SOME WORDS ABOUT GRAVITATIONAL ENTROPY AND PENROSE'S WEYL CURVATURE CONJECTURE

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Phys. Rev. D 102, 023539 (2020), arXiv:2004.10222 [gr-qc]

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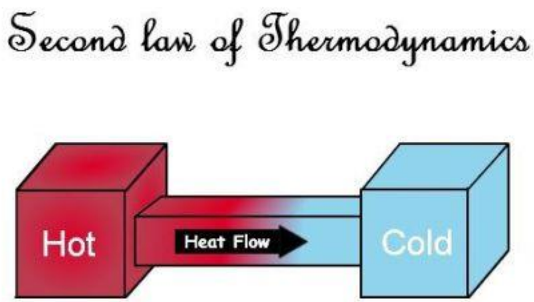
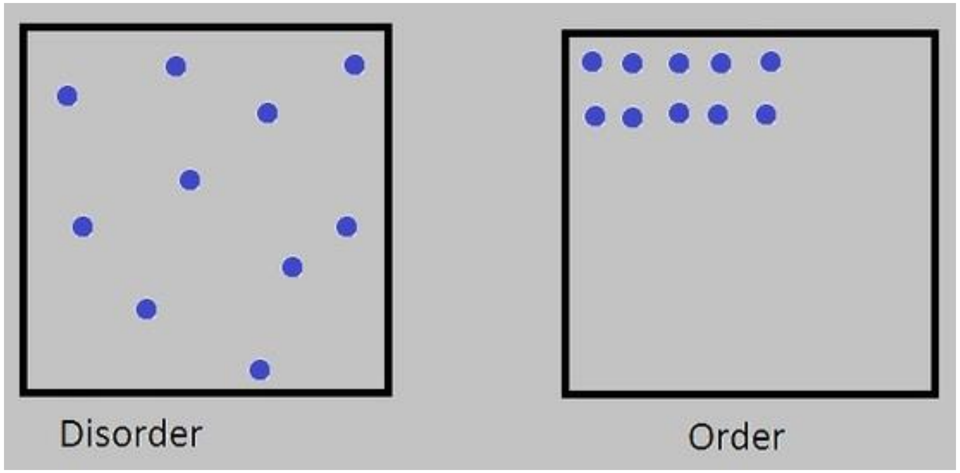
MANY DIFFERENT APPROACHES TO THE CONCEPT OF ENTROPY

- From thermodynamics: entropy as the arrow of time, entropy cannot decrease in time (Clausius);



- From statistical mechanics: as a measure of disgregation and as a quantification of the number of possible different microscopic realizations of the same macroscopic system (Maxwell, Boltzmann, Gibbs);

- From information theory: from a probabilistic perspective (von Neumann, Shannon);



- Can we assign a notion of entropy to the gravitational field?

THE PIONEERING WORKS OF HAWKING AND …

Commun. math. Phys. 31, 161–170 (1973)

The Four Laws of Black Hole Mechanics

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B. Carter and S. W. Hawking
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Received January 24, 1973

Abstract. Expressions are derived for the mass of a stationary axisymmetric solution of the Einstein equations containing a black hole surrounded by matter and for the difference in mass between two neighboring such solutions. Two of the quantities which appear in these expressions, namely the area A of the event horizon and the "surface gravity" κ of the black hole, have a close analogy with entropy and temperature respectively. This analogy suggests the formulation of four laws of black hole mechanics which correspond to and in some ways transcend the four laws of thermodynamics.

Commun. math. Phys. 25, 152—166 (1972)

Black Holes in General Relativity

S. W. Hawking
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Received October 15, 1971

Abstract. It is assumed that the singularities which occur in gravitational collapse are not visible from outside but are hidden behind an event horizon. This means that one can still predict the future outside the event horizon. A black hole on a spacelike surface is defined to be a connected component of the region of the surface bounded by the event horizon. As time increase, black holes may merge together but can never bifurcate. A black hole would be expected to settle down to a stationary state. It is shown that a stationary black hole must have topologically spherical boundary and must be axisymmetric if it is rotating. These results together with those of Israel and Carter go most of the way towards establishing the conjecture that any stationary black hole is a Kerr solution. Using this conjecture and the result that the surface area of black holes can never decrease, one can place certain limits on the amount of energy that can be extracted from black holes.

HAWKING: BLACK HOLE ENTROPY IS GIVEN BY THE HORIZON AREA + NEVER DECREASE AREA THEOREMS

THIS IS THE THERMODYNAMICAL APPROACH TO ENTROPY

… AND BEKENSTEIN

PHYSICAL REVIEW D VOLUME 7, NUMBER 8 15 APRIL 1973

Black Holes and Entropy*

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(Received 2 November 1972)

There are a number of similarities between black-hole physics and thermodynamics. Most striking is the similarity in the behaviors of black-hole area and of entropy: Both quantities tend to increase irreversibly. In this paper we make this similarity the basis of a thermodynamic approach to black-hole physics. After a brief review of the elements of the theory of information, we discuss black-hole physics from the point of view of information theory. We show that it is natural to introduce the concept of black-hole entropy as the measure of information about a black-hole interior which is inaccessible to an exterior observer. Considerations of simplicity and consistency, and dimensional arguments indicate that the black-hole entropy is equal to the ratio of the black-hole area to the square of the Planck length times a dimensionless constant of order unity. A different approach making use of the specific properties of Kerr black holes and of concepts from information theory leads to the same conclusion, and suggests a definite value for the constant. The

BEKENSTEIN: BLACK HOLE ENTROPY AS (SHANNON) INFORMATION ENTROPY

REMARKABLY THE SAME RESULT AS HAWKING WAS OBTAINED: BLACK HOLE ENTROPY IS HORIZON AREA

Testing the black-hole area law with GW150914

Maximiliano Isi,[1,*] Will M. Farr,[2,3,†] Matthew Giesler,[4] Mark A. Scheel,[5] and Saul A. Teukolsky[4,5]

COMPLETELY DIFFERENT PHYSICAL ARGUMENTS WERE USED

… BUT WHAT IS THIS ENTROPY ACTUALLY REFERRING TO?

PHYSICAL REVIEW D VOLUME 9, NUMBER 12 15 JUNE 1974

Generalized second law of thermodynamics in black-hole physics*

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(Received 17 September 1973)

crease by an amount S. Actually, the increase in S_{bh} may be even larger because any information that was available about the body to start with will also be lost down the black hole. Therefore, if we denote by ΔS_e the change in common entropy in the black-hole exterior ($\Delta S_e \equiv -S$), then we expect that

$$\Delta S_{bh} + \Delta S_e = \Delta(S_{bh} + S_e) > 0 . \qquad (19)$$

BLACK HOLE ENTROPY IS THE ENTROPY OF THE PURE GRAVITATIONAL FIELD, AND IT SHOULD NOT BE CONFUSED WITH THE ENTROPY OF A MATTER FIELD OUTSIDE THE EVENT HORIZON

SCHWARZSCHILD IS AN EMPTY SPACETIME, BUT NEVERTHELESS IT COMES WITH A NONZERO ENTROPY

WHEELER: *IN GENERAL RELATIVITY WE CAN HAVE MASS WITHOUT HAVING MATTER*

BLACK HOLE ENTROPY AS AN AREA

- FROM COLLEGE PHYSICS TEXTBOOKS: ENTROPY IS AN EXTENSIVE VARIABLE AND IT IS EXPECTED TO SCALE AS THE VOLUME OF THE PHYSICAL SYSTEM IT REFERS TO

- AN OTHER POSSIBLE WAY OF RE-INTERPRETING THE HAWKING NEVER DECREASE AREA THEOREMS: THE CHRISTODOULOU-RUFFINI IRREDUCIBLE MASS

Reversible and Irreversible Transformations in Black-Hole Physics*

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(Received 17 September 1970)

PHYSICAL REVIEW D VOLUME 4, NUMBER 12 15 DECEMBER 1971

Reversible Transformations of a Charged Black Hole*

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(Received 1 March 1971; revised manuscript received 26 July 1971)

ON THE IRREDUCIBLE BLACK HOLE MASS FORMULA

where

$$r_+ = M + (M^2 - J^2M^{-2} - Q^2)^{1/2}. \tag{8.50}$$

With these values, (8.47) reduces in this case to the exact differential of the mass formula of Christodoulou and Ruffini (1970, 1971) which takes the form

$$M = \{(M_0 + \frac{1}{4}Q^2M_0^{-1})^2 + \frac{1}{4}J^2M^{-2}M_0^{-2}\}^{1/2} \tag{8.51}$$

where M_0 is an integration constant, given by the expression

$$M_0^2 = \frac{1}{16\pi}\mathscr{A} = \frac{1}{4}\{2M^2 - Q^2 + 2(M^4 - M^2Q^2 - J^2)^{1/2}\} \tag{8.52}$$

where $\mathscr{A}$ is the surface area of the black hole (i.e. the integral of surface area over a spacelike 2-dimensional section of $\mathscr{H}^+$).

Although the basic variation formula (8.38) for a star was derived by considering *reversible* processes, we can immediately deduce that it will hold for *any* process if we assume that the equilibrium configuration is a well defined function of the distribution of J, Q, S and $N^{(s)}$ over the rotating matter rings, which will be the case under a wide range of natural conditions. Now in the case when a central black hole is present our experience with the Kerr solutions leads to the *generalized no-hair conjecture* according to which the system as a whole should have just *two additional degrees of freedom in the non-electromagnetic case, and three in the electromagnetic case* (leaving a mathematically conceivable magnetic monopole moment out of account) *in addition to the degrees of freedom* (determining the distribution of $J, Q, S, N^{(s)}$) *associated directly with the external matter rings.*

Now the formula (8.47) derived by considering reversible variations, involves just one additional degree of freedom, namely J_H, associated with the hole in the non-electromagnetic case, and just two, namely J_H and Q_H in the electromagnetic case. However it has been shown by Hawking (1971) (in the manner which he describes in the accompanying course) that the result discovered by Christodoulou and Ruffini in the special case of the Kerr – Newman black holes must be true in general, i.e. the surface area $\mathscr{A}$ of the hole must *always remain constant in any transformation which is reversible.* Taking $\mathscr{A}$ itself as the additional degree of freedom, it therefore follows from the generalized no hair conjecture that the mass variation in a completely general (not necessarily reversible) change between neighbouring black hole equilibrium states should be given by

$$\begin{aligned} dM = {} & \mathscr{T}\, d\mathscr{A} + \Omega_H \left(dJ_H + \int \delta J_F \right) + \int \Omega\, dJ_M \\ & + \Phi^H\, dQ_H + \int \Phi^S \delta Q + \int \overline{\Theta}\delta S + \int \bar{\mu}^{(s)} \delta N_{(s)} \end{aligned} \tag{8.53}$$

where the form of the coefficient $\mathscr{T}$ (which has the dimensions of surface tension) remains to be determined. Since the $\mathscr{T}\, d\mathscr{A}$ contribution (unlike all the other terms) can only be produced by a non-reversible transformation there is

Brandon Carter, Republication of: Black hole equilibrium states Part II: General theory of stationary black hole states, Golden Oldie, GRG (2020) 42:653-744

THE HOLOGRAPHIC PRINCIPLE

A black hole is fully described by the area of its horizon:

- Gerard't Hooft, Dimensional Reduction in Quantum Gravity, Conf. Proc. C 930308 (1993) 284, [arXiv:gr-qc/9310026].
- Leonard Susskind, The World as a Hologram, Jour. Math. Phys. 36 (1995) 6377, [arXiv:hep-th/9409089].

What matters is the boundary

ADDING COSMOLOGICAL MOTIVATIONS: THE WEYL CURVATURE HYPOTHESIS BY ROGER PENROSE

- IT CONJECTURES THAT THE WEYL TENSOR SHOULD BE A GOOD MEASURE OF GRAVITATIONAL ENTROPY;

- IT IS EXPECTED THAT THE BIG BANG SINGULARITY SHOULD COME WITH ZERO WEYL CURVATURE, WHEREAS BIG CRUNCHES AND BLACK HOLE SINGULARITIES DUE TO GRAVITATIONAL COLLAPSE SHOULD HAVE LARGE WEYL CURVATURE;

- DURING THE COLLAPSE OF A STAR OF MASS M, ENTROPY INCREASES BY A FACTOR OF $10^{20} (M/M_o)^{1/2}$

- SINCE WEYL CURVATURE QUANTIFIES TIDAL DEFORMATIONS, THIS IS JUST THE STATEMENT THAT WE EXPECT BLACK HOLE AND BIG CRUNCH SINGULARITIES TO EXHIBIT VERY MESSY AND CHAOTIC CURVATURE BEHAVIOR, PERHAPS LIKE THOSE IN THE BKL DESCRIPTION.

- RIEMANN CURVATURE CAN BE DECOMPOSED INTO WEYL AND RICCI CURVATURE. RICCI CURVATURE IS GIVEN BY EINSTEIN EQUATIONS ONCE THE MATTER CONTENT IS KNOWN, WHILE WEYL CURVATURE CAN BE NONZERO ALSO IN VACUUM.

IMPLEMENTING THE WEYL CURVATURE HYPOTHESIS IS NOT A SIMPLE TASK

- Clifton-Ellis-Tavakol, Class. Quant. Grav. 30 (2013) 125009.

- It has been adopted by several authors for describing the formation of astrophysical structures (galaxies, filaments, voids, overdensities,...) in late-time cosmology (assuming dust) both using exact and approximate formalisms.

- Density of the gravitational entropy: $T_{\rm grav}\dot{s}_{\rm grav} = -dV\sigma_{ab}\left(\pi^{ab}_{\rm grav} + \frac{(\rho c^2 + p)}{3\rho_{\rm grav}}E^{ab}\right)$

It is not a measure of the "pure" gravitational field because it depends directly also on ρ and p (e.g. on the matter content).

PHYSICAL REVIEW D **102,** 023539 (2020)

Thermodynamics of shearing massless scalar field spacetimes is inconsistent with the Weyl curvature hypothesis

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IMPLEMENTING THE WEYL CURVATURE HYPOTHESIS IS NOT A SIMPLE TASK

- The proposal of considering an entropy density proportional to the square of the Weyl curvature works for 5-dimensional Schwarzschild and Schwarzschild-anti-de Sitter black holes, but not for the Reissner-Nordström spacetime
- Li-Li-Song, EPJC 76 (2016) 111
- $S = \int_V C_{abcd}C^{abcd}dV$ does not admit a general applicability in black hole physics

- It was proposed to consider $S = \int_V \frac{C_{abcd}C^{abcd}}{R_{ab}R^{ab}}dV$ when studying isotropic cosmological singularities, but this proposal is directly sensitive to the matter content of the spacetime via the Ricci tensor
- Pelavas-Coley, Int. Jour. Theor. Phys. 45 (2006) 1258

FORMULATION OF THE QUESTION WE WANT TO ANSWER:

- Does an appropriate quantity χ function <u>only</u> of the Weyl curvature such that

$$S = \int_V \chi dV = \frac{A_H}{4}$$

exist for static and spherically-symmetric (possibly distorted) black holes

$$ds^2 = -f(r)[1 + h(r)]dt^2 + \frac{[1 + h(r)]dr^2}{f(r)} + r^2 d\Omega^2$$

$$h(r) = \sum_{k=0}^{\infty} \epsilon_k \left(\frac{M}{r}\right)^k ,$$

in 4 and 5 dimensions?

[For this matric ansatz see Yunes-Stein, PRD 83 (2011) 104002, Johannsen-Psaltis, PRD 83 (2011) 124015.]

OUR ANSWER: YES

Working with the Newman-Penrose formalism we can compute

$$\Psi_2 = \frac{r^2(1+h)^2 f'' + r^2 f(1+h)h'' + r(1+h)(rh' - 2h - 2)f' - f(h')^2 r^2 + 2(1+h)^2(f-1-h)}{12(1+h)^3 r^2},$$

$$D\Psi_2 = \frac{[r^2(1+h)^2 f'' + r^2 f(1+h)h'' + r(1+h)(rh' - 2h - 2)f' - (h')^2 f r^2 + 2(1+h)^2(f-1-h)]\sqrt{2f}}{8(1+h)^{7/2} r^3},$$

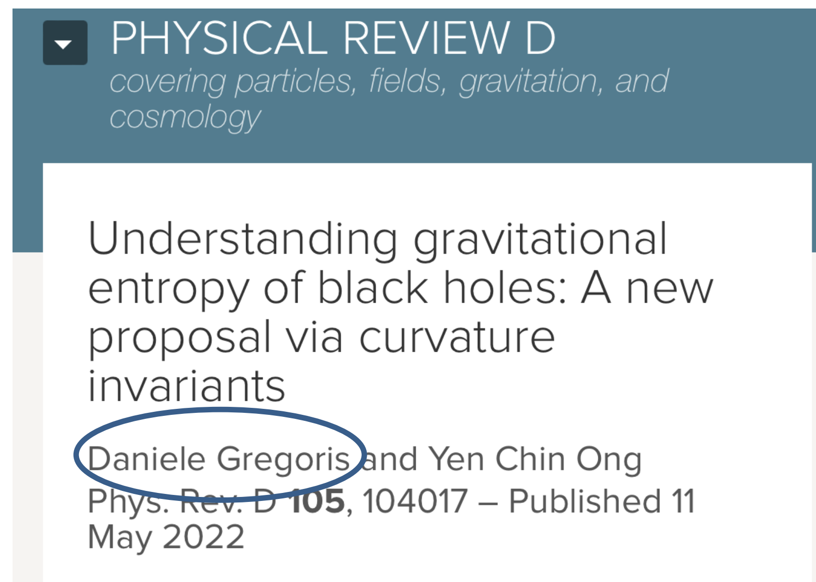
Therefore

$$S = \frac{1}{3\sqrt{2}} \int_0^{r_H} \int_\Omega \left| \frac{D\Psi_2}{\Psi_2} \right| r^2 \sqrt{\frac{1+h}{f}}\, dr\, d\Omega = \frac{A_H}{4} = 4\pi M_{\text{irr}}^2$$

Spatial hypersurface volume element

Remarks:

- Our formalism is fully based on the Weyl curvature: it is an appropriate result for a density of gravitational entropy;
- We have not made assumptions on $f(r)$: our formalism comes with a general applicability to all black hole spacetimes regardless of whether they are empty space solutions or not;
- Our formalism can be applied also to Bardeen regular black holes


PHYSICAL REVIEW D
covering particles, fields, gravitation, and cosmology

Understanding gravitational entropy of black holes: A new proposal via curvature invariants

Daniele Gregoris and Yen Chin Ong

Phys. Rev. D **105**, 104017 – Published 11 May 2022

PHYSICAL CONSIDERATIONS

What we learnt about black hole entropy in general relativity:

- Black hole entropy and irreducible mass are related to tidal effects;
- Black hole entropy and irreducible mass are a property of the focusing of light rays because we can use the expression for the Newman-Penrose spin coefficient $\rho \propto \frac{D\Psi_2}{\Psi_2}$.

a (real) convergence ρ and shear σ. The proper 2-area δA of an element of horizon changes according to

$$\frac{d\delta A}{dv} = -2\rho\delta A\,, \qquad (2)$$

where v is the affine parameter of a typical local generator. In turn ρ satisfies

$$\frac{d\rho}{dv} = \rho^2 + |\sigma|^2 + 4\pi T_{\delta\gamma} l^\delta l^\gamma\,, \qquad (3)$$

where $T_{\delta\gamma}$ is the stress-energy tensor of the matter at the horizon, and $l^\delta = dx^\delta/dv$ is the (null) tangent vector to the local generator (as well as the outgoing normal to the horizon). We assume the weak energy condition[7]: $T_{\delta\gamma} l^\delta l^\gamma \geq 0$.

If we calculate $d^2\delta A/dv^2$ from (2), eliminate first derivatives with (2) and (3), integrate (over area) for given v, and then over v from v to $v = \infty$, we get

$$\frac{dA}{dv} = 2\int_v^\infty dv' \int_H (4\pi T_{\delta\gamma} l^\delta l^\gamma + |\sigma|^2 - \rho^2)\delta A(v')\,. \qquad (4)$$

JOURNAL OF MATHEMATICAL PHYSICS VOLUME 6, NUMBER 9 SEPTEMBER 1965

The Gravitational Compass*

P. Szekeres†
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(Received 7 October 1964; final manuscript received 25 February 1965)

ENTROPY GROWS DURING THE FORMATION OF A BLACK HOLE

Take the Tolman metric IV

$$ds^2 = -A(r)dt^2 + \frac{dr^2}{B(r)} + r^2(d\theta^2 + \sin^2 d\phi^2)\,, \qquad A(r) \neq B(r)$$

$$A(r) = Y^2\left(1 + \frac{r^2}{X^2}\right)\,, \qquad B(r) = \frac{\left(1 - \frac{r^2}{R^2}\right)\left(1 + \frac{r^2}{X^2}\right)}{1 + \frac{2r^2}{X^2}}\,,$$

with X, Y and R being some constants. It describes a sphere of compressible fluid in hydrostatic equilibrium supported by a pressure vanishing at the boundary of the configuration.

We compute

$$\Psi_2 = -\frac{(X^2 + 2R^2)r^2}{3R^2(X^2 + 2r^2)^2}$$

and

$$\int_0^R \left|\frac{2(X^2 - 2r^2)}{r(X^2 + 2r^2)}\right| r^2 dr = \left[r^2 - X^2 \ln(X^2 + 2r^2)\right]\Big|_0^R = R^2 - X^2 \ln\left(1 + 2\left(\frac{R}{X}\right)^2\right) < R^2\,.$$

Thus, our method shows that the entropy within this stellar configuration is smaller than what it would be for a black hole. This is consistent with the increase of entropy during black hole formation phase.

OPEN PROBLEMS

Open question about gravitational entropy in general relativity:

- If we try to compute gravitational entropy according to our recipe in some inhomogeneous universe, do we obtain a function which is increasing in time in the same intervals in which spatial shear effects are? If yes, ours would be a good tool for investigating the formation of astrophysical structures.

Open question about black hole entropy beyond general relativity:

- Ours is a purely geometrical result because we have never used that $f(r)$ should arise as a solution of the Einstein field equations. Thus, if we apply our formula to some black hole which possesses the same symmetries but it is a solution in some modified gravity theory we still get a result which is an area.

- However, it has been argued that in modified gravitational theories, the entropy does not obey anylonger to an area law;

- So, in principle, a different combinations of curvature quantities should be adopted as a density of gravitational entropy;

- So: what is the physical foundation of black hole entropy in modified theories of gravity?