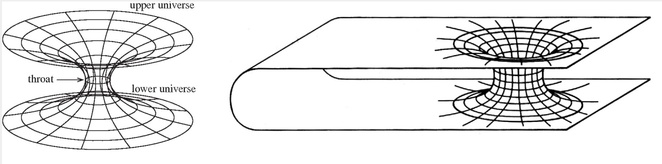
Universidad Autónoma Metropolitana, México

July 11, 2024

Outline

Wormholes and Their Issues

Wormholes. Hypothetical space-times with a non-trivial tropology that would allow communication between distant regions of the same universe, or even between differente universes.

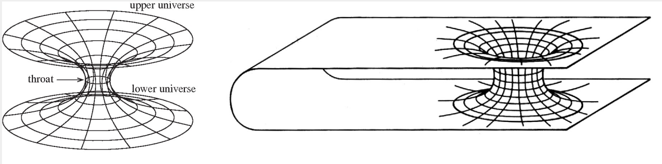


[1] M. S. Morris and K. S. Thorne, American Journal of Physics **56**, 395 (1988).

[2] K. A. Bronnikov and J. C. Fabris, Classical and Quantum Gravity **14**, 831 (1997).

Wormholes and Their Issues

Wormholes. Hypothetical space-times with a non-trivial tropology that would allow communication between distant regions of the same universe, or even between differente universes.



One of the main issues regarding their realistic existence is:

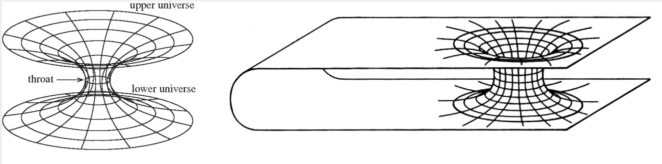
- Regular, spherically symmetric wormholes violate the null energy condition in standard General Relativity[1].

[1] M. S. Morris and K. S. Thorne, American Journal of Physics **56**, 395 (1988).

[2] K. A. Bronnikov and J. C. Fabris, Classical and Quantum Gravity **14**, 831 (1997).

Wormholes and Their Issues

Wormholes. Hypothetical space-times with a non-trivial tropology that would allow communication between distant regions of the same universe, or even between differente universes.



One of the main issues regarding their realistic existence is:

- Regular, spherically symmetric wormholes violate the null energy condition in standard General Relativity[1].

Ring wormholes[2] are capable of avoiding the previous problem, but introduce another mayor obstacle: a curvature singularity bounding their throats.

[1]M. S. Morris and K. S. Thorne, American Journal of Physics **56**, 395 (1988).

[2]K. A. Bronnikov and J. C. Fabris, Classical and Quantum Gravity **14**, 831 (1997).

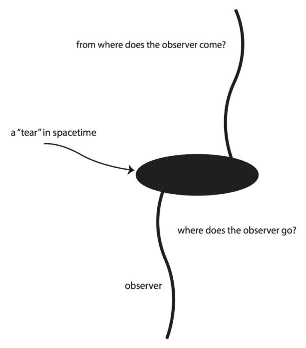
The Problem With Singularities

Curvature singularities:

- Divergent or undefined curvature invariants.
- Can cause geodesic incompleteness.

Incomplete curves in spacetime

- Singularities do not belong to the spacetime manifold.
- Curves are helpful tools to identify missing regions (holes).
- **A spacetime with incomplete geodesics is singular.**



A geodesic $\gamma(\lambda)$ is complete if it is defined for all values $\lambda \in \mathbb{R}$ of its affine parameter λ.

A spacetime is geodesically incomplete if it contains at least an incomplete geodesic.

The Electromagnetic Dipole Wormhole

The Lagrangian that generates this wormhole solution is

$$\mathscr{L} = R - 2\varepsilon\nabla_\mu\Phi\nabla^\mu\Phi - e^{-2\alpha\Phi}F_{\mu\nu}F^{\mu\nu}, \tag{1}$$

Field Equations

$$\begin{aligned} R_{\mu\nu} =& 2\varepsilon\nabla_\mu\Phi\nabla_\nu\Phi + 2e^{-2\alpha\Phi}\left(F_{\mu\rho}F_\nu^{\ \rho} - \frac{1}{4}g_{\mu\nu}F_{\delta\gamma}F^{\delta\gamma}\right), \\ \nabla_\mu(e^{-2\alpha\Phi}F^{\mu\nu}) = 0, &\quad \nabla^\mu\nabla_\mu\Phi = \frac{\alpha}{2}e^{-2\alpha\Phi}F_{\delta\gamma}F^{\delta\gamma}, \end{aligned}$$

where:

- R is the Ricci scalar,
- $F_{\mu\nu}$ is the electromagnetic field tensor,
- Φ is the scalar field of a zero spin particle,
- $\varepsilon = +1$ for a dilatonic field and $\varepsilon = -1$ for a phantom field,
- α is a coupling constant. Interesting cases are $\alpha^2 = 1$ (a low-energy string theory), and $\alpha^2 = 3$ (a 5D Kaluza-Klein theory).

Its line element in oblate spheroidal coordinates $\{t, x, y, \varphi\}$ is [3]

$$ds^2 = -(dt + \Omega d\varphi)^2 + e^K \Delta \left(\frac{L^2 dx^2}{\Delta_1} + \frac{dy^2}{1 - y^2} \right) + \Delta_1 (1 - y^2) d\varphi^2, \qquad (2)$$

with $\Delta = L^2(x^2 + y^2)$, $\Delta_1 = L^2(x^2 + 1)$, and $\Omega = aLx(1 - y^2)/\Delta$. Also,

$$K = \frac{k}{L^4} \frac{\left[1 - y^2\right] \left[8x^2 y^2 (x^2 + 1) - (1 - y^2)(x^2 + y^2)^2\right]}{(x^2 + y^2)^4}, \qquad \text{where: } k = \frac{a^2}{8} \left(1 - \frac{4\varepsilon}{\alpha^2} \right).$$

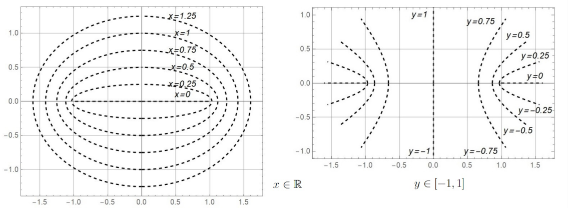


Figure: Profiles for different fixed values of x (left) and y (right). Here $L = 1$.

Asymptotically:

For spherical coordinates $\{r, \theta, \varphi\}$,

$$Lx \to r, \quad y \to \cos\theta.$$

[3] G. Miranda, J. C. Del Aguila, and T. Matos, Phys. Rev. D **99**, 124045 (2019).

The Electromagnetic Dipole Wormhole

	k	
α^2	Dilatonic Field ($\varepsilon = 1$)	Ghost Field ($\varepsilon = -1$)
1	$-3a^2/8$	$5a^2/8$
3	$-a^2/24$	$7a^2/24$

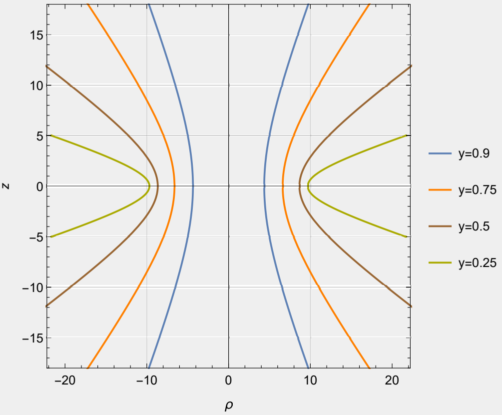


Figure: Embedding profiles of the electromagnetic dipole wormhole.

Principal Features:

- It is axially symmetric and stationary ($\partial/\partial t$ y $\partial/\partial\varphi$ are Killing vectors).
- Angular momentum: $J = a$.
- $L > 0$ is related to the size of the throat.
- Is asymptotically flat (if $a = 0 \rightarrow$ flat space-time).
- Its throat is located at $x = 0$.
- Curvature singularity σ of radius L at $x = y = 0$ bounding the throat.

Principal Features:

- It is axially symmetric and stationary ($\partial/\partial t$ y $\partial/\partial\varphi$ are Killing vectors).
- Angular momentum: $J = a$.
- $L > 0$ is related to the size of the throat.
- Is asymptotically flat (if $a = 0 \to$ flat space-time).
- Its throat is located at $x = 0$.
- Curvature singularity σ of radius L at $x = y = 0$ bounding the throat.

Scalar field and four-potential

$$\Phi = \underbrace{\frac{ay}{\alpha L^2(x^2+y^2)}}_{\text{Dipole-like scalar field}}, \quad A = -\frac{e^{\alpha\Phi}}{2}\left[(1-e^{-\alpha\Phi})dt + \Omega d\varphi\right]. \tag{3}$$

In an orthonormal frame $\{\hat{t}, \hat{x}, \hat{y}, \hat{\varphi}\}$:

$$F_{\hat{\mu}\hat{\nu}} = \underbrace{\frac{a}{L^3x^3}\begin{bmatrix} 0 & y & -\sqrt{1-y^2}/2 & 0 \\ -y & 0 & 0 & -\sqrt{1-y^2}/2 \\ \sqrt{1-y^2}/2 & 0 & 0 & -y \\ 0 & \sqrt{1-y^2}/2 & y & 0 \end{bmatrix}}_{\text{Electromagnetic dipole}} + \mathcal{O}\left(\frac{1}{x^4}\right).$$

Null Energy Condition (NEC)

Orthonormal vector basis $\{V_I^\mu\}$**:** $V_0 = \frac{\partial}{\partial t}$, $V_1 = \frac{e^{-K/2}}{L}\sqrt{\frac{\Delta_1}{\Delta}}\frac{\partial}{\partial x}$,

$(I = 0, 1, 2, 3)$, $V_2 = e^{-K/2}\sqrt{\frac{1-y^2}{\Delta}}\frac{\partial}{\partial y}$, $V_3 = \frac{1}{\sqrt{\Delta_1(1-y^2)}}\left(\frac{\partial}{\partial \varphi} - \Omega\frac{\partial}{\partial t}\right)$.

The stress-energy measured by an arbitrary null observer with tangent

$$k^\mu = \sum_I A_I V_I^\mu, \quad \text{where } -A_0^2 + A_1^2 + A_2^2 + A_3^2 = 0,$$

is given by

$$\begin{aligned} T_{\mu\nu}k^\mu k^\nu =& \frac{a^2 e^{-K}}{2L^6(x^2+y^2)^4}\left[A_0^2 + A_3^2\right]\left[y^2(1-y^2) + x^2(1+3y^2)\right] \\ &+ \frac{a^2 e^{-K}}{2L^6(x^2+y^2)^5}\left[a^2(A_1F_1 + A_2F_2)^2 - 8k(A_1F_2 - A_2F_1)^2\right], \end{aligned} \quad (4)$$

with $F_1 = (x^2 - y^2)\sqrt{1-y^2}$ and $F_2 = 2xy\sqrt{x^2+1}$.

Null Energy Condition (NEC)

Orthonormal vector basis $\{V_I^\mu\}$: $V_0 = \dfrac{\partial}{\partial t}$, $V_1 = \dfrac{e^{-K/2}}{L}\sqrt{\dfrac{\Delta_1}{\Delta}}\dfrac{\partial}{\partial x}$,

$(I = 0, 1, 2, 3)$, $V_2 = e^{-K/2}\sqrt{\dfrac{1-y^2}{\Delta}}\dfrac{\partial}{\partial y}$, $V_3 = \dfrac{1}{\sqrt{\Delta_1(1-y^2)}}\left(\dfrac{\partial}{\partial \varphi} - \Omega\dfrac{\partial}{\partial t}\right)$.

The stress-energy measured by an arbitrary null observer with tangent

$$k^\mu = \sum_I A_I V_I^\mu, \quad \text{where} \; -A_0^2 + A_1^2 + A_2^2 + A_3^2 = 0,$$

is given by

$$\begin{aligned} T_{\mu\nu}k^\mu k^\nu =& \frac{a^2 e^{-K}}{2L^6(x^2+y^2)^4}\left[A_0^2 + A_3^2\right]\left[y^2(1-y^2) + x^2(1+3y^2)\right] \\ &+ \frac{a^2 e^{-K}}{2L^6(x^2+y^2)^5}\left[a^2(A_1F_1 + A_2F_2)^2 - 8k(A_1F_2 - A_2F_1)^2\right], \end{aligned} \tag{4}$$

with $F_1 = (x^2 - y^2)\sqrt{1-y^2}$ and $F_2 = 2xy\sqrt{x^2+1}$.

If $k > 0$, the NEC is not guaranteed to be satisfied.

If $k \leq 0$, then $T_{\mu\nu}k^\mu k^\nu > 0$ (the NEC holds).

Analyzing the Curvature Singularity

General form of the curvature scalars in this wormhole

$$R_X = \frac{e^{-\delta K} F(x,y)}{(x^2+y^2)^{\beta}}, \tag{5}$$

where:

- $\beta, \delta \in \mathbb{Z}^+$. Recall that,

$$K = \frac{k}{L^4} \frac{\left[1-y^2\right]\left[8x^2y^2(x^2+1)-(1-y^2)(x^2+y^2)^2\right]}{(x^2+y^2)^4}.$$

- $F(x,y)$ is a polynomial with degree less than that of $(x^2+y^2)^{\beta}$.

Directional singularity. The limit $x, y \to 0$ is not well defined (it depends on the direction of approach).

General form of the curvature scalars in this wormhole

$$R_X = \frac{e^{-\delta K} F(x,y)}{(x^2+y^2)^{\beta}}, \tag{5}$$

where:

- $\beta, \delta \in \mathbb{Z}^+$. Recall that,

$$K = \frac{k}{L^4}\,\frac{\left[1-y^2\right]\left[8x^2y^2(x^2+1)-(1-y^2)(x^2+y^2)^2\right]}{(x^2+y^2)^4}.$$

- $F(x,y)$ is a polynomial with degree less than that of $(x^2+y^2)^{\beta}$.

Directional singularity. The limit $x, y \to 0$ is not well defined (it depends on the direction of approach).

What happens to geodesics in the neighborhood of the ring singularity? Do they become incomplete?

Geodesics Approaching the Singularity

Constants of motion:

- Conjugate momenta $p_0 = -\mathcal{E}$ y $p_3 = \mathcal{L}$.
- $\kappa = g^{\mu\nu} p_\mu p_\nu = \begin{cases} 0 & \text{for null geodesics,} \\ -1 & \text{for time-like geodesics.} \end{cases}$
- This wormhole does not admit a non-trivial quadratic first integral.

Consider the Hamiltonian of freely-falling particles: $2\mathcal{H} = \kappa$.

$$e^K \Delta \left(\frac{L^2 \dot{x}^2}{\Delta_1} + \frac{\dot{y}^2}{1 - y^2} \right) = \kappa + \mathcal{E}^2 - \frac{(\Omega\mathcal{E} + \mathcal{L})^2}{\Delta_1 (1 - y^2)}. \quad (6)$$

Geodesics Approaching the Singularity

Constants of motion:

- Conjugate momenta $p_0 = -\mathcal{E}$ y $p_3 = \mathcal{L}$.
- $\kappa = g^{\mu\nu} p_\mu p_\nu = \begin{cases} 0 & \text{for null geodesics,} \\ -1 & \text{for time-like geodesics.} \end{cases}$
- This wormhole does not admit a non-trivial quadratic first integral.

Consider the Hamiltonian of freely-falling particles: $2\mathcal{H} = \kappa$.

$$e^K \Delta \left(\frac{L^2 \dot{x}^2}{\Delta_1} + \frac{\dot{y}^2}{1 - y^2} \right) = \kappa + \mathcal{E}^2 - \frac{(\Omega\mathcal{E} + \mathcal{L})^2}{\Delta_1 (1 - y^2)}. \tag{6}$$

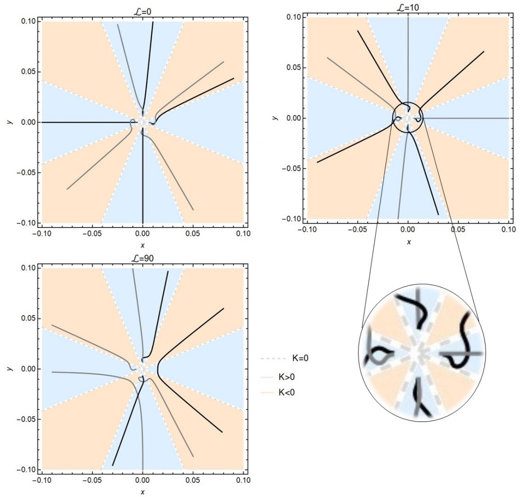
- Incomplete geodesics within the throat ($x = \dot{x} = \ddot{x} = 0$) can be found for $k > 0$ and $|y| \ll 1$ ($\mathcal{L} = 0$):

$$\lambda = \pm \int y e^{-k/2L^4 y^4} dy, \qquad \text{finite } \lambda \text{ when } y \to 0.$$

- On the other hand, if $k < 0$: $\dot{y}, \ddot{y} \to 0$ and infinite λ when $y \to 0$.

The integrability of general geodesics is not guaranteed.

Numerical solutions that are directed toward the singularity.



[4] J. Wheeler, Les Houches Lectures (1964).

[5] J. C. Del Aguila, and T. Matos, Phys. Rev. D **107**, 064047 (2023)

Numerical solutions that are directed toward the singularity.

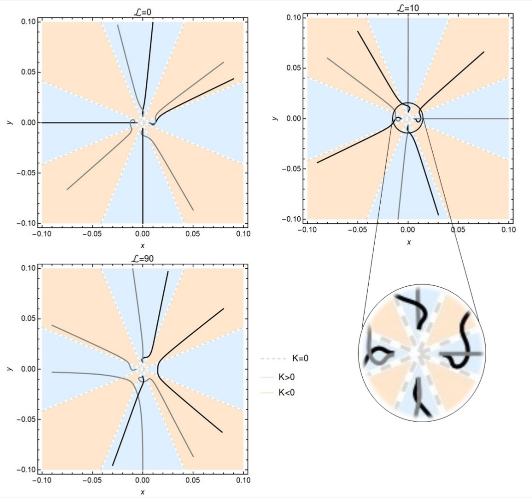
Infinite affine parameter needed to reach the singularity.

Vanishing curvature when approaching the singularity from regions with $K > 0$ (blue region).

Complete geodesics despite unbounded curvature.

Wheeler's "bag of gold" singularity.

- Infinite volume bounded by a finite superficial area[4].



[4] J. Wheeler, Les Houches Lectures (1964).

[5] J. C. Del Aguila, and T. Matos, Phys. Rev. D **107**, 064047 (2023)

Conclusions

The geodesic completeness of a wormhole with a ring singularity σ was studied. It is supported by an electromagnetic field and a (phantom or dilatonic) scalar field.

	Geodesics that encounter σ			
k	**Conditions**	**Curvature**	**Regularity**	**NEC**
Positive	Constrained to $x = 0$ (finite λ required, incomplete)	Unbounded	Singular (singularity visible to observers in the throat)	Can be violated
Negative	Infinite λ required to reach σ	Vanishing	Complete causal geodesics	$T_{\mu\nu} k^\mu k^\nu \geq 0$

Wormholes supported by a phantom field ($k > 0$) are physically problematic:

- Violation of the NEC.
- Geodesic incompleteness.

For wormholes with a dilatonic scalar field and a weak coupling ($\alpha^2 \leq 4 \to k < 0$):

- The ring singularity does not necessarily imply geodesic incompleteness.
- The null energy condition holds.

Other Comments

Interpretation of the singularity through a five-dimensional analysis using Kaluza-Klein theory (not presented here):

- Geodesiscs require an infinite parameter to reach the singularity due to the radius of the fifth dimension becoming infinite near the singular region.
- Endless paths lead to the singularity.

Other Comments

Interpretation of the singularity through a five-dimensional analysis using Kaluza-Klein theory (not presented here):

- Geodesiscs require an infinite parameter to reach the singularity due to the radius of the fifth dimension becoming infinite near the singular region.
- Endless paths lead to the singularity.

The phantom version of this wormhole is not a physically viable spacetime, but its dilatonic counterpart cannot be discarded yet.

Is the dilatonic wormhole a physically realistic model?

- What is the effect of the singularity on observers with limited acceleration?
- Incompleteness of accelerated time-like curves is still a possibility.

Thank You!

[6] Contact information: jcdelaguila@xanum.uam.mx