Beyond Center for Fundamental Science, ASU

Quantum gravity effects in spacetimes with a fundamental length scale

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Contents

Why Pixelate?

The Consistency Conundrum

Mass & Energy Dictate Geometry

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R \quad = \quad \frac{8\pi G}{c^4} T_{\mu\nu}$$

But Matter & Energy are Quantized

- A quantum $T_{\mu\nu}$ implies a pixelated $R_{\mu\nu}$
- What does that mean? How can we describe this?
- A fundamental roadblock for quantum gravity.
- Are the continuum and even real numbers just an idealization of reality?

Where Spacetime Breaks Down?

Minimum Compton Wavelength $\lambda_c \geq r_S$

$$r_S = \frac{2GM}{c^2} \qquad r_S = \lambda_c \qquad \lambda_c = \frac{\hbar}{Mc}$$

Smallest measurable length without an event horizon

At the limit $l_P = \lambda_c = r_S$, eliminate M, ignore factor of 2

“Planck Length" $l_P = \sqrt{\frac{G\hbar}{c^3}}$

Actual length scale may be bigger

Ref: **Hossenfelder2012**

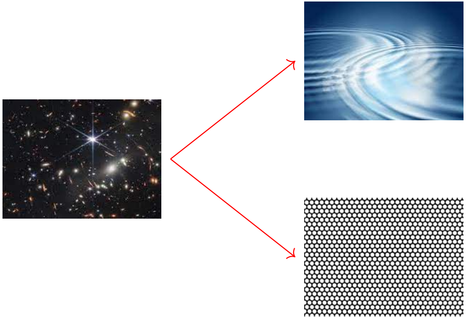
Quantum Fields & Renormalization

$$\int_0^{\Lambda} \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{p^2 - m^2 + i\epsilon}$$

- Λ defines the cut-off
- Fundamental scale would imply a $\Lambda \propto l_P^{-1}$
- Do quantum fields "work" better in pixelated spacetime?
  - Renormalization not needed to eliminate singularities!
- Non-zero minimum length = no singularities

Ref: **Snyder1947**

Hypothesis



1. Spacetime at the smallest scale is discrete, pixelated
2. Continuum is more graph like, and only a manifold in the infra red limit
3. Can this pixelation have measurable consequences for QFT?

A Phenomenology Framework

Modified Propagators to QFT

Modified propagators (massless case) lead to new Feynman rules

Modified propagators and corresponding equation of motion

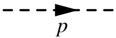
$$E^2 = p^2 + \kappa\eta^2 p^4 \leftrightarrow \frac{\partial^2\phi}{\partial t^2} - \frac{\partial^2\phi}{\partial x^2} + \kappa\eta^2\frac{\partial^4\phi}{\partial x^4} = 0$$

$$\eta \propto \frac{1}{M_P} \qquad n(k) = \frac{\sqrt{1+\kappa\eta^2 k^2}}{1+2\kappa\eta^2 k^2}$$

Leading order in inverse Planck Mass

Vacuum is now dispersive
$\kappa < 1,\ n(k) > 1$, and $\kappa > 1,\ n(k) < 1$

Altered Feynman rules for propagator



$$= \frac{1}{p_0^2 - p^2(1+\kappa\eta^2 p^2) + i\epsilon}$$

- Doubly Special Relativity (DSR) proposes modified propagators as a consequence of fundamental length
- DSR is a framework to admit an observer independent fundamental length whilst preserving Lorentz symmetries
- Propagators can be parameterized by Planck scale and for sub and super luminal propagation

Ref: Amelino-Camelia 2002

Curved Momentum Space and Measure

Lorentz covariant DSR interpreted as non-trivial momentum space geometry. Non trivial momentum space geometry has a long history tracing back to Snyder (Snyder 1947) and Born (Born 1938).

On-shell relation carries extra powers of $\vec{\mathrm{p}}$

Minimal length implies $[x, p]$ is a function of p,

or geometry is non-commutative

$\left[x^i, x^j\right] \neq 0$

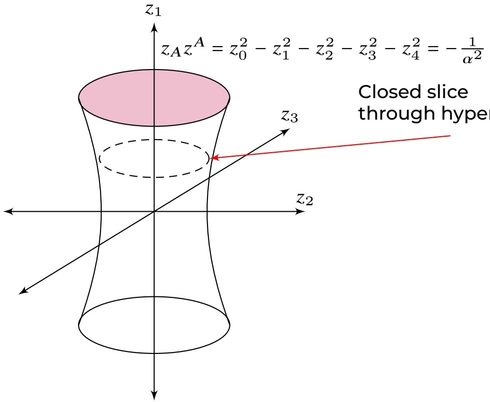
Integrals over off-shell momentum need modification of momentum integrals

$\int \mathrm{d}^n p \to \int \sqrt{-g}\mathrm{d}^n p$ to restore Lorentz invariance?

$$g^{\mathcal{M}}_{\mu\nu} p^\mu p^\nu \neq p_0^2 - \sum_i p_i^2$$

Momentum space is not flat

Choosing a Measure



- Leading order on-shell and $\sqrt{-g}$:

$$\boxed{\sqrt{-g} = 1 + \kappa\eta^2 p_0^2 \ , E^2 = p^2 + \kappa\eta^2 p^4 .}$$

- Momentum space de Sitter, curvature radius α
- Embedding in 5D Minkowski restricted to surface $z_A z^A = -\frac{1}{\alpha^2}$
- For n dimensions define $\alpha^2 = \frac{2\kappa\eta^2}{n}$ and:

$$z_0 = \left(\alpha^2\right)^{-\frac{1}{2}} \sinh\left[\left(\alpha^2\right)^{\frac{1}{2}} p_0\right],$$

$$z_i = \cosh\left[\left(\alpha^2\right)^{\frac{1}{2}} p_0\right] p_i \text{ ,with}$$

$$\sum_{i=1}^{i=n} p_i^2 = \left(\alpha^2\right)^{-1} .$$

- Embedding in 5D Minkowski restricted to surface $z_A z^A = -\frac{1}{\alpha^2}$
- Gives leading order non-diagonal momentum space metric:

$$g^{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta_{ij} - \alpha^2\left[p_0^2\delta_{ij} + p_i p_j\right] \end{pmatrix},$$

$$g_{\mu\nu} = \begin{pmatrix} 1 & 0 \\ 0 & -\delta_{ij} + \alpha^2\left(p_0^2\delta_{ij} + p_i p_j\right) \end{pmatrix}.$$

With thanks to G.Rosati, Ref: Amelino-Camelia et al. 2012

Putting it Together

Position space propagators and density of states

Position space propagators (Hadamard) computed to leading order in $\kappa\eta^2$

↓

$$D(t,x;t',x') = \int_{-\infty}^{\infty} \frac{\mathrm{d}^n p}{(2\pi)^n} \frac{e^{-i[p_0(t-t')-\vec{\mathrm{p}}\cdot(\vec{\mathrm{x}}-\vec{\mathrm{x}}')]}}{p_0^2-p^2(1+\kappa\eta^2 p^2)}$$

$$D^{(1)}(\sigma^2) = -\frac{1}{2\pi^2}\left\{\frac{1}{\sigma^2} - \frac{\kappa\eta^2}{\sigma^4}\right\},$$
$$\sigma^2 = t^2 - r^2 - i\epsilon$$

↓

"Point Splitting"

↓

$$\langle 0|T_{\mu\nu}|0\rangle = \left\langle 0\middle|\left(\partial_\mu\phi\partial_\nu\phi - \tfrac{1}{4}g_{\mu\nu}\partial_\mu\phi\partial^\mu\phi\right)\middle|0\right\rangle,$$
$$\langle 0|(\partial_x\phi)^2|0\rangle = \tfrac{1}{2}\lim_{x'\to x}\partial_x\partial_{x'}D^{(1)}(x,x')$$

Density of states integrals to leading order in $\kappa\eta^2$

↓

$$\rho(\omega) = \frac{\omega^2\sqrt{-g}\,\mathrm{d}\omega}{2\pi^2}$$

$$\log Z = \frac{1}{2\pi^2}\int_0^\infty \omega^2 \log Z\sqrt{-g}\,\mathrm{d}\omega,$$
$$\langle E\rangle = -\frac{\partial \log Z}{\partial\beta}$$

↓

"First Order Metric Determinant"

↓

$$\langle E\rangle = -\frac{1}{2\pi^2}\frac{\partial}{\partial\beta}\int_0^\infty(\omega^2+\kappa\eta^2\omega^4)\log Z\sqrt{-g}\,\mathrm{d}\omega,$$

Ref: P. C. Davies and Tee 2024

Vacuum Results

Free Space - Two ways

Values from both methods (almost) agree

Thermalized propagator renormalized by point splitting

$$\langle T_{\mu\nu} \rangle = \left(\frac{\pi^2}{30\beta^4} + \frac{\kappa\eta^2\pi^4}{126\beta^6} \right) \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & \frac{1}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{3} \end{bmatrix}$$

Thermal vacuum partition function with modified measure

$$\langle E \rangle = \frac{\pi^2}{30\beta^4} + \frac{8\kappa\eta^2\pi^4}{126\beta^6}$$

Slightly higher value for Density of States due to unrealistic upper integral limit

Adding a boundary - Casimir

Obtained values follow similar pattern

Propagator with boundary enforced by images renormalized by point splitting

$$\langle T_{\mu\nu}\rangle = \frac{-\pi^2}{1440a^4}\left(1 - \frac{5\kappa\eta^2\pi^2}{42a^2}\dots\right)\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & -1 & 0 & 0\\ 0 & 0 & -1 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

Density of States has larger contribution due to upper limit

Regularized $3+1$ Hamiltonian with “Fierz” regulator

$$\langle T_{tt}\rangle = \frac{-\pi^2}{1440a^4}\left(1 - \frac{12\kappa\eta^2\pi^2}{42a^2}\right)$$

$\kappa < 0$ entails negative energy density.
Combination of measure and propagator should resolve this (see next section)

Ref: P. C. Davies and Tee 2024; Brown and Maclay 1969; Fierz 1959

Accelerating Detectors

The Davies-Fulling-Unruh Detector

Uniformly accelerated particle detector in a vacuum

Particle detector minimally coupled to scalar QFT uniformly accelerated with inverse acceleration α

$$x = y = 0, \; z = (t^2 + \alpha^2)^{\frac{1}{2}}$$

With proper time τ

$$z = \alpha \cosh \frac{\tau}{\alpha}, \; t = \alpha \sinh \frac{\tau}{\alpha}$$

Transition rate of detector given by:

$$c^2 \sum_E |\langle E| \, m(0) \, |E_0\rangle|^2 \int_{-\infty}^{\infty} \mathrm{d}(\Delta\tau) \; e^{i(E-E_0)\Delta\tau} D^+(\Delta\tau)$$

Strictly positive detector response function:

$$\mathcal{F}(E) = \int_{-\infty}^{\infty} \mathrm{d}(\Delta\tau) \; e^{i(E-E_0)\Delta\tau} D^+(\Delta\tau)$$

Ref: Unruh 1976; P. C. Davies 1975; DeWitt 1979

Negative Transition Probabilities

Computed without modified measure

Correction term not strictly positive!

$$\mathcal{F}(E) = \left\{\left(1 + \frac{\kappa\eta^2}{6\alpha^2}\right)\frac{E-E_0}{e^{2\pi(E-E_0)\alpha}-1} + \left(\frac{\kappa\eta^2}{6}\right)\frac{(E-E_0)^3}{e^{2\pi(E-E_0)\alpha}-1}\right\}$$

Recall $\kappa < 0$ for sub-luminal propagation

Introduction of measure flips sign of propagator correction, resolving tension!

$$\mathcal{F}(E) = \left\{\left(1 - \frac{\kappa\eta^2}{6\alpha^2}\right)\frac{E-E_0}{e^{2\pi(E-E_0)\alpha}-1} - \left(\frac{\kappa\eta^2}{6}\right)\frac{(E-E_0)^3}{e^{2\pi(E-E_0)\alpha}-1}\right\}$$

Ref: P. C. W. Davies and Tee 2023

Final Remarks

Directions and Final Remarks

Puzzles & Lines of Inquiry

- There are many curved spacetime models we can attack with this framework
- In unpublished (ongoing) work, accelerating mirrors appear not to resolve negative probability tension with modified measure!
- Other results on gravitational lensing (Tee and Jafari 2022) indicate cosmological detection of leading order $\kappa\eta^2$ corrections still ambitious
- Work continues to detect other inconsistencies (Hawking radiation, 1-loop QFT)
- We are still working to connect eigenmode sums from the implied equation of motion $\frac{\partial^2 \phi}{\partial t^2} - \frac{\partial^2 \phi}{\partial x^2} + \kappa\eta^2 \frac{\partial^4 \phi}{\partial x^4} = 0$ to our modified propagator
- Fundamental thermodynamic quantities such as the partition function need reconciliation with pixelation and in particular “pixelgenesis” for expanding cosmologies

Thank You & Questions