Relational dynamics in group field theory with the Page-Wootters formalism

based on arXiv:2407.03432 [AC, S Gielen]

Tuesday 9th July, 2024

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Outline

1. GFT in a nutshell
2. Canonical quantisation of GFT
   - Algebraic and deparametrised approaches
3. New relational quantum dynamics
   - Parametrised theory
   - Quantum kinematics
   - Dynamics: Dirac quantisation and Page–Wootters
4. Conclusions
   - What to do with this?

Basics

- GFT for *simplicial* gravity coupled to a massless scalar field χ (matter)

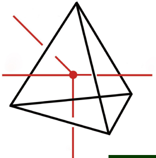
$$\varphi : SU(2)^4 \times \mathbb{R} \to \mathbb{C} \quad \text{(or } \mathbb{R}\text{)}$$

$$\varphi(g_I, \chi) \equiv \varphi(g_1, \dots, g_4, \chi)$$

- Peter–Weyl: mode decomposition

$$\varphi(g_I, \chi) = \sum_J \varphi_J(\chi) D^J(g_I)$$

$D^J(g_I)$ convolution of Wigner D-matrices with $SU(2)$ intertwiner(s), J multi-index



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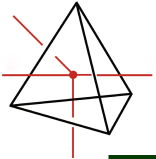


Motivation

- Spin foam models [Reisenberger, Rovelli, De Pietri, Krasnov, ...]

$$Z_{\text{GFT}} = \int \mathcal{D}\varphi \mathcal{D}\bar{\varphi}\; e^{-S[\varphi,\overline{\varphi}]} = \sum_\Gamma \lambda^{n_V(\Gamma)} \mathcal{A}_\Gamma$$

λ coupling, $\mathcal{A}_\Gamma$ spinfoam amplitude

- Feynman expansion generates a sum over graphs Γ dual to simplicial complexes (triangulations of spacetime)
- Path integral provides a sum over all possible geometries and topologies

Canonical quantisation of GFT [Oriti, Gielen, Sindoni, Wilson-Ewing, Marchetti, ...]

- Shift perspective: *Hilbert space quantisation* by postulating [Oriti]

$$\left[\hat{\varphi}_J(\chi), \hat{\varphi}^\dagger_{J'}(\chi')\right] = \delta_{JJ'}\delta(\chi - \chi')$$

- Abstract Fock space (we call this **algebraic approach**)

$\hat{\varphi}_J(\chi)|0\rangle = 0\,, \qquad |$  $\rangle = \hat{\varphi}^\dagger_J(\chi)|0\rangle$

- Construct algebraic operators such as number and scalar field

$$\hat{N} = \sum_J \int \mathrm{d}\chi\, \hat{\varphi}^\dagger_J(\chi)\hat{\varphi}_J(\chi)\,, \qquad \hat{\chi} = \frac{1}{\langle\hat{N}\rangle}\sum_J \int \mathrm{d}\chi\, \chi\, \hat{\varphi}^\dagger_J(\chi)\, \hat{\varphi}_J(\chi)$$

- No dynamics so far! Physical states should satisfy $\frac{\delta S[\hat{\varphi},\hat{\varphi}^\dagger]}{\delta\hat{\varphi}^\dagger_J(\chi)}|\Psi\rangle = 0$
- Recent focus on "peaked coherent states" [Marchetti, Oriti] which depend on a parameter χ_0 such that $\langle\hat{\chi}\rangle = \chi_0 \Rightarrow$ dynamics wrt (*quantum*) clock expectation values [**tempus post quantum**]
- But no clear link with constraint and classical theory!

- Work with $\varphi \in \mathbb{R}$ and choose χ as time variable *before* quantisation
- From the GFT action (free theory)

$$S_0[\varphi] = \frac{1}{2} \int \mathrm{d}\chi \sum_J \varphi_{-J}(\chi) \left(K_J^{(0)} + K_J^{(2)} \partial_\chi^2 \right) \varphi_J(\chi)$$

one can define canonical momenta $\pi_J(\chi)$ of the field $\varphi_J(\chi)$ and find a relational Hamiltonian (which gives evolution via PB $\{\cdot, H\}$)

$$H = -\frac{1}{2} \sum_J \left[\frac{\pi_J(\chi)\pi_{-J}(\chi)}{K_J^{(2)}} + K_J^{(0)} \varphi_J(\chi)\varphi_{-J}(\chi) \right]$$

- Now perform genuine canonical quantisation $[\hat{\varphi}_J(\chi), \hat{\pi}_{J'}(\chi)] = \mathrm{i}\delta_{JJ'}$
- Define again number operator (using $\left[\hat{a}_J(\chi), \hat{a}^\dagger_{J'}(\chi)\right] = \delta_{JJ'}$)

$$\hat{N}(\chi) = \sum_J \hat{a}^\dagger_J(\chi)\hat{a}_J(\chi)$$

- Evolution is clear $\hat{N}(\chi) = e^{\mathrm{i}\hat{H}\chi}\, \hat{N}(0)\, e^{-\mathrm{i}\hat{H}\chi}$ but χ naive classical time parameter! (What about clock-covariance?) [**tempus ante quantum**]

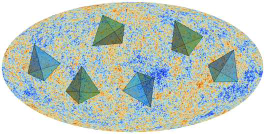
- Both approaches yield compelling results that recover cosmology
- We are ignoring interactions: this can describe homogeneous settings
- One finds the volume dynamics gives quantum-corrected FLRW

Bouncing FLRW cosmology

$$\left(\frac{1}{\langle \hat{V}(\chi)\rangle}\frac{\mathrm{d}\langle \hat{V}(\chi)\rangle}{\mathrm{d}\chi}\right)^2 = 4\omega^2\left(1+\frac{vE}{\langle \hat{V}(\chi)\rangle}-\frac{v^2Q^2}{\langle \hat{V}(\chi)\rangle^2}\right)$$

Agreement with GR $(V'/V)^2 = 12\pi G$ at late times requires identification $\omega^2 = 3\pi G$

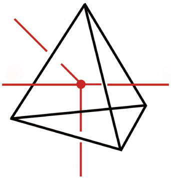
- Here $\hat{V}(\chi)$ is the GFT volume operator ($\sim$ number of ◬), v is volume of GFT quanta, E and Q^2 initial conditions
- Spacetime emerges as a collection of excited states (analogy: condensate physics)



- Can we define properly the relational dynamics encoded in $\hat{V}(\chi)$?

The goal: clarify dynamics in canonical GFT

What is a satisfactory definition of relational quantum dynamics in quantum gravity? E.g., for the simple GFT number operator $\hat{N}(\chi)$



Leverage complementary strengths of GFT approaches

- Clear classical phase space and Hamiltonian formulation
- Genuine canonical quantisation in a gauge-independent way
- Hilbert space formulation in terms of constraints (i.e., Dirac)
- Physical observables evolving in a *tempus post quantum* setting

In a sense, we wish connect to known quantisation methods

Parametrisation of classical GFT

[AC, S. Gielen – arXiv:2407.03432]

- Parametrise the theory $S_0[\varphi]$ so that both φ and χ depend on a fiducial parameter τ (here focus on a single J)

$$S[\varphi,\chi] = \frac{1}{2}\int \mathrm{d}\tau \left[|K^{(0)}|\varphi^2(\partial_\tau\chi) + |K^{(2)}|\frac{(\partial_\tau\varphi)^2}{(\partial_\tau\chi)}\right]$$

- Defining conjugate momenta π_φ and p_χ one can rewrite the action as

$$S[\varphi,\chi] = \int \mathrm{d}\tau\,(\pi_\varphi\partial_\tau\varphi + p_\chi\partial_\tau\chi - NC)$$

with N Lagrange multiplier (Lapse) and the constraint

$$C = p_\chi + H(\varphi,\pi_\varphi) = 0$$

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- We know how to define *relational Dirac Observables* (DO) following [Rovelli, Dittrich, Thiemann, Tambornino,...] such that $\{F, C\} \approx 0$

$$F_{f_\varphi,\chi}(\chi_0) \approx \sum_{n=0}^{\infty} \frac{(\chi_0-\chi)^n}{n!}\{f_\varphi, H_\varphi\}_n$$

with f_φ GFT space phase function

- Promote to operators on $\mathcal{H}_{\text{kin}} = \mathcal{H}_\chi \otimes \mathcal{H}_\varphi$ such that

$$[\hat{\varphi}, \hat{\pi}_\varphi] = \mathrm{i}\,, \qquad [\hat{\chi}, \hat{p}_\chi] = \mathrm{i}$$

Geometry sector (GFT)

- Quantum Hamiltonian

$$\hat{H}_\varphi = \frac{1}{2}\left(\frac{\hat{\pi}_\varphi^2}{|K^{(2)}|} - |K^{(0)}|\hat{\varphi}^2\right)$$

- Eigenvalue problem

$$\hat{H}_\varphi |\psi_\pm^E\rangle = E|\psi_\pm^E\rangle$$

solved by Parabolic Cylinder functions ($\pm$ denotes degeneracy)

- Spectral decomposition

$$\hat{H}_\varphi = \int_\pm \mathrm{d}E\; E\; |\psi_\pm^E\rangle\langle\psi_\pm^E|$$

Matter sector (scalar field)

- Isomorphic to particle on a line

$$\hat{\chi}|\chi\rangle = \chi|\chi\rangle$$
$$\hat{p}_\chi|p_\chi\rangle = p_\chi|p_\chi\rangle$$

with $\langle\chi|\chi'\rangle = \delta(\chi - \chi')$ (same for p_χ)

- "Time operator"

$$\hat{\chi} = \int \mathrm{d}\chi\; \chi\; |\chi\rangle\langle\chi|$$

- Clock Hamiltonian decomposition

$$\hat{H}_\chi = \hat{p}_\chi = \int \mathrm{d}p_\chi\; p_\chi\; |p_\chi\rangle\langle p_\chi|$$

Dirac quantisation (as in other known systems, e.g., LQC)

- Constraint $\hat{C} = \hat{p}_\chi \otimes \mathbb{I}_\varphi + \mathbb{I}_\chi \otimes \hat{H}_\varphi$ with physical states

$$\hat{C}|\Psi_{\mathsf{phys}}\rangle = 0$$

- Use group averaging $\delta(\hat{C}) = \frac{1}{2\pi}\int \mathrm{d}\alpha\, e^{\mathrm{i}\alpha\hat{C}}$ to define inner product

$$\langle\Psi_{\mathsf{phys}}|\Psi_{\mathsf{phys}}\rangle_{\mathsf{phys}} := \langle\Psi_{\mathsf{kin}}|\delta(\hat{C})|\Psi_{\mathsf{kin}}\rangle_{\mathsf{kin}}$$

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- *Frozen formalism* (and the "problem of time")

$$\hat{U}_{\chi\varphi}(\alpha)|\Psi_{\mathsf{phys}}\rangle = |\Psi_{\mathsf{phys}}\rangle \qquad \text{with} \qquad \hat{U}_{\chi\varphi}(\alpha) = e^{-\mathrm{i}\alpha\hat{p}_\chi} \otimes e^{-\mathrm{i}\alpha\hat{H}_\varphi}$$

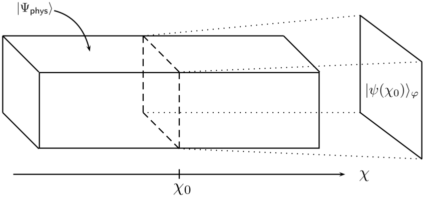
- But we can quantise relational DO! One now chooses the (quantum) clock $\hat{\chi}$ and defines dynamics wrt its eigenvalues [Höhn, Lock, Smith]

- In particular we define the DO associated to the GFT number $\mathfrak{N} = \mathfrak{a}^\dagger\mathfrak{a}$

$$\hat{N}_D(\chi_0) := \frac{1}{2\pi}\int \mathrm{d}\alpha\, \hat{U}_{\chi\varphi}(\alpha)\left(|\chi_0\rangle\langle\chi_0| \otimes \hat{\mathfrak{N}}\right)\hat{U}^\dagger_{\chi\varphi}(\alpha)$$

- Aiming for a "tempus post quantum" setting, we apply the **Page-Wootters formalism** (thanks to the "*trinity of relational quantum dynamics*" [Höhn, Lock, Smith]) for the **first time in quantum gravity**
- PW is equivalent to Dirac quantisation [Höhn, Lock, Smith]
- Conditioned state $|\psi(\chi_0)\rangle_\varphi := \Big(\langle\chi_0| \otimes \mathbb{I}_\varphi\Big)|\Psi_{\mathsf{phys}}\rangle$
- Physical state → *history state* $|\Psi_{\mathsf{phys}}\rangle = \int \mathrm{d}\chi_0 \, |\chi_0\rangle_\chi \otimes |\psi(\chi_0)\rangle_\varphi$
- Inner product (*a priori* different from Dirac quantisation)

$$\langle\Psi_{\mathsf{phys}}|\Psi_{\mathsf{phys}}\rangle_{\mathsf{PW}} := \langle\Psi_{\mathsf{phys}}|\Big(|\chi_0\rangle\langle\chi_0| \otimes \mathbb{I}_\varphi\Big)|\Psi_{\mathsf{phys}}\rangle_{\mathsf{kin}}$$



Result and interpretation for QG [arXiv:2407.03432]

- One finds that the expectation value of Dirac observable in the physical inner product matches with the Page–Wootters evaluation

$$
\begin{aligned}
N_D(\chi_0) &:= \langle \Psi_{\text{phys}} | \hat{N}_D(\chi_0) | \Psi_{\text{phys}} \rangle_{\text{phys}} \\
&= \langle \Psi_{\text{phys}} | \hat{\mathfrak{N}} | \Psi_{\text{phys}} \rangle_{\text{PW}} \\
&= {}_{\varphi}\langle \psi(\chi_0) | \hat{\mathfrak{N}} | \psi(\chi_0) \rangle_{\varphi} \\
&= {}_{\varphi}\langle \psi | \, \hat{U}^{\dagger}_{\varphi}(\chi_0) \, \hat{\mathfrak{N}} \, \hat{U}_{\varphi}(\chi_0) \, | \psi \rangle_{\varphi}
\end{aligned}
$$

- Moreover, it gives the deparametrised result back!

- Strengthens deparametrised: quantisation and choice of clock commute
- GFT observables interpreted as conditional on the clock
- Consistent framework that allows to answer relevant questions about quantum geometry at any specific value of relational time
- GFT quanta are seen as associated to the same clock reading (synchronisation issues [Marchetti, Kotecha, Oriti])

- **Single reparametr. invariance** $\mathcal{H}_{\mathsf{kin}} = \mathcal{H}_\chi \otimes \mathcal{H}_\varphi^{\mathsf{tot}}$ with $\mathcal{H}_\varphi^{\mathsf{tot}} = \bigotimes_J \mathcal{H}_{\varphi_J}$

$$N_D^{\mathsf{tot}}(\chi_0) = {}_{\varphi_{\mathsf{tot}}}\langle\psi|\Big[\bigotimes_J \hat{U}^\dagger_{\varphi_J}(\chi_0)\Big]\hat{\mathfrak{N}}_{\mathsf{tot}}\Big[\bigotimes_J \hat{U}_{\varphi_J}(\chi_0)\Big]|\psi\rangle_{\varphi_{\mathsf{tot}}}$$
$$= {}_{\varphi_{\mathsf{tot}}}\langle\psi|\sum_J \hat{U}^\dagger_{\varphi_J}(\chi_0)\,\hat{\mathfrak{a}}^\dagger_J\hat{\mathfrak{a}}_J\,\hat{U}_{\varphi_J}(\chi_0)\,|\psi\rangle_{\varphi_{\mathsf{tot}}}$$

Multiple modes and "multi-fingered time" [arXiv:2407.03432]

Single reparametr. invariance $\mathcal{H}_{\text{kin}} = \mathcal{H}_\chi \otimes \mathcal{H}_\varphi^{\text{tot}}$ with $\mathcal{H}_\varphi^{\text{tot}} = \bigotimes_J \mathcal{H}_{\varphi_J}$

$$N_D^{\text{tot}}(\chi_0) = {}_{\varphi_{\text{tot}}}\langle\psi| \Big[\bigotimes_J \hat{U}^\dagger_{\varphi_J}(\chi_0) \Big] \hat{\mathfrak{N}}_{\text{tot}} \Big[\bigotimes_J \hat{U}_{\varphi_J}(\chi_0) \Big] |\psi\rangle_{\varphi_{\text{tot}}}$$
$$= {}_{\varphi_{\text{tot}}}\langle\psi| \sum_J \hat{U}^\dagger_{\varphi_J}(\chi_0)\, \hat{\mathfrak{a}}^\dagger_J \hat{\mathfrak{a}}_J\, \hat{U}_{\varphi_J}(\chi_0)\, |\psi\rangle_{\varphi_{\text{tot}}}$$

Multiple reparametr. invariances (i.e., super-Hamiltonian $\sum_J N_J C_J$)

- Here we have $\mathcal{H}_{\text{kin}} = \mathcal{H}_\chi^{\text{tot}} \otimes \mathcal{H}_\varphi^{\text{tot}} = \bigotimes_J \left(\mathcal{H}_{\chi_J} \otimes \mathcal{H}_{\varphi_J} \right)$
- multi-constraint Dirac $\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{phys}} = \langle\Psi_{\text{kin}}| \prod_J \delta(\hat{C}_J) |\Psi_{\text{kin}}\rangle_{\text{kin}}$
- multi-clock PW $\langle\Psi_{\text{phys}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{PW}} = \langle\Psi_{\text{phys}}| \Big(\bigotimes_J |\chi_J^0\rangle\langle\chi_J^0| \Big) \otimes \mathbb{I}_\varphi^{\text{tot}} |\Psi_{\text{phys}}\rangle_{\text{kin}}$

Evolution along different "single-mode times"

$$\langle\Psi_{\text{phys}}|\hat{N}_D^{\text{tot}}(\{\chi_J^0\})|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{phys}} = \langle\Psi_{\text{phys}}|\hat{\mathfrak{N}}_{\text{tot}}|\Psi_{\text{phys}}\rangle^{\text{M}}_{\text{PW}}$$
$$= {}^{\text{M}}_{\varphi_{\text{tot}}}\langle\psi| \sum_J \hat{U}^\dagger_{\varphi_J}(\chi_J^0)\, \hat{\mathfrak{a}}^\dagger_J \hat{\mathfrak{a}}_J\, \hat{U}_{\varphi_J}(\chi_J^0)\, |\psi\rangle^{\text{M}}_{\varphi_{\text{tot}}}$$

Conclusion: proper relational dynamics in GFT

- **Acknowledge pros and cons of literature**
  - We leverage complementary strengths of existing approaches!
- **Clear phase space and kinematical structure**
  - Relational DO and genuine canonical quantisation
  - In particular DO (Poisson-)commute with the constraint
- **Quantum dynamics: Dirac and Page–Wootters**
  - Constrained quantisation in terms of Hilbert spaces $\mathcal{H}_{\text{phys}}$, $\mathcal{H}_{\text{kin}}$
  - Quantum DO evolving wrt eigenvalues of "time operator"
  - Describe quantum geometry at specific "time slices" of $|\Psi_{\text{phys}}\rangle$
  - Multiple modes under control (realisation of multi-fingered time)
- **Fully covariant framework**
  - Quantise *without* the need to single out a clock beforehand
  - One can now study changes of quantum reference frame
- **Full field theory?** (Infinite tensor products and functional techniques)