The Causal Set Path Integral and an Emerging Continuum

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Can causal sets approximate the continuum?

Two questions:

– start with spacetime manifold, approximate by causal set

– start with causal set, find suitable "smoothed" spacetime

First direction is in good shape:

– "Poisson sprinkling" of points approximates manifold
(at some density $\Leftrightarrow$ some discreteness scale)

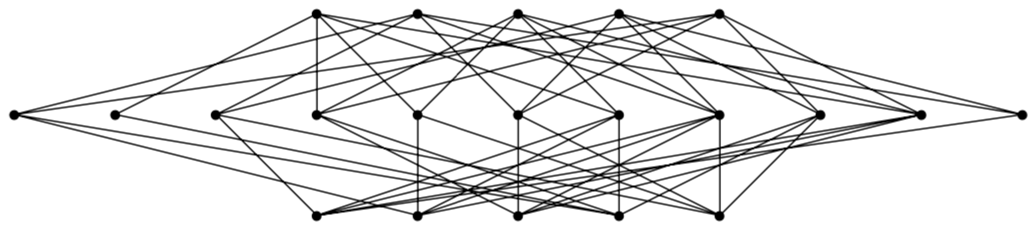
– Can reconstruct coarse-grained topology, volume, curvature,
d'Alembertian, Greens functions, etc.

– Open questions about defining sets that are "close" to each other

– Locality can be tricky

But. . .

most causal sets are nothing at all like manifolds

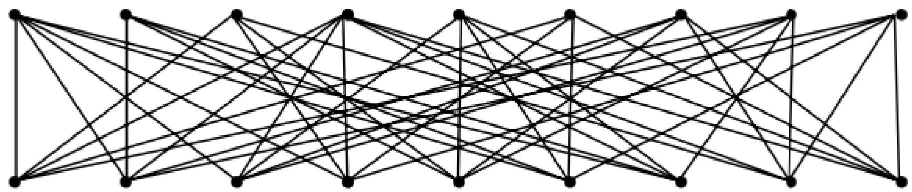


KR order

- Almost all causal sets are Kleitman-Rothschild orders (three layers/moments of time, . . .)

$$\frac{\text{\# of KR orders with } n \text{ elements}}{\text{\# of causal sets with } n \text{ elements}} = 1 + \mathcal{O}\left(\frac{1}{n}\right)$$

If these are excluded by hand. . .



2-layer set

- Almost all remaining causal sets are two-layer sets
- Then four-layer, five-layer, . . .
- Manifoldlike causal sets are of measure zero

Causal set path integral

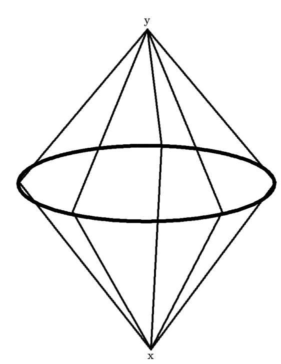
Path sum: Choose set Ω of causal sets

$$Z[\Omega] = \sum_{C \in \Omega} \exp \frac{i}{\hbar} I[C],$$

How do we construct a discrete “Einstein-Hilbert action”?

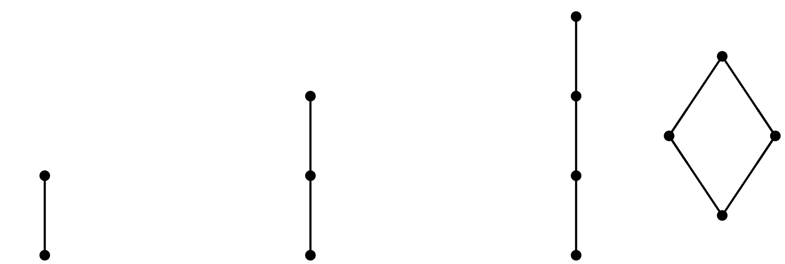
Basic ingredient: causal diamond/Alexandrov interval/order interval:

$$I(x, y) = \{z | x \prec z \prec y\}$$

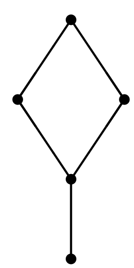


For continuum spacetimes, causal diamond volumes depend on curvature
– to make invariants from a causal set, count causal diamonds
– use to reconstruct geometry of sprinkled set

Invariant $N_J(C)$: number of (open) intervals in set C with exactly J points



$\in N_0$ $\in N_1$ $\in N_2$



$N_0\!=\!5,\ N_1\!=\!2,\ N_2\!=\!1,\ N_3\!=\!1$

Benincasa-Dowker-Glaser action:

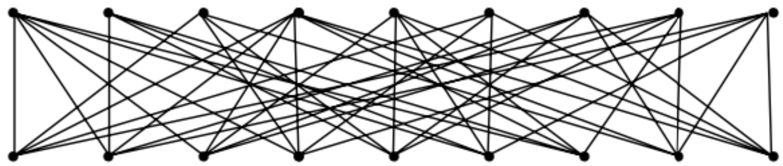
$$\frac{1}{\hbar}\boldsymbol{I}_{BDG}(\boldsymbol{C}) = \boldsymbol{\beta}_4 \left(\frac{\boldsymbol{\ell}}{\boldsymbol{\ell_p}}\right)^{\mathbf{2}} (\boldsymbol{n} - \boldsymbol{N}_\mathbf{0} + \mathbf{9}\boldsymbol{N}_\mathbf{1} - \mathbf{16}\boldsymbol{N}_\mathbf{2} + \mathbf{8}\boldsymbol{N}_\mathbf{3})$$

For “sprinkled” causal set, $\boldsymbol{I}_{BDG}$ approximates Einstein-Hilbert action

Does the BDG action suppress non-manifoldlike causal sets?

- Result 1 (S. Carlip and S. P. Loomis):

  For 2-layer sets, $\mathcal{Z} \sim 2^{-cn^2}$ for a large range of coupling constants (for $\ell > 1.136\ell_p$)



Sketch of proof:

– for two layers, only $N_0 \neq 0$, so $I_{BDG} \sim (n - N_0)$

– write $N_0 = pN_{max} = \frac{pn^2}{4}$

$$\Rightarrow \ \mathcal{Z} \sim \sum_p \mu_n(p)\, e^{-i\beta p n^2}$$

– use combinatorial arguments to bound measure $\mu_n(p)$

– approximate sum as integral, use steepest descent (carefully!)

Very strong suppression

For Planck size discreteness scale,
a region $1\,\mathrm{cm}^3 \times 1\,\mathrm{ns}$ has $n \sim 10^{133}$

$\Rightarrow$ suppression factor of $\sim 2^{-10^{266}}$

- Result 2 (A. Mathur, A. A. Singh, and S. Surya):
  - For a very large class of layered causal sets, same suppression *but* with "link action": $I_{link} \sim (n - N_0)$

Reminder: $I_{BDG} \sim \; n - N_0 + 9N_1 - 16N_2 + 8N_3$

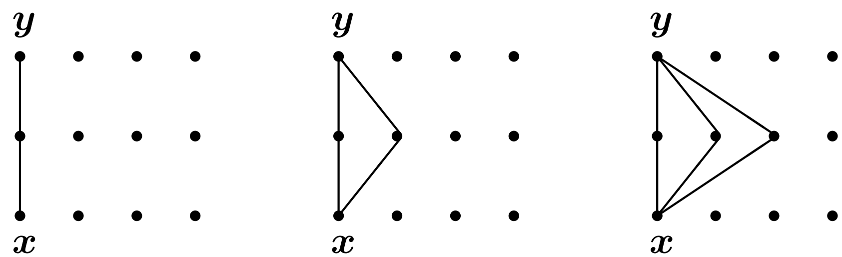
$$I_{link} \sim n - N_0 + \cancel{9N_1} - 1\cancel{6N_2} + \cancel{8N_3}$$

Proof: same as before, but more complicated combinatorics for $\mu_n(p)$

• Result 3 (P. Carlip, S. Carlip, and S. Surya):

For KR orders, I_{link} is almost always equal to I_{BDG}

Basic argument:



– intervals with *at least* one element are common

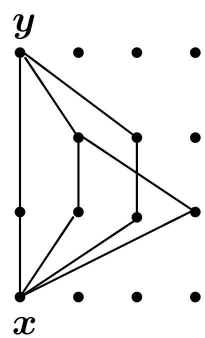
– intervals with *only* one element are very rare

⇒ for large KR orders, $N_{J>0}$, subdominant in action

(in fact, $N_0 \sim n^2$ while $N_{J>0} \sim n$)

• Result 4 (P. Carlip, S. Carlip, and S. Surya):
Same is true for almost all layered causal sets

Basic argument:



– with more layers, many more possible paths
– intervals with only small numbers of points are very rare

Note: this result only holds for layered causal sets

For sets obtained from a sprinkled manifold, $N_J \sim n^{2-\frac{2}{d}}$ for all J

Path integral suppresses

– a very large class of “bad” causal sets

– but *not* manifoldlike causal sets!

Some remaining problems:

- There are probably other “bad” causal sets
  Can they be classified, and are they suppressed?
- BDG action was derived from manifold Einstein-Hilbert action
  Can it be obtained from first principles?