MIXMASTER UNIVERSE

IN A 2D NON-COMMUTATIVE GUP FRAMEWORK

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STRUCTURE OF THE TALK

- General GUP framework
- Classical symplectic structures of GUP theories

Cosmological models:

- Bianchi I model
- Bianchi II model
- Mixmaster model

GENERALIZED UNCERTAINTY PRINCIPLE THEORIES

- Quantum non-relativistic theories based on a deformation of the ordinary Heisenberg's uncertainty principle (HUP).
- They are related to String theory, in a low-energy regime.
- Deformation of the usual Heisenberg algebra between the quantum position and momentum operators.

Possible main consequences:

- Nonzero minimal uncertainty in position

⬇

Minimum length

- Non-commutativity of the coordinate operators

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Non-commutative "geometry"

WHY GUP THEORIES?

Implement GUP framework
in the quantization scheme of cosmological models

- Homogenous cosmology: gravitational field with finite number of degrees of freedom.
- One-particle quantization schemes are well-grounded in this context.

Modified Heisenberg algebras can introduce novel relevant features in the dynamics of the early Universe (minimal structures, non-commutativity).

These features, in principle, comes from some more general theory.

CLASSICAL SYMPLECTIC STRUCTURE

Goal: construct a classical theory such that we can interpret the set *(q, p)* as coordinate on *2d* smooth manifold, where we can define a **symplectic form** that induces the **Poisson structure**:

$$\{q_i, p_j\} = \delta_{ij} f(q,p),$$
$$\{q_i, q_j\} = L_{ij}(q,p),$$
$$\{p_i, p_j\} = 0.$$

2-FORM MATRIX

$$\omega_{ab} = \begin{pmatrix} 0 & -\frac{1}{f}\mathrm{id}_d \\ \frac{1}{f}\mathrm{id}_d & \frac{1}{f^2}L \end{pmatrix}$$

REQUEST

- Nondegeneracy:

$$\det(\omega_{ab}) = f^{-2d} \neq 0$$

- Closure:

$$d\omega = 0$$

CLOSURE CONDITION IN 3D

Closure condition: set of equations for *L(q,p)* and *f(q,p).*

- *f(q,p)* independent from q ⟶ *f(p)*
- General form for *L(q,p)*:

$$L_{12}(q,p) = S_{12}(p) - P(p)q_1 + Q(p)q_2,$$
$$L_{23}(q,p) = S_{23}(p) - R(p)q_2 + P(p)q_3,$$
$$L_{13}(q,p) = S_{13}(p) - R(p)q_1 + Q(p)q_3.$$

- Gradient equation:

$$\begin{cases} \frac{\partial f}{\partial p_1} = Q(p_1,p_2,p_3) \\ \frac{\partial f}{\partial p_2} = P(p_1,p_2,p_3) \\ \frac{\partial f}{\partial p_3} = R(p_1,p_2,p_3) \end{cases}$$

+

- If $S_{ij} = 0$:Integrability conditions for (Q,P,R)
- If $S_{ij} \neq 0$ and (Q,P,R) irrotational: equations that constrains the S_{ij}

BIANCHI I MODEL – DEFORMED CASE

BIANCHI I HAMILTONIAN IN MISNER VARIABLES

$$H_I = N e^{-3\alpha}\left(-p_\alpha^2 + p_+^2 + p_-^2\right) \approx 0$$

REDUCTION AND IMPOSITION OF GUP SYMPLECTIC STRUCTURE

$$\tilde{H}_I = \sqrt{p_+^2 + p_-^2}$$

$$\{\gamma_\pm, p_\pm\} = 1 + \beta(p_+^2 + p_-^2)$$

$$\{\gamma_+, \gamma_-\} = -2\beta(\gamma_+ p_- - \gamma_- p_+)$$

$$\{p_\pm, p_\pm\} = 0$$

DYNAMICS

$$|\dot{\gamma}| = 1 + \beta\left(p_+^2 + p_-^2\right)$$

$$\gamma_\pm(\alpha) = \frac{p_\pm}{\sqrt{p_+^2 + p_-^2}}\left[1 + \beta\left(p_+^2 + p_-^2\right)\right]\alpha + \gamma_\pm^0$$

BIANCHI II MODEL - DEFORMED CASE

REDUCED BIANCHI II HAMILTONIAN IN MISNER VARIABLES

$$\tilde{H}_{II} = \sqrt{p_+^2 + p_-^2 + e^{4\alpha - 8\gamma_+}}$$

Receding potential in α time

CONSTANT OF MOTION IN THE STANDARD CASE

$$\begin{cases} \omega := \tilde{H}_{II} - \frac{p_+}{2} \\ p_- \end{cases}$$

CONSTANT OF MOTION IN THE DEFORMED CASE

$$\begin{cases} \Omega := \tilde{H}_{II} - \frac{1}{2\sqrt{1+\beta p_-^2}} \tan^{-1}\left(\frac{\sqrt{\beta} p_+}{\sqrt{1+\beta p_-^2}} \right) \\ p_- \end{cases}$$

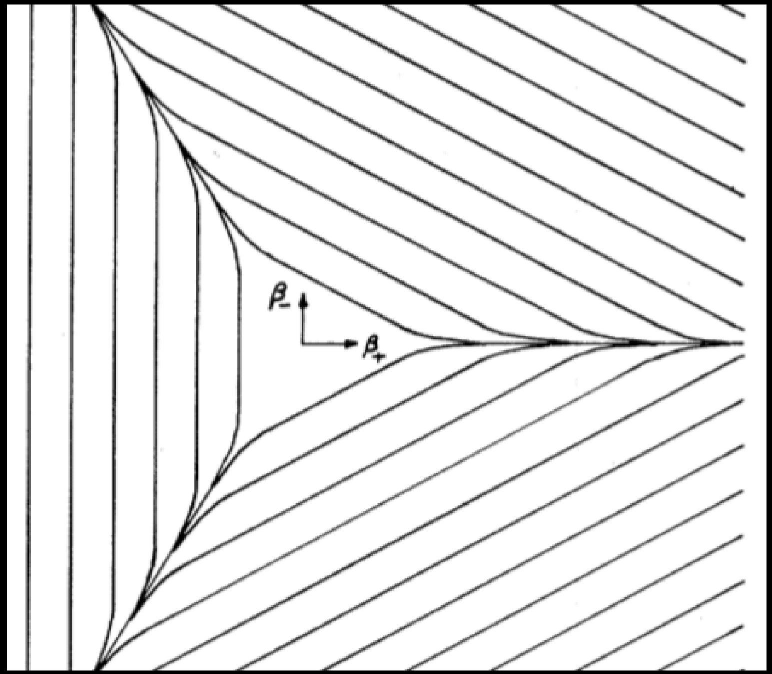
THE MIXMASTER MODEL

REDUCED BIANCHI IX HAMILTONIAN IN MISNER VARIABLES

$$\tilde{H}_{IX} = \Big(p_+^2 + p_-^2 + e^{4\alpha - 8\gamma_+} + e^{4\alpha}\big(e^{-8\gamma_+} + 2e^{4\gamma_+}(\cosh(4\sqrt{3}\gamma_-) - 1) + e^{4\gamma_+}\cosh(2\sqrt{3}\gamma_-)\big)\Big)^{1/2}$$

MIXMASTER APPROXIMATION TOWARDS THE INITIAL SINGULARITY

$$\tilde{H}_{M_{IX}} = \big(p_+^2 + p_-^2 + e^{4\alpha - 8\gamma_+} + e^{4\alpha + 4\gamma_+ + 4\sqrt{3}\gamma_-} + e^{4\alpha + 4\gamma_+ - 4\sqrt{3}\gamma_-}\big)^{1/2}$$



Dynamics of the Universe: *2d* point particle bouncing against the walls of an equilateral triangular potential well.

- Inside the well and far away from the walls: free particle (Bianchi I dynamics)
- Potential-dominated region: bounces described by Bianchi II dynamics

THE MIXMASTER MODEL – STANDARD PICTURE

- Free motion represents a Kasner epoch characterized by $|\dot{\gamma}| = 1$
- Wall velocity with respect to the particle's trajectory: $|\dot{\gamma}_{wall}| = 1/2$
- Every bounce against a single wall represents a transition from one Kasner epoch to another.
- Bianchi II conserved quantities: $p_-,\quad \tilde{H}_{II} - p_+/2$
- **BKL map**:

$$\sin\theta_f - \sin\theta_i = \frac{1}{2}\sin(\theta_f + \theta_i), \quad |\theta_i| < |\theta_{max}| = \frac{\pi}{3}$$

- If $|\theta_i| > |\theta_{max}|$ the bounce will occur against another wall: infinite sequences of bounces, with decreasing energy.
- Ergodic and chaotic motion.

METHOD OF CONSISTENT POTENTIAL

It is a method to establish the range of validity of velocity term dominated (VTD) solution.

EOM solved only considering velocity dominated terms

↓

Velocity term dominated solution in the neglected potential

↙ ↘

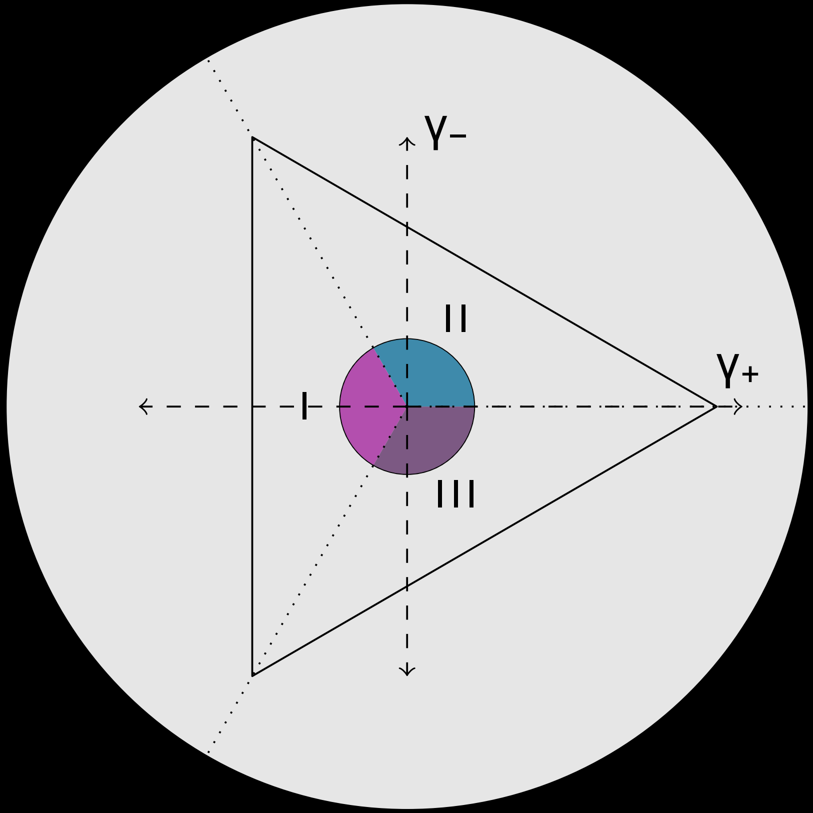
Exponentially small terms: valid solution (Kasner epoch)

Exponentially growing terms: not valid solution (bounce against the wall)

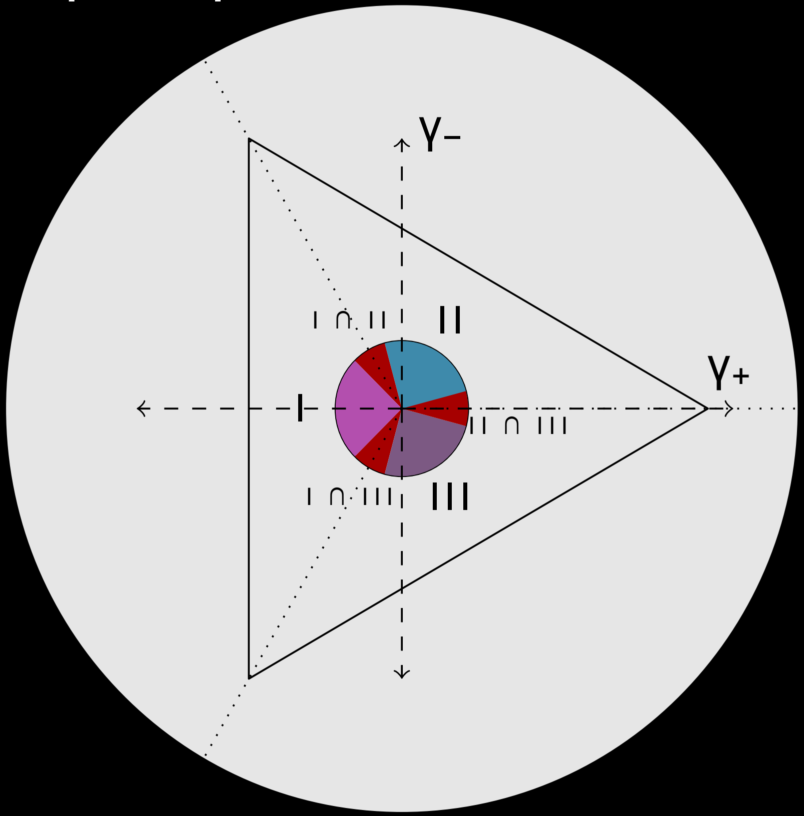
STANDARD MIXMASTER MODEL: only one potential at a time, for every trajectory, becomes relevant (except for the corners).

METHOD OF CONSISTENT POTENTIAL

- In the deformed case the CP method points out that there exists trajectories for which **two** potential walls at a time are relevant.
- The overlapping regions depend on the initial energy of every run: amplified corners.
- The picture of bounces against a **single** wall is in principle not available.

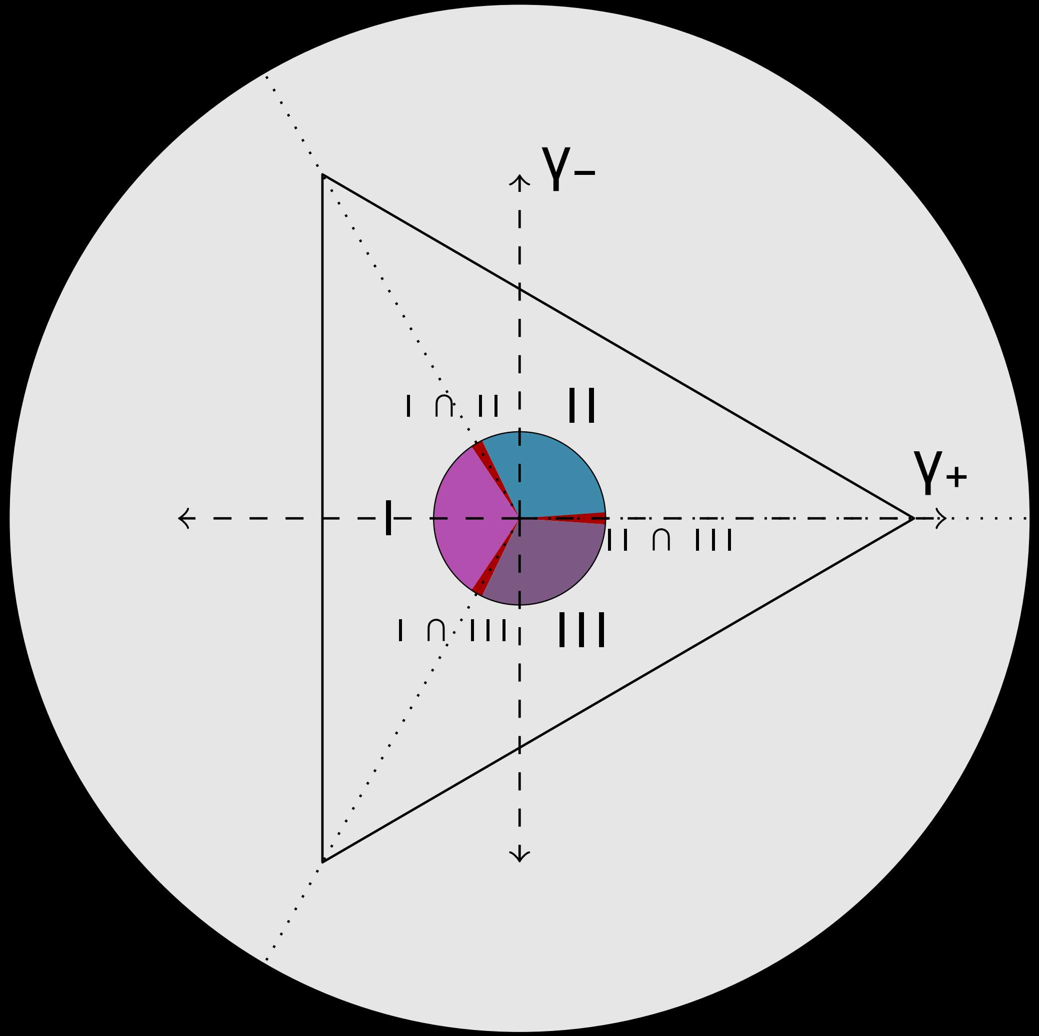


MCP in the standard case



MCP in the deformed case

METHOD OF CONSISTENT POTENTIAL



MCP in the deformed case towards the singularity

- **Towards the initial singularity**, the ratio of the value of the two involved potential walls in these overlapping regions, is comparable only in an increasingly narrower band pointing at the corner.
- For all the practical purpose, **only one potential wall at a time is relevant** and we can recover in the deformed case the bounce-against-a-single-potential-wall picture.

THE MIXMASTER MODEL – DEFORMED PICTURE

- Free motion represents a Kasner epoch characterized by $|\dot{\gamma}| = 1 + \beta(p_+^2 + p_-^2)$
- Wall velocity with respect to the particle's trajectory: $|\dot{\gamma}_{wall}| = 1/2$
- Every bounce against a single wall represents a transition from one Kasner epoch to another.
- **BKL map**:

$$H_i\sqrt{1 + H_i^2\sin^2\theta_i} + \frac{1}{2}\tan^{-1}\left(\frac{H_i\cos\theta_i}{\sqrt{1 + H_i^2\sin^2\theta_i}}\right) =$$

$$H_f\sqrt{1 + H_i^2\sin^2\theta_i} - \frac{1}{2}\tan^{-1}\left(\sqrt{\frac{H_f^2 - H_i^2\sin^2\theta_i^2}{1 + H_i^2\sin^2\theta_i^2}}\right),$$

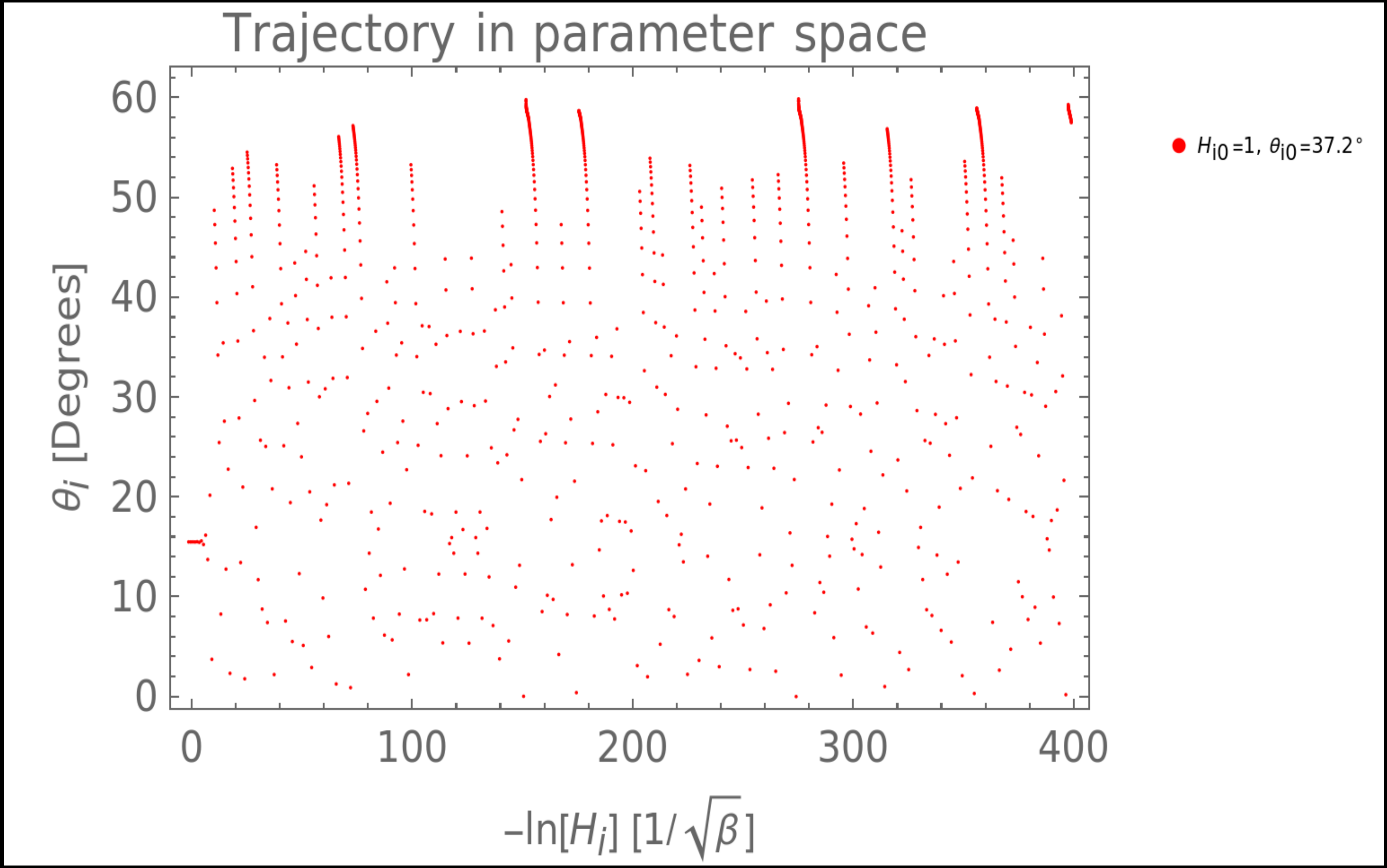
$$\theta_f = \sin^{-1}\left(\frac{H_i}{H_f}\sin\theta_i\right), \quad |\theta_i| < |\theta_{max}| = \cos^{-1}\left(\frac{1}{2(1 + H_i^2)}\right)$$

THE MIXMASTER MODEL – DEFORMED PICTURE

New features of the **BKL map**:

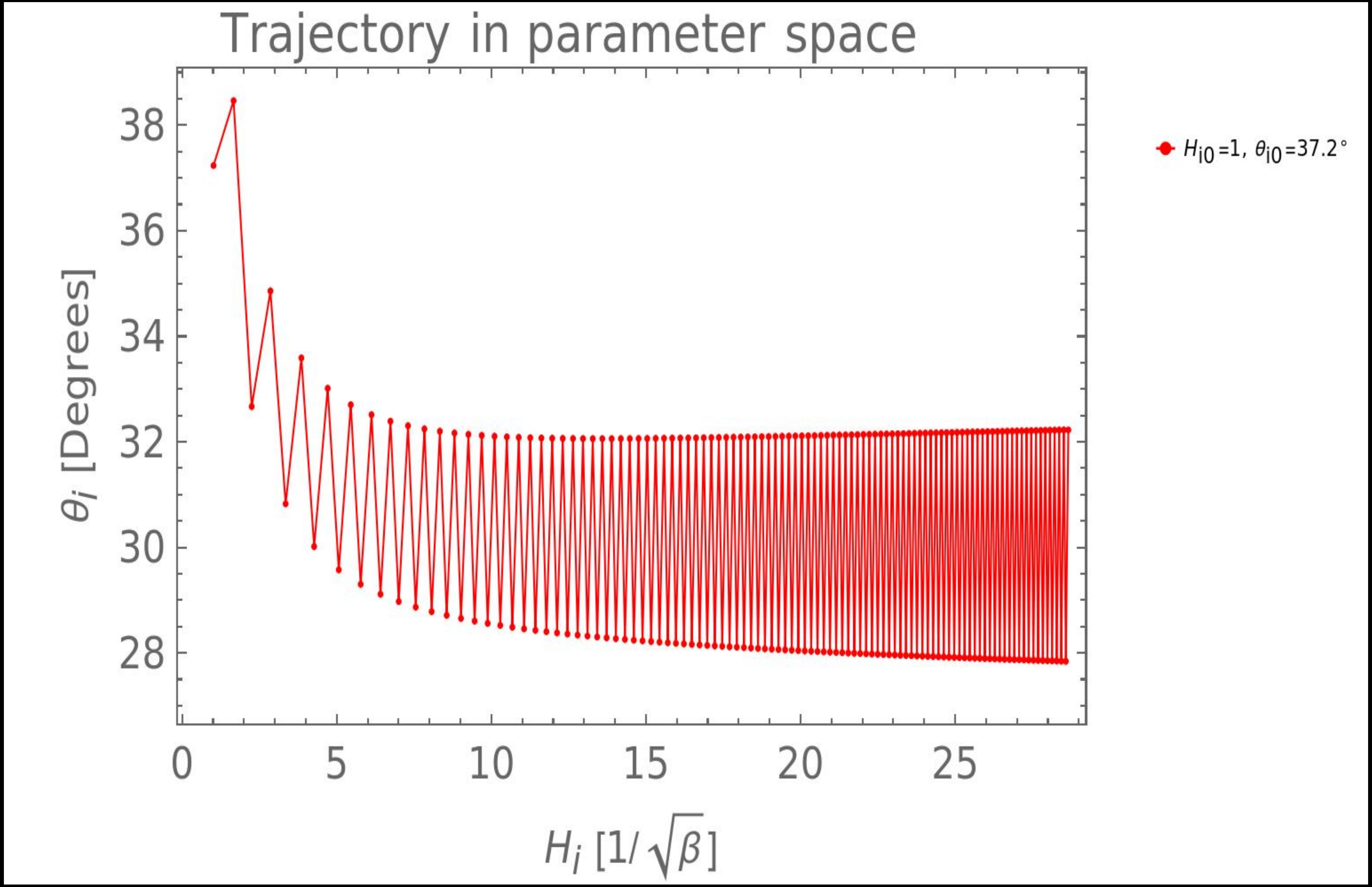
- Much more complicated map.
- Not solvable analytically.
- The **reflection law** depends on the initial angle and on the **initial energy**.
- The energy is **increasing** at every bounce.
- The maximum angle for the bounce with respect a single wall is always **greater than π/3** and it is dependent on the initial energy.
- If the initial angle is greater than the maximum angle, the bounce will occur on a different wall.
- **Infinite** sequence of bounces.
- Is the system still **ergodic**?

THE MIXMASTER MODEL – STANDARD PICTURE



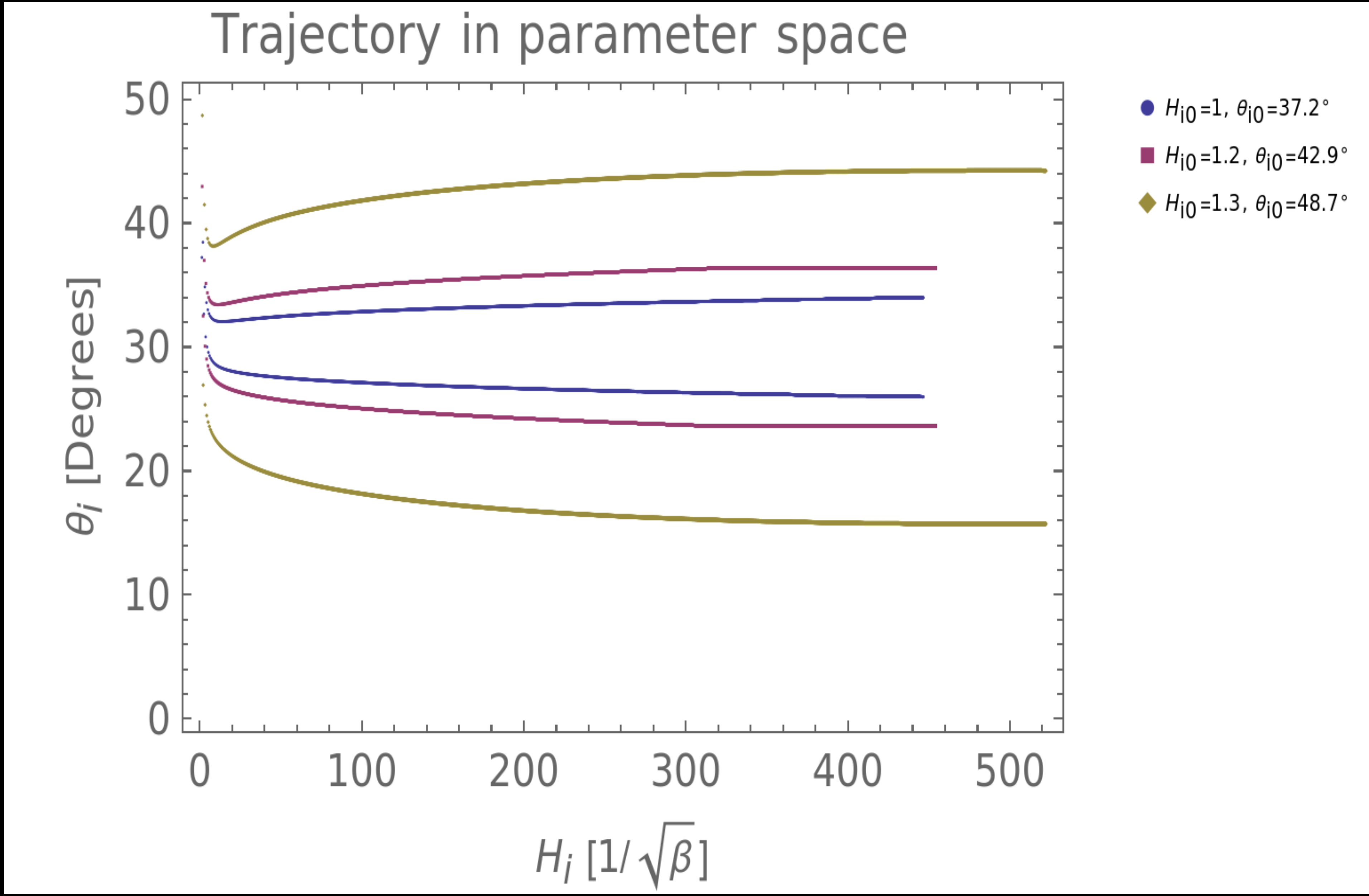
Trajectory of the **standard Mixmaster Universe** for the first 1000 bounces in the space of parameter, for a generic initial condition.

THE MIXMASTER MODEL – DEFORMED PICTURE



Trajectory of the **deformed Mixmaster Universe** for the first 200 bounces in the space of parameter, for a generic initial condition.

THE MIXMASTER MODEL – DEFORMED PICTURE



Trajectory of the **deformed Mixmaster Universe** for the first 50000 bounces in the space of parameter, for different initial conditions.

CONCLUSIONS AND FURTHER DEVELOPMENTS

- GUP theories could provide useful physical insights in the context of cosmology.
- GUP theories can be implemented classically in a rigorous way, characterizing the **symplectic structure** of the theory.
- In this framework we can study the dynamics of several cosmological models in the **minisuperspace** Hamiltonian formulation.
- We were able to repeat the analysis of Mixmaster Universe in the deformed case, for a chosen GUP theory.
- The GUP non-commutative effects change drastically the behavior of the Universe towards the singularity, probably **removing ergodicity** of the system.
- **Quantum analysis** is needed to fully explore the GUP consequences.

THANK YOU FOR YOUR ATTENTION