Boson stars and their relatives in semiclassical gravity

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Boson stars and their relatives in semiclassical gravity

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Outline

1. Scalar field dark matter (as motivation)
2. Semiclassical description of the Einstein-Klein-Gordon system
3. Boson stars and their relatives
4. Conclusions

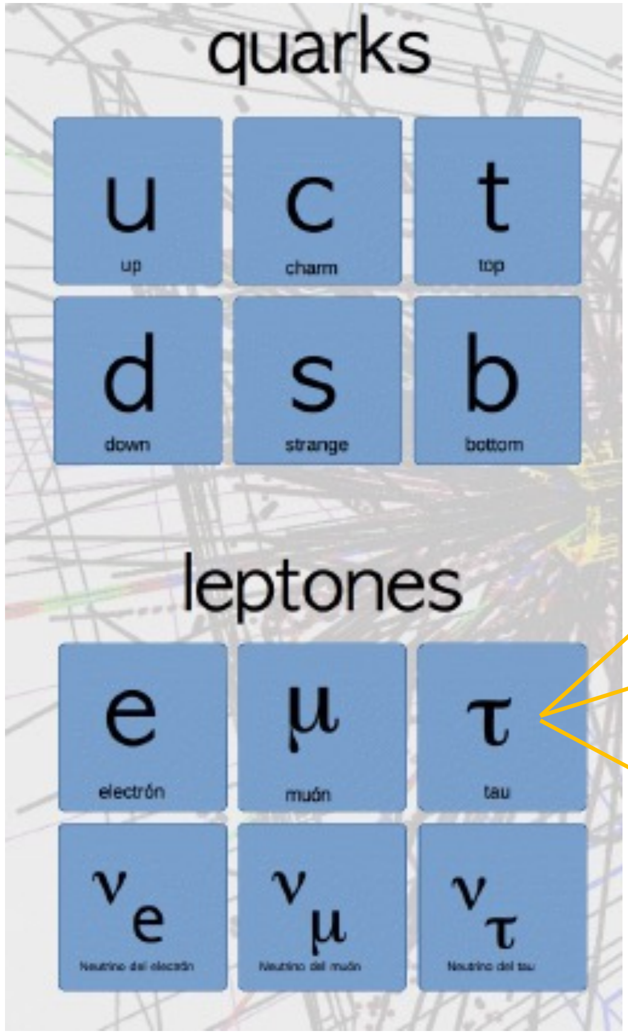
Dark matter: Could it be that simple?

Fundamental interactions

1. Strong force
2. Electromagnetic force
3. Weak force
4. Gravitational force

Quarks:Too much interaction

Could it be that simple?

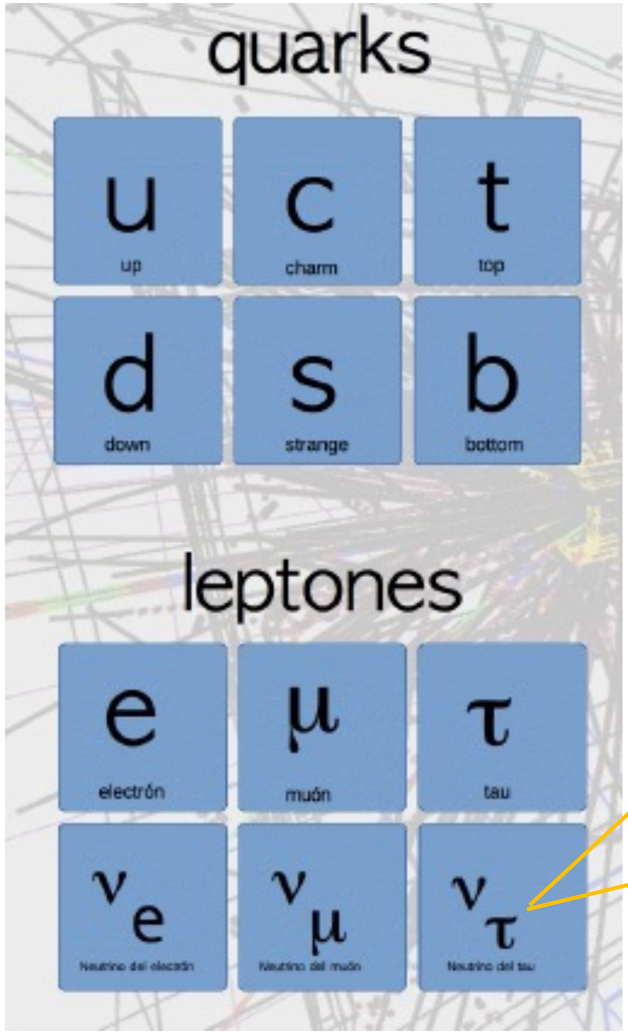


Fundamental interactions

1. Strong force
2. Electromagnetic force
3. Weak force
4. Gravitational force

Charged leptons: Good enought

Could it be that simple?

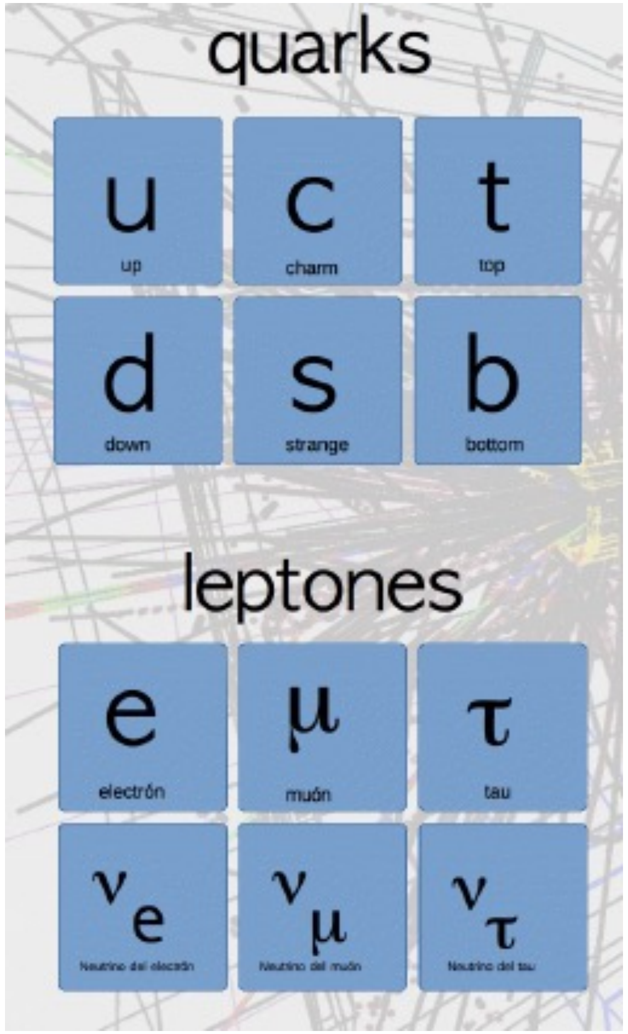


Fundamental interactions

1. Strong force
2. Electromagnetic force
3. Weak force
4. Gravitational force

Neutrinos: Difficult to see, but observable

Could it be that simple?



Fundamental interactions

1. Strong force
2. Electromagnetic force
3. Weak force
4. Gravitational force

Is dark matter the particle that interacts only through gravitational interactions?

The darkest scenario:

- Dark matter interacts only through gravitational interactions:
- 1) Forget how to detect it on Earth
- 2) Main properties: The mass and the spin
- 3)Consider the case of a bosonic particle that interact only through gravitation

Scalar field as dark matter?

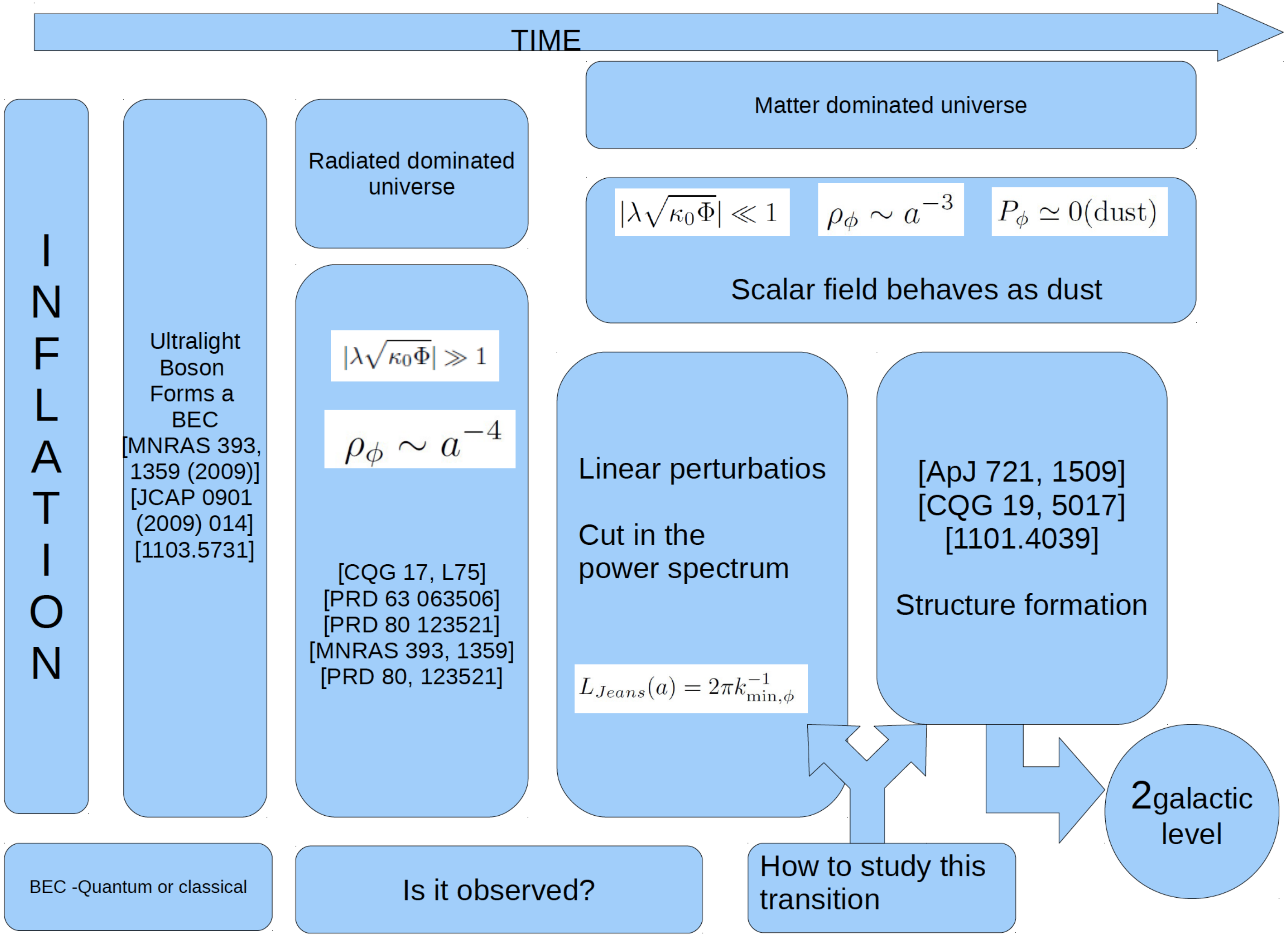
- A different approach: The Scalar Field Dark Matter model (SFDM)
  The Dark Matter is modeled by a scalar field with a ultra-light associated particle. ($m \sim 10^{-23}$eV)
  - At cosmological scales it behaves as cold dark matter
    T. Matos, L.A. Urena-Lopez, Class. Quant. Grav. **17** L75 (2000),
    V. Sahni and L.M. Wang, Phys. Rev D **62**, 103517 (2000).
  - At galactic scales, it does not have its problems: neither a cuspy profile, nor a over-density of satellite galaxies.

Ultralight scalars as cosmological dark matter
Lam Hui Jeremiah P. Ostriker ,Scott Tremaine, Edward Witten
Phys.Rev. D95 (2017) no.4, 043541

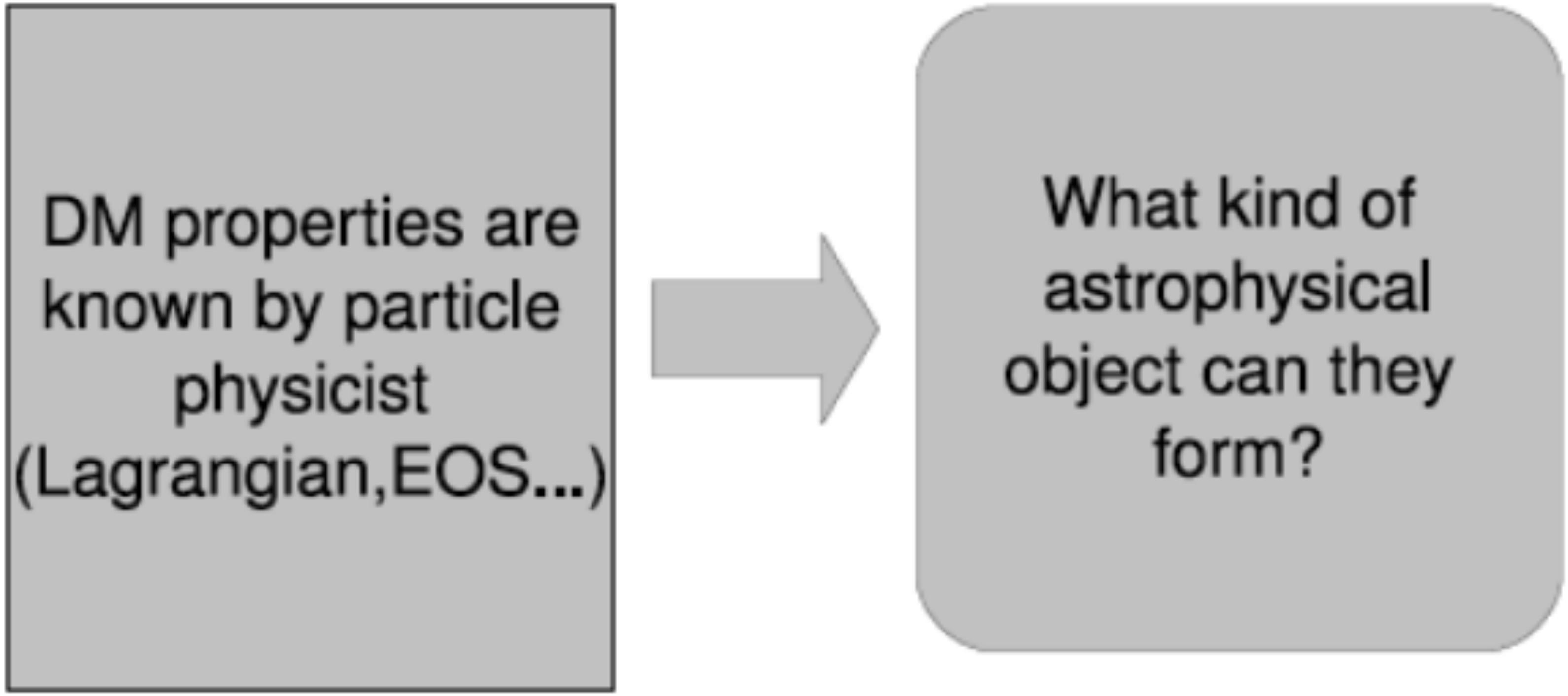
$$\mathcal{L}_{\mathrm{L-SFDM}} = \mathcal{L}_{\mathrm{GR}} + \mathcal{L}_{B} + \mathcal{L}_{\Lambda} - \sqrt{-g}[\Phi^{,\mu}\Phi_{,\mu} + 2\mathrm{V}(\Phi)]$$

An ultra-light boson as dark matter?

$$\mathcal{L}_{\mathrm{L-SFDM}} = \mathcal{L}_{\mathrm{GR}} + \mathcal{L}_B + \mathcal{L}_\Lambda - \sqrt{-g}[\Phi^{,\mu}\Phi_{,\mu} + 2\mathrm{V}(\Phi)]$$



Can the dark matter halo be a self-gravitating object made of ultralight spin-zero bosons?


DM properties are known by particle physicist (Lagrangian,EOS...)
What kind of astrophysical object can they form?

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Systems of Self-Gravitating Particles in General Relativity and the Concept of an Equation of State*

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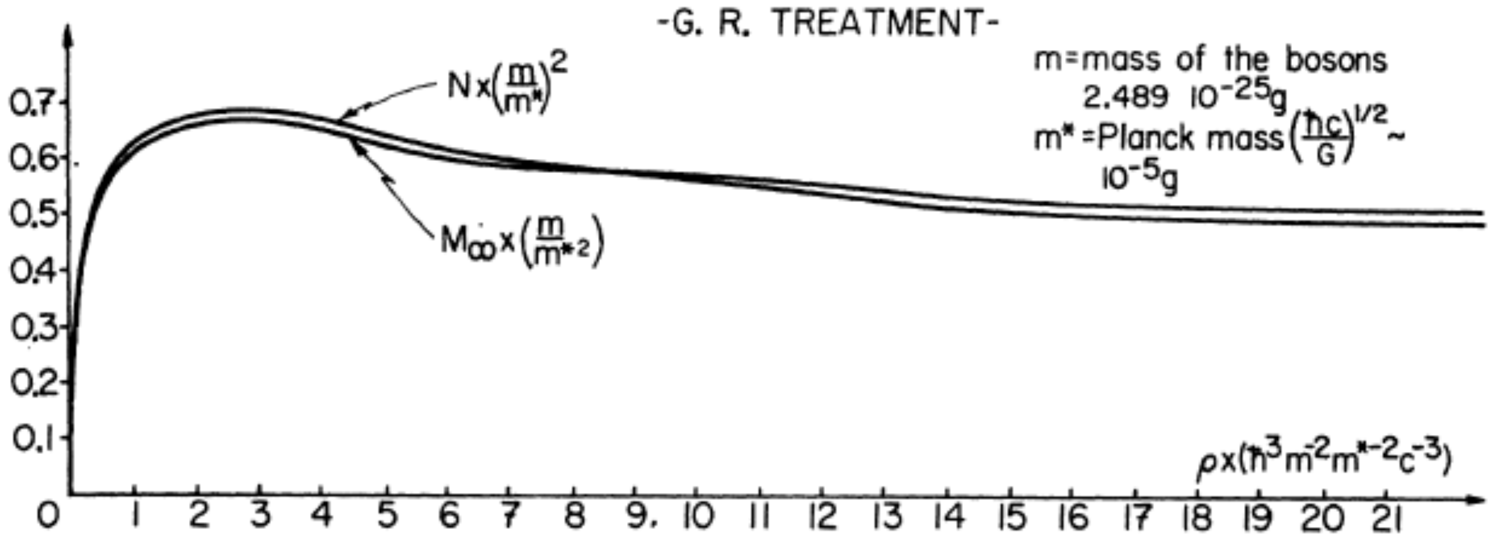
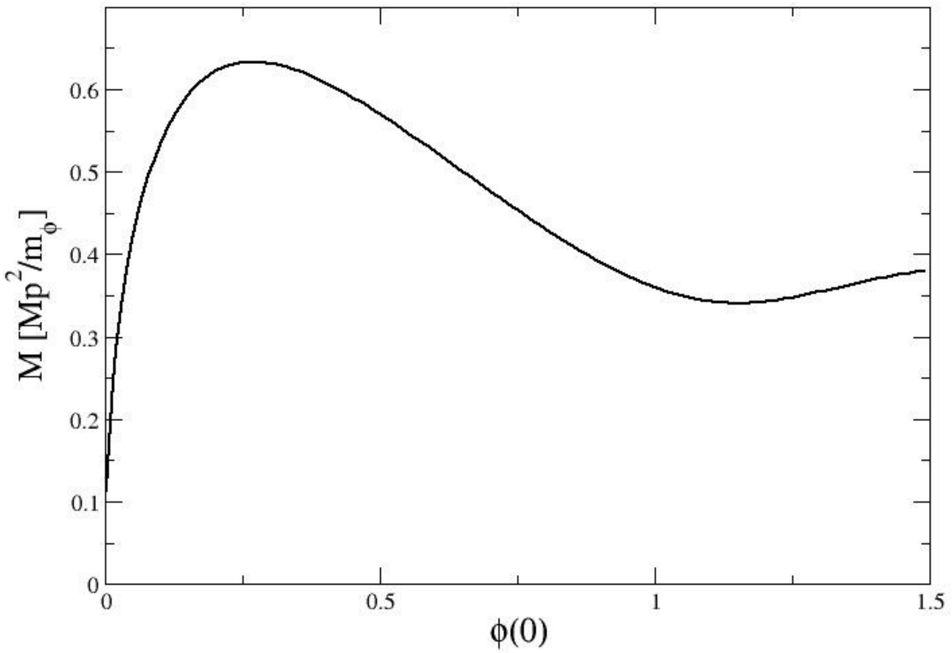
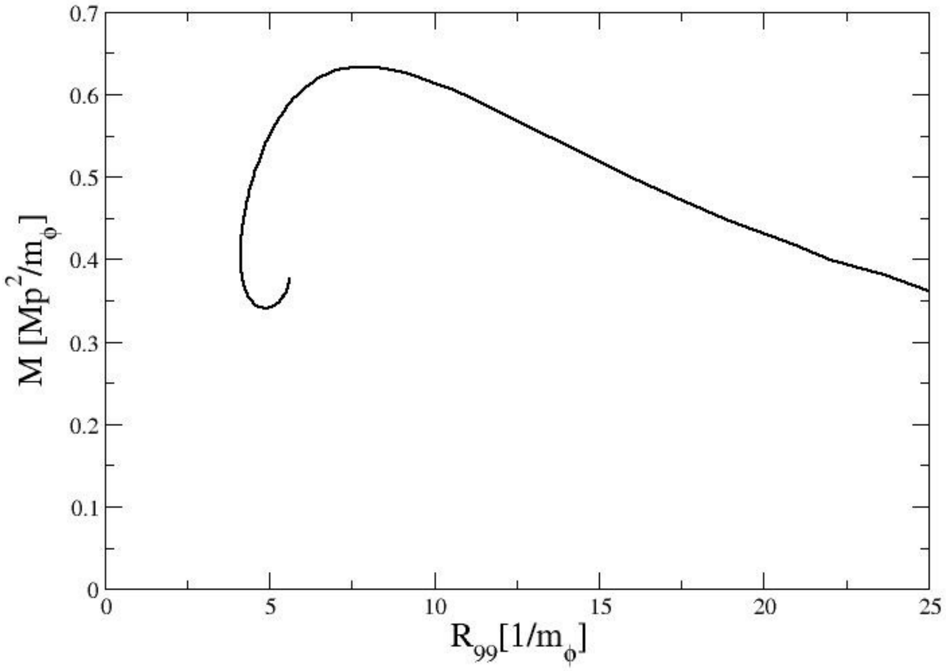
(Received 4 February 1969)

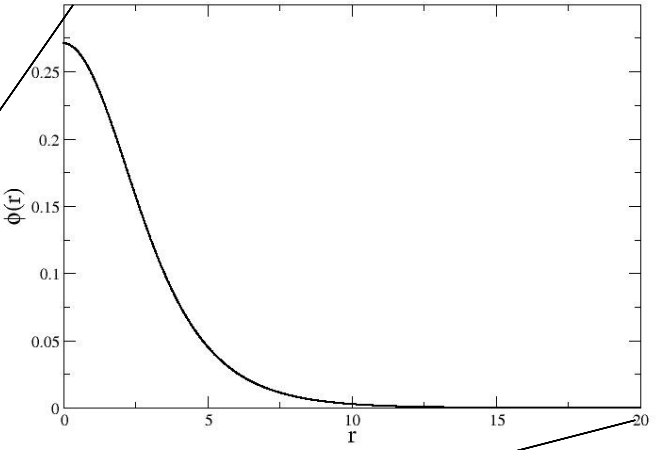


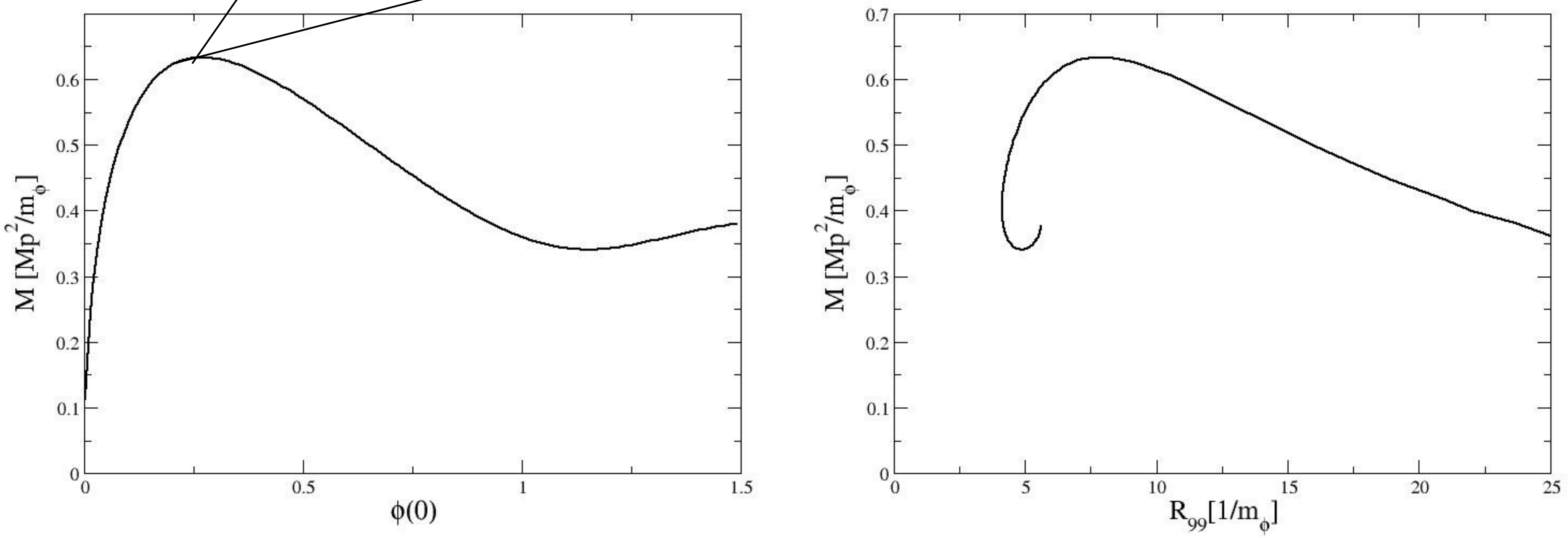
Possible boson star configurations

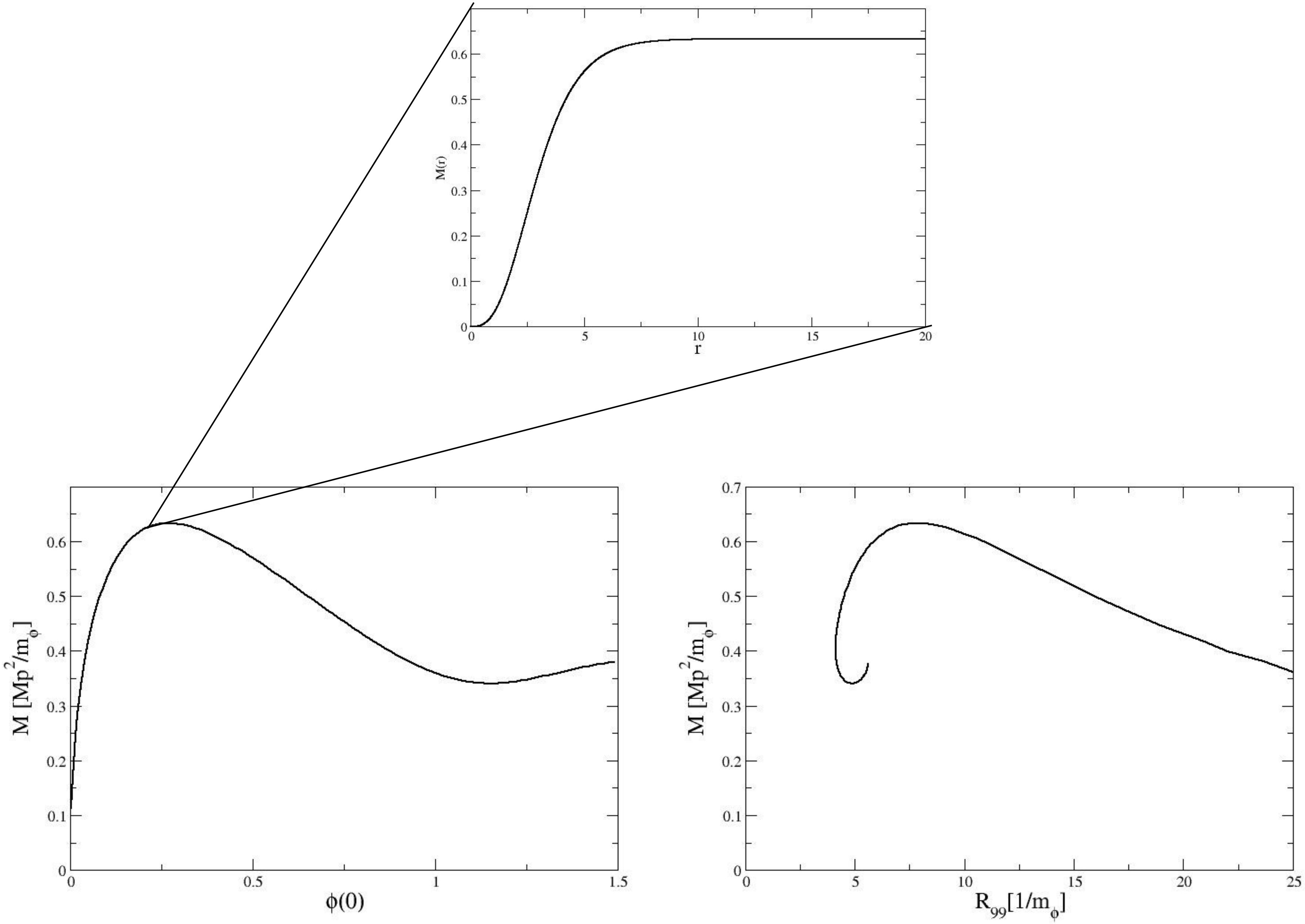




Possible bos rations

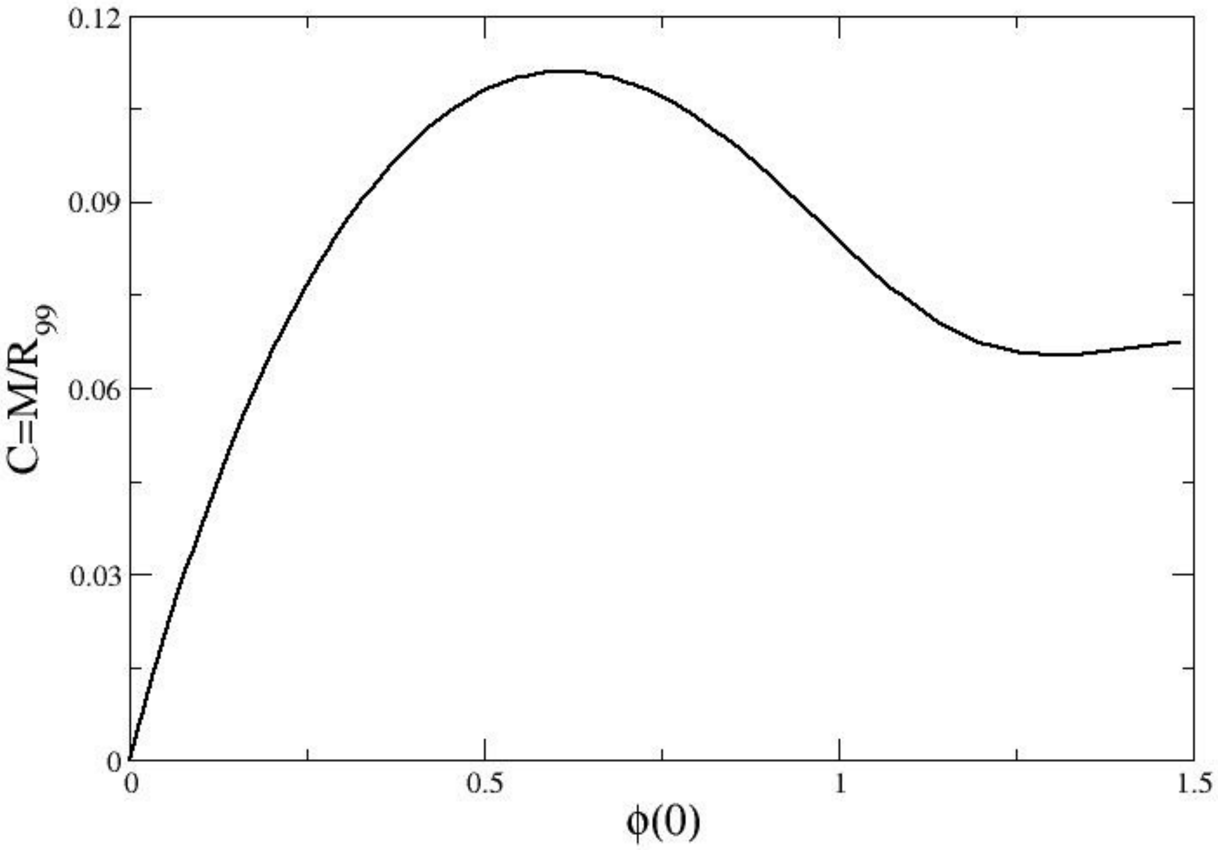






M(r)
r
M [Mp²/m_φ]
φ(0)
M [Mp²/m_φ]
R_99[1/m_φ]

Compactness



Typical compactness:

Sun= .00001

Dark matter halo=.0001

Neutron Star=0.2

Semiclassical description of the Einstein-Klein-Gordon system

$$G_{\mu\nu} = 8\pi G \langle \hat{T}_{\mu\nu} \rangle$$

The program:

- Step 1:

(i) Consider a stationary, globally hyperbolic background spacetime on which the free quantum fields are defined. The assumption of stationarity allows one to introduce a preferred space of "positive-norm" solutions of the matter field equations, hence a preferred vacuum state. This in turn provides a well-defined theory for the quantum fields, which we describe in terms of a Fock space representation. In particular, field operators can be written (formally) as linear combinations of creation and annihilation operators, wherein the "coefficients" are mode functions $f_I(x)$ that solve the *classical complex* field equations.

Paso 1: Espacio de Hilbert

Inner product

$$(\phi_1, \phi_2) := \int_{\Sigma_t} j^\mu n_\mu d\gamma = -i \int_{\Sigma_t} [\phi_1 (\pounds_n \phi_2^*) - (\pounds_n \phi_1) \phi_2^*] d\gamma.$$

Conmutation relations

$$[\hat{\phi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = i\delta^{(3)}(\vec{x} - \vec{y}),$$

$$[\hat{\phi}(t, \vec{x}), \hat{\phi}(t, \vec{y})] = [\hat{\pi}(t, \vec{x}), \hat{\pi}(t, \vec{y})] = 0,$$

A single real quantum scalar field

$$\hat{\phi}(x) = \sum_I [\hat{a}_I f_I(x) + \hat{a}_I^\dagger f_I^*(x)],$$

Hilbert's space

$$|N_1, N_2, \ldots\rangle = \frac{(\hat{a}_1^\dagger)^{N_1}}{\sqrt{N_1!}} \frac{(\hat{a}_2^\dagger)^{N_2}}{\sqrt{N_2!}} \ldots |0\rangle,$$

- Step 2:

(ii) Compute the expectation value $\langle \hat{T}_{\mu\nu} \rangle$ of the stress energy-momentum tensor operator with respect to a given state in the Fock space. In order to do so, we need a regularization and renormalization prescription that removes the ill-defined ultraviolet behavior of the theory, leading to sensible finite outcomes. To achieve this, in this work, we impose normal ordering. More sophisticated approaches include, e.g., adiabatic subtraction [69] and Pauli-Villars renormalization [70–72], although we expect the differences between such methods and ours to be suppressed in the limit of large occupation numbers, as we consider in our configurations, which we also assume to be far from the Planck scale. More generally, we can also compute a statistical average by tracing $\hat{T}_{\mu\nu}$ with a density operator. This offers the interesting possibility of considering, for instance, thermal states with a given temperature.

Step 2: Semiclasical gravity

Semiclassical

$$G_{\mu\nu} = 8\pi G\langle \hat{T}_{\mu\nu}\rangle .$$

Normal ordering

$$:\hat{a}_I \hat{a}_I^\dagger : = \hat{a}_I^\dagger \hat{a}_I$$

Second quantization

$$\hat{T}_{\mu\nu} = \frac{1}{2}\sum_{I,J}[\hat{a}_I \hat{a}_J T_{\mu\nu}(f_I, f_J) + \hat{a}_I^\dagger \hat{a}_J T_{\mu\nu}(f_I^*, f_J) + \mathrm{H.c.}].$$

Averages

$$\langle N_1, N_2, \ldots | \hat{T}_{\mu\nu} | N_1, N_2, \ldots \rangle = \sum_I N_I T_{\mu\nu}(f_I, f_I^*),$$

- Step 3:

(iii) Solve the semiclassical Einstein equations $G_{\mu\nu} = 8\pi G\langle\hat{T}_{\mu\nu}\rangle$ sourced by the expectation value (or statistical average) of the (renormalized) stress energy-momentum tensor. This step takes into account the backreaction of the quantum fields on the classical geometry.

Static space-time

$$ds^2 = -\alpha^2(\vec{x})dt^2 + \gamma_{ij}(\vec{x})dx^i dx^j;$$

$$\partial_t^2\phi - \alpha D^i(\alpha D_i\phi) + \alpha^2 {m_0}^2\phi = 0,$$

$$f_I(t,\vec{x}) = \frac{1}{\sqrt{2\omega_I}} e^{-i\omega_I t} u_I(\vec{x}),$$

$$\langle u_1, u_2\rangle := \int_\Sigma u_1^*(\vec{x}) u_2(\vec{x}) \frac{d\gamma}{\alpha(\vec{x})}, \quad u_1, u_2 \in Y.$$

$$G_{\mu\nu} = 8\pi G\langle \hat{T}_{\mu\nu}\rangle.$$

$$R^{(3)} = 16\pi G\rho,$$

$$R_{ij}^{(3)} - \frac{1}{\alpha}D_i D_j \alpha = 4\pi G[\gamma_{ij}(\rho - S) + 2S_{ij}],$$

where:

$$\rho = \sum_I \frac{N_I}{2\omega_I}\left[|Du_I|^2 + \left(\frac{{\omega_I}^2}{\alpha^2} + {m_0}^2\right)|u_I|^2\right],$$

$$j_k = \sum_I \frac{N_I}{2}\frac{i}{\alpha}[(D_k u_I)u_I^* - u_I(D_k u_I^*)],$$

$$S_{ij} = \sum_I \frac{N_I}{2\omega_I}\left\{(D_i u_I)(D_j u_I^*) + (D_j u_I)(D_i u_I^*) - \gamma_{ij}\left[|D_i u_I|^2 - \left(\frac{{\omega_I}^2}{\alpha^2} - {m_0}^2\right)|u_I|^2\right]\right\},$$

Static-spherically symmetric

$$\gamma_{ij}dx^i dx^j = \gamma^2 dr^2 + r^2 d\Omega^2, \qquad \gamma = \left(1 - \frac{2GM}{r}\right)^{-1/2},$$

Harmonic expansion

$$u_I(\vec{x}) = v_{n\ell}(r) Y^{\ell m}(\vartheta, \varphi), \qquad I = (n\ell m),$$

Inner product

$$\int_0^\infty v_{n\ell}(r) v^*_{n'\ell}(r) \frac{\gamma(r)}{\alpha(r)} r^2 dr = \delta_{nn'}.$$

$$-\frac{\alpha}{\gamma r^2}\left(\frac{\alpha r^2}{\gamma} v'_{n\ell}\right)' + \alpha^2 \left[\frac{\ell(\ell+1)}{r^2} + {m_0}^2\right] v_{n\ell} = (\omega_{n\ell})^2 v_{n\ell}$$

Einstein-Klein-Gordon system

$$\frac{2GM'}{r^2} = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}} \left[\frac{|v'_{n\ell}|^2}{\gamma^2} + \left(\frac{(\omega_{n\ell})^2}{\alpha^2} + {m_0}^2 + \frac{\ell(\ell+1)}{r^2} \right) |v_{n\ell}|^2 \right], \tag{44a}$$

$$\frac{1}{\alpha\gamma r^2} \left(\frac{r^2\alpha'}{\gamma} \right)' = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}} \left[\left(2\frac{(\omega_{n\ell})^2}{\alpha^2} - {m_0}^2 \right) |v_{n\ell}|^2 \right], \tag{44b}$$

$$\frac{(\alpha\gamma)'}{r\alpha\gamma^3} = \sum_{n\ell} \frac{\kappa_\ell N_{n\ell m}}{\omega_{n\ell}} \left[\frac{|v'_{n\ell}|^2}{\gamma^2} + \frac{(\omega_{n\ell})^2}{\alpha^2} |v_{n\ell}|^2 \right], \tag{44c}$$

Normalized functions: $$\psi_{n\ell} = \sqrt{\frac{N_{n\ell m}}{\omega_{n\ell}}} v_{n\ell}.$$

$$\psi''_{n\ell} = -\left[\gamma^2 + 1 - (2\ell+1) r^2 \gamma^2 \left(\frac{\ell(\ell+1)}{r^2} + {m_0}^2\right)(\psi_{n\ell})^2\right] \frac{{\psi_{n\ell}}'}{r} - \left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - {m_0}^2\right)\gamma^2 \psi_{n\ell},$$

$$\gamma' = \sum_{n\ell} \frac{2\ell+1}{2} r\gamma \left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2} + {m_0}^2\right)\gamma^2 (\psi_{n\ell})^2 + (\psi'_{n\ell})^2\right] - \left(\frac{\gamma^2 - 1}{2r}\right)\gamma,$$

$$\alpha' = \sum_{n\ell} \frac{2\ell+1}{2} r\alpha \left[\left(\frac{(\omega_{n\ell})^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - {m_0}^2\right)\gamma^2 (\psi_{n\ell})^2 + (\psi'_{n\ell})^2\right] + \left(\frac{\gamma^2 - 1}{2r}\right)\alpha,$$

This leads to:

$$
\begin{aligned}
\psi''_{n\ell} &= -\left[\gamma^2 + 1 - (2\ell+1)r^2\gamma^2\left(\frac{\ell(\ell+1)}{r^2} + m_0^2\right)\psi_{n\ell}^2\right]\frac{\psi'_{n\ell}}{r} - \left(\frac{\omega_{n\ell}^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - m_0^2\right)\gamma^2\psi_{n\ell} \\
\frac{d\gamma}{dr} &= \sum_{n\ell}\frac{2\ell+1}{2}r\gamma\left[\left(\frac{\omega_{n\ell}^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2} + m_0^2\right)\gamma^2\psi_{n\ell}^2 + \psi'^2_{n\ell}\right] - \left(\frac{\gamma^2-1}{2r}\right)\gamma, \\
\frac{d\alpha}{dr} &= \sum_{n\ell}\frac{2\ell+1}{2}r\alpha\left[\left(\frac{\omega_{n\ell}^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - m_0^2\right)\gamma^2\psi_{n\ell}^2 + \psi'^2_{n\ell}\right] + \left(\frac{\gamma^2-1}{2r}\right)\alpha,
\end{aligned}
$$

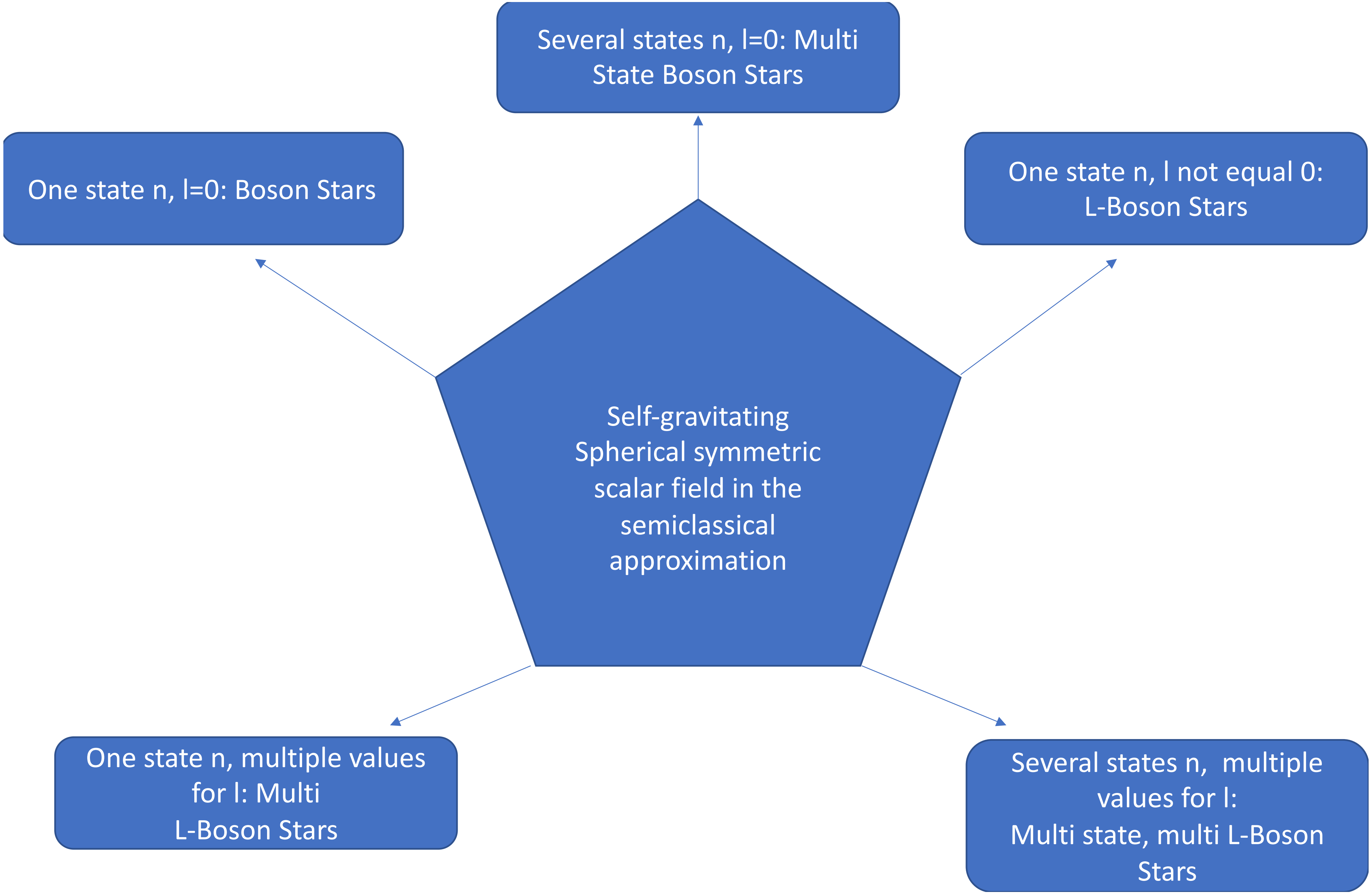
$n = 1, \ell = 0$ — Boson stars

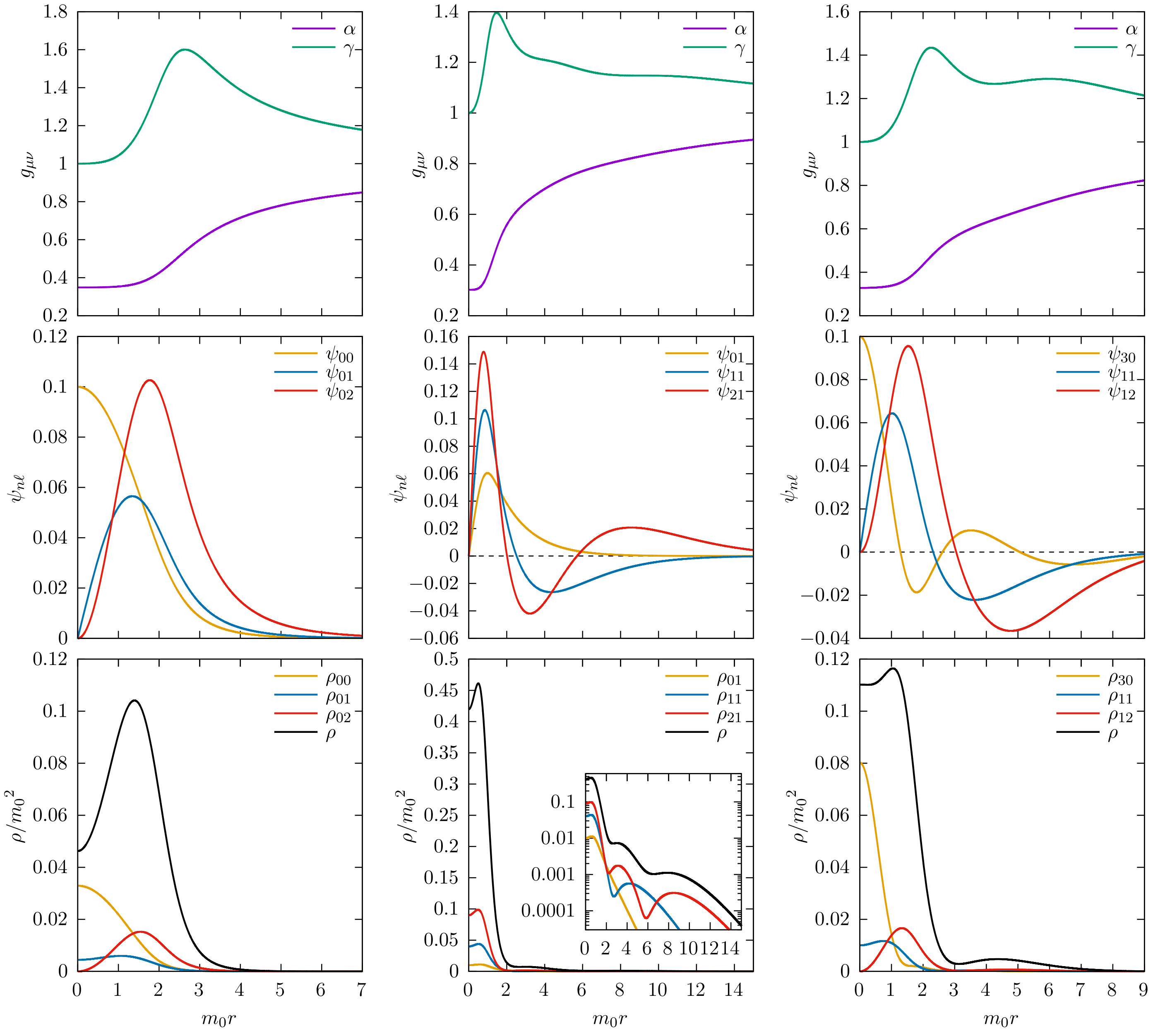
$n = 1, 2, 3..., \ell = 0$ — Multistate Boson stars

$n = 1, \ell = 1, 2, 3...$ — $\ell-$Boson Star

Boson stars relatives

Name	n	ℓ	Relativistic	Newtonian
Multi-ℓ multistate boson star	$n_1, n_2, \ldots, n_p$	$\ell_1, \ell_2, \ldots, \ell_q$	Sec. V	$\cdots$
Multistate ℓ-boson star	$n_1, n_2, \ldots, n_p$	ℓ_1	Sec. V	$\cdots$
Multistate boson star	$n_1, n_2, \ldots, n_p$	0	[49–51]	[74]
Boson star	n_1	0	[1–6, 8]	[73, 121]
ℓ-Boson star	n_1	ℓ_1	[75–78]	[81–83]
Multi-ℓ boson star	n_1	$\ell_1, \ell_2, \ldots, \ell_q$	Sec. V	[83]


Several states n, l=0: Multi State Boson Stars
One state n, l=0: Boson Stars
One state n, l not equal 0: L-Boson Stars
Self-gravitating Spherical symmetric scalar field in the semiclassical approximation
One state n, multiple values for l: Multi L-Boson Stars
Several states n, multiple values for l: Multi state, multi L-Boson Stars



(a) Multi-ℓ boson star

(b) Multi-state ℓ-boson star

(c) Multi-ℓ multi-state boson star

Name	n	ℓ	$\psi^0_{n\ell}/{m_0}^{\ell}$	$\omega_{n\ell}/m_0$	${m_0}^2 N_{nl}$	${m_0}^2 N$	Fig.
	0	0	0.1	0.5278	0.0195		
Multi-ℓ boson star	0	1	0.2	0.6453	0.0243	0.8134	1(a)
	0	2	0.4	0.7736	0.1442		
	0	1	0.3	0.7438	0.0289		
Multistate ℓ-boson star	1	1	0.6	0.8235	0.1150	1.3884	1(b)
	2	1	0.9	0.8792	0.3189		
	3	0	0.1	0.8679	0.0133		
Multi-ℓ multistate boson star	1	1	0.3	0.7497	0.0439	1.2745	1(c)
	1	2	0.5	0.8247	0.2259		

Conclusions

- Boson stars might arise as self-gravitating compact objects made of ultra-ligh spin zero DM particles

- There are many realization of Boson Stars relatives (such as Multistate boson stars, L-Boson stars, and more)

- They could be more compact, with a richer structure and because their stability, if they form, they can be astrophysical candidates.

Extra-slides (stability)

Multi-State Boson stars (MSBS)

$$\hat{\Phi} = \sum_{nlm} \hat{b}_{nlm} \Phi_{nlm}(t, \mathbf{x}) + \hat{b}^{\dagger}_{nlm} \Phi^{*}_{nlm}(t, \mathbf{x})$$

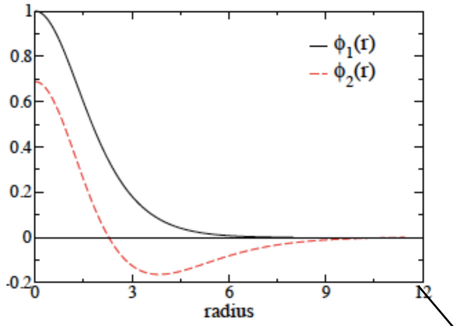
$$\hat{T}_{ab} = \partial_a \hat{\Phi} \partial_b \hat{\Phi} - \frac{1}{2} g_{ab} (g^{cd} \partial_c \hat{\Phi} \partial_d \hat{\Phi} + \mu^2 |\hat{\Phi}|^2)$$

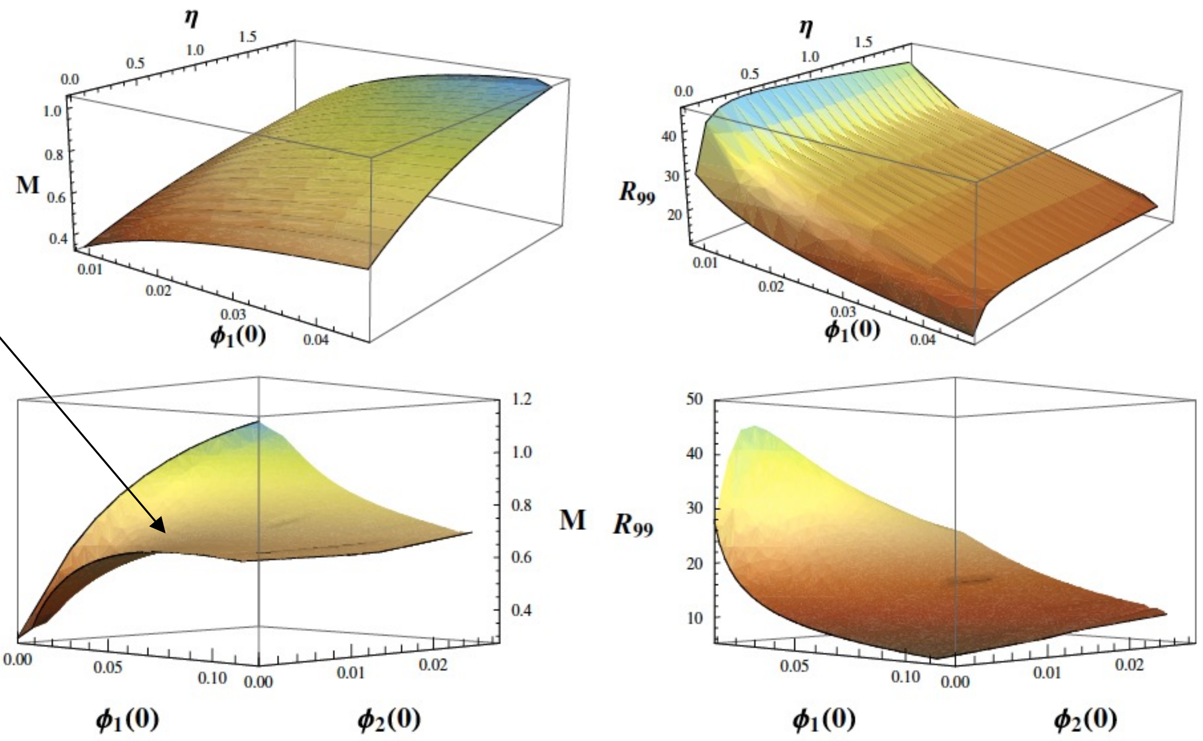
$$G_{ab} = 8\pi \langle Q | \hat{T}_{ab} | Q \rangle$$

$$ds^2 = -\alpha^2(r) dt^2 + a^2(r) dr^2 + r^2 d\Omega .$$

$$\begin{aligned}
\partial_r a &= \frac{a}{2} \left\{ -\frac{a^2 - 1}{r} + 4\pi r \sum_{n=1}^{\mathscr{I}} \left[\left(\frac{\omega_n^2}{\alpha^2} + m^2 \right) a^2 \phi_n^2 + \Phi_n^2 \right] \right\} , \\
\partial_r \alpha &= \frac{\alpha}{2} \left\{ \frac{a^2 - 1}{r} + 4\pi r \sum_{n=1}^{\mathscr{I}} \left[\left(\frac{\omega_n^2}{\alpha^2} - m^2 \right) a^2 \phi_n^2 + \Phi_n^2 \right] \right\} , \\
\partial_r \phi_n &= \Phi_n , \\
\partial_r \Phi_n &= -\left\{ 1 + a^2 - 4\pi r^2 a^2 m^2 \left(\sum_{s=1}^{\mathscr{I}} \phi_s^2 \right) \right\} \frac{\Phi_n}{r} - \left(\frac{\omega_n^2}{\alpha^2} - m^2 \right) \phi_n a^2 .
\end{aligned}$$

n=2





$$\eta = \frac{N^{(2)}}{N^{(1)}}.$$

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Multistate boson stars

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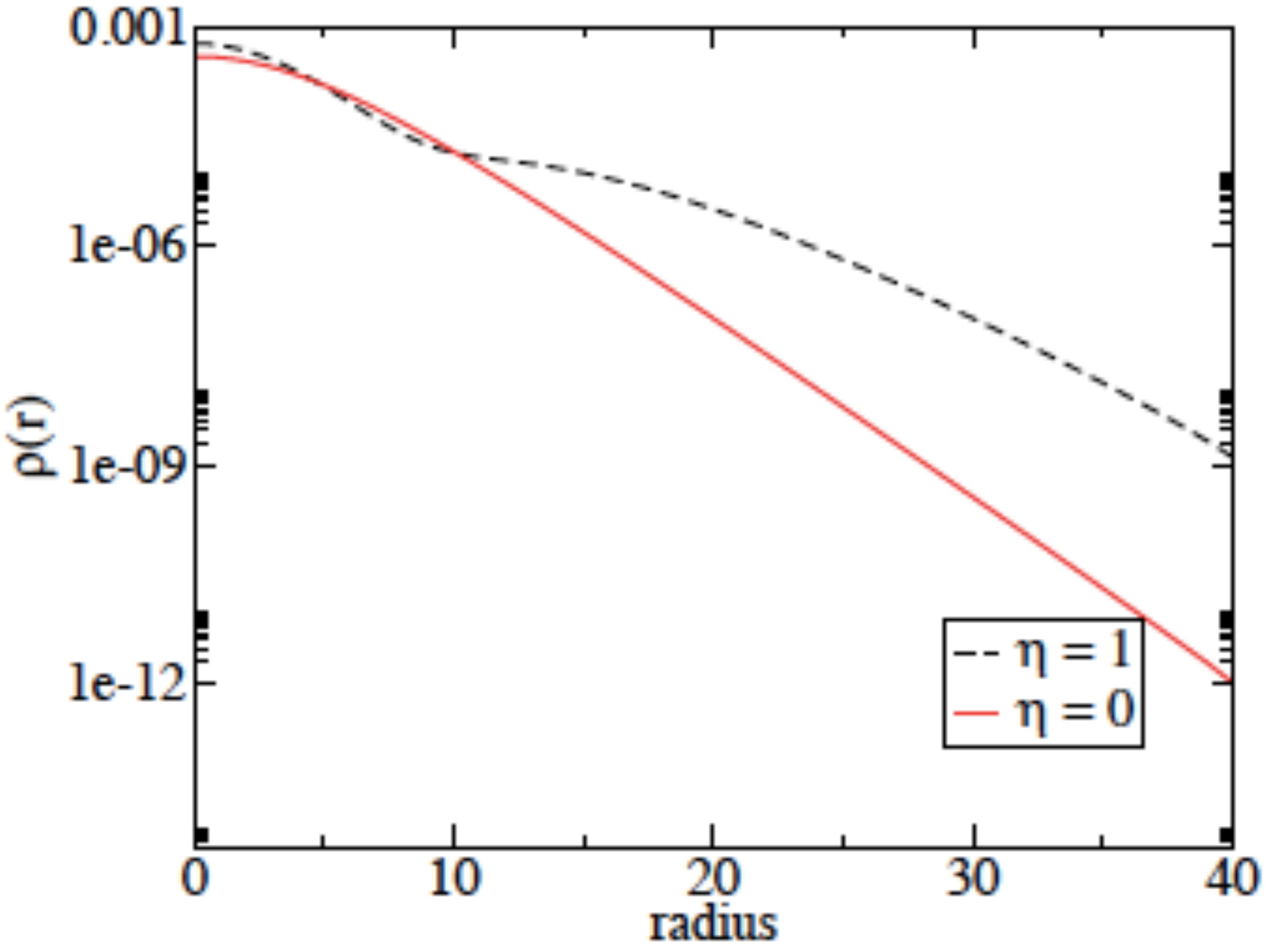


0.001
1e-06
1e-09
1e-12
ρ(r)
0
10
20
30
40
radius
η = 1
η = 0

$\ell-$Boson Star

$$\sum_{m=-\ell}^{\ell} |Y^{\ell m}(\vartheta,\varphi)|^2 = \frac{2\ell+1}{4\pi}$$

$$T_{\mu\nu} = \frac{1}{2}\sum_i \left[\nabla_\mu \Phi_i^* \nabla_\nu \Phi_i + \nabla_\mu \Phi_i \nabla_\nu \Phi_i^* \right.$$
$$\left. -g_{\mu\nu}\left(\nabla_\alpha \Phi_i^* \nabla^\alpha \Phi_i + \mu^2 \Phi_i^* \Phi_i\right)\right]$$

$$\Phi_{\ell m}(t,r,\vartheta,\varphi) = \phi_\ell(t,r) Y^{\ell m}(\vartheta,\varphi)$$

$$ds^2 = -\alpha^2 dt^2 + \gamma^2 dr^2 + r^2 d\Omega^2, \quad \gamma^2 := \frac{1}{1-\frac{2M}{r}},$$

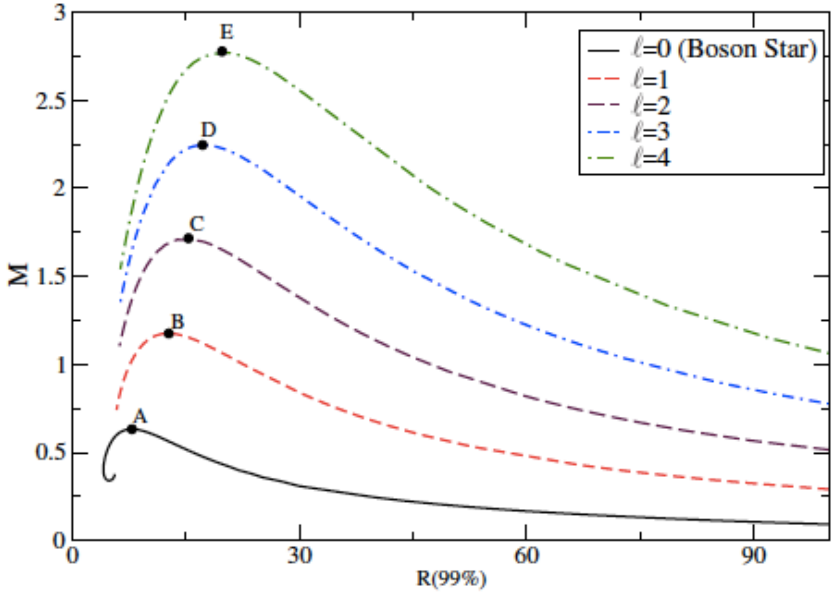
$$u_\ell := \psi_\ell / r^\ell$$

$\ell-$Boson Star

$$\gamma' = \frac{2\ell+1}{2} r\gamma \left[\left(\frac{\omega^2}{\alpha^2} + \frac{\ell(\ell+1)}{r^2} + \mu^2 \right) \gamma^2 u_\ell^2 r^{2\ell} + (u_\ell' r^\ell + \ell u_\ell r^{\ell-1})^2 \right] - \left(\frac{\gamma^2 - 1}{2r} \right) \gamma,$$

$$\alpha' = \frac{2\ell+1}{2} r\alpha \left[\left(\frac{\omega^2}{\alpha^2} - \frac{\ell(\ell+1)}{r^2} - \mu^2 \right) \gamma^2 u_\ell^2 r^{2\ell} + (u_\ell' r^\ell + \ell u_\ell r^{\ell-1})^2 \right] + \left(\frac{\gamma^2 - 1}{2r} \right) \alpha,$$

$$u_\ell'' = \left(\mu^2 - \frac{\omega^2}{\alpha^2} \right) \gamma^2 u_\ell - (\gamma^2 + 2\ell + 1) \frac{u_\ell'}{r} + \ell^2 (\gamma^2 - 1) \frac{u_\ell}{r^2} + (2\ell+1) \left(\mu^2 + \frac{\ell(\ell+1)}{r^2} \right) \gamma^2 (r u_\ell' + \ell u_\ell) u_\ell^2 r^{2\ell},$$



Letters

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Letter

ℓ-boson stars

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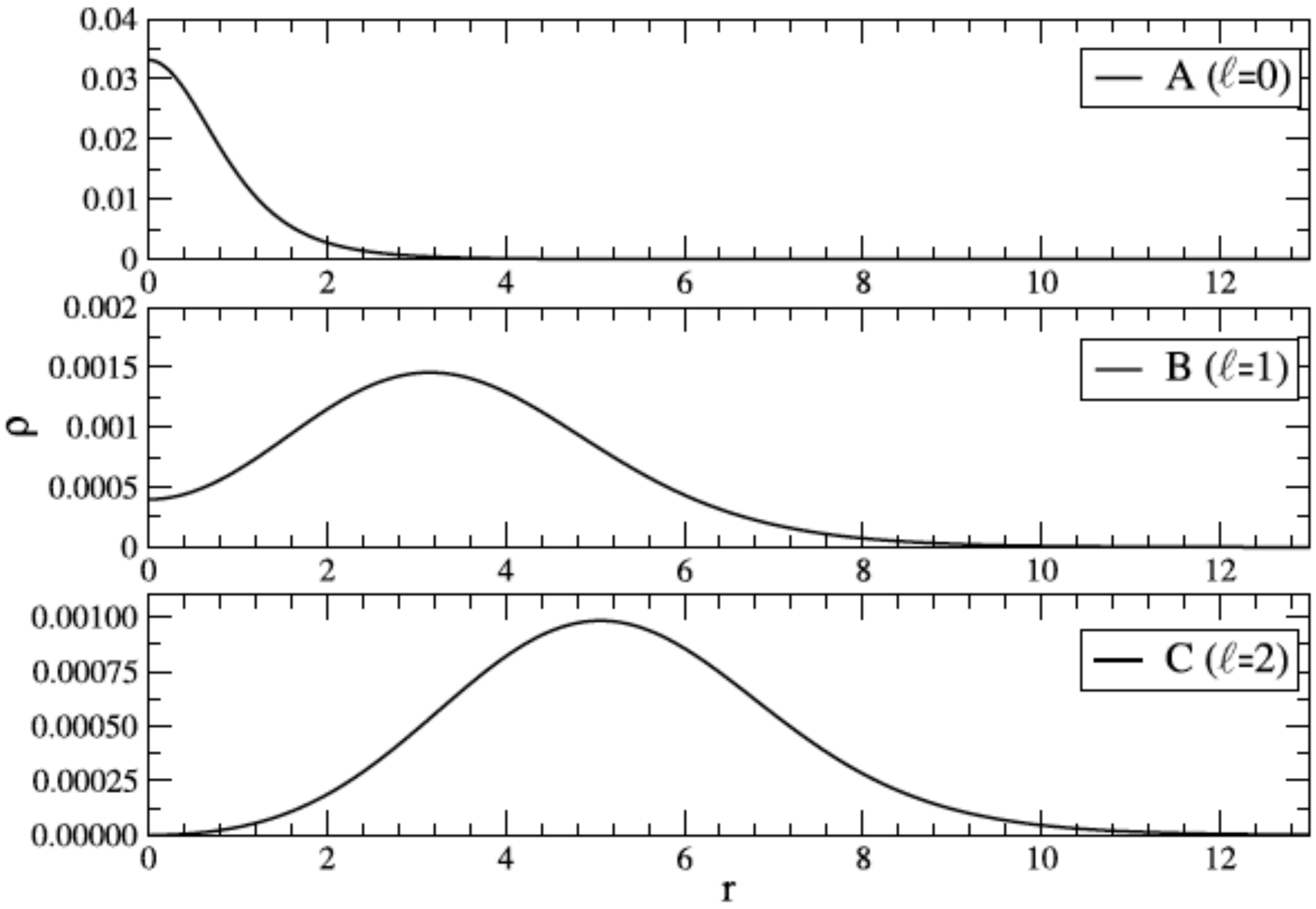


Configuration	M	$R(99\%)$	ω	$M/R(99\%)$
A ($\ell = 0$)	0.63	7.89	0.854	0.08
B ($\ell = 1$)	1.18	12.75	0.836	0.09
C ($\ell = 2$)	1.72	15.35	0.832	0.11
D ($\ell = 3$)	2.25	17.22	0.820	0.13
E ($\ell = 4$)	2.78	19.80	0.819	0.14

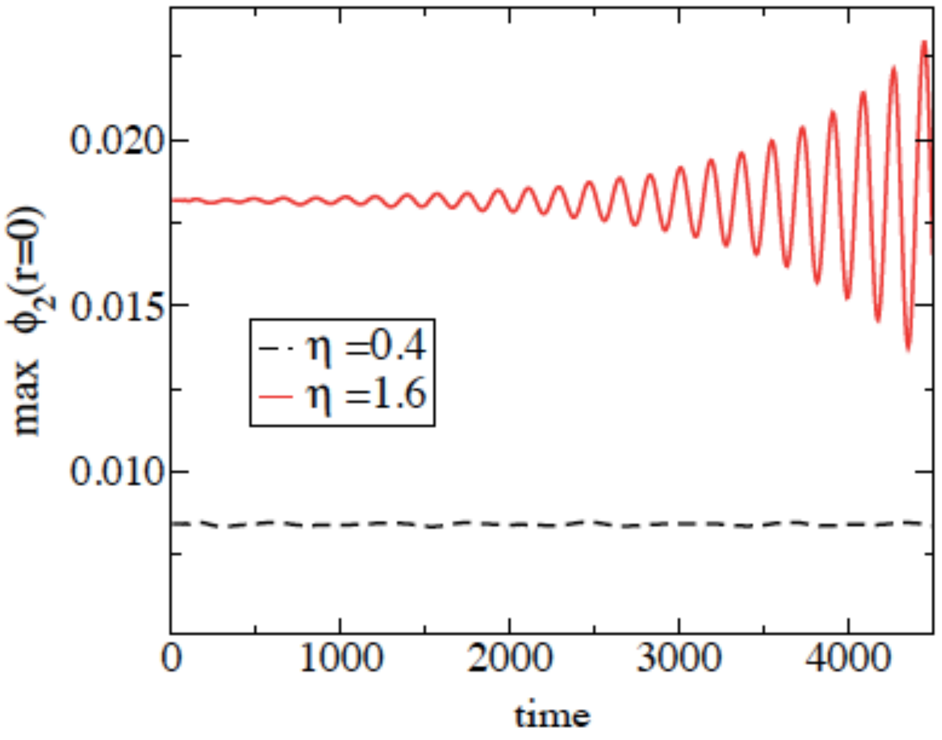
Astrophysical realization of boson stars relatives demands stability

Numerical perturbation analysis: Multistate boson stars

Boson stars in excited states are unstable under numerical perturbations.

Multistate boson stars, even with particles in the excited states, can be stable

$$\eta = \frac{N^{(2)}}{N^{(1)}}.$$

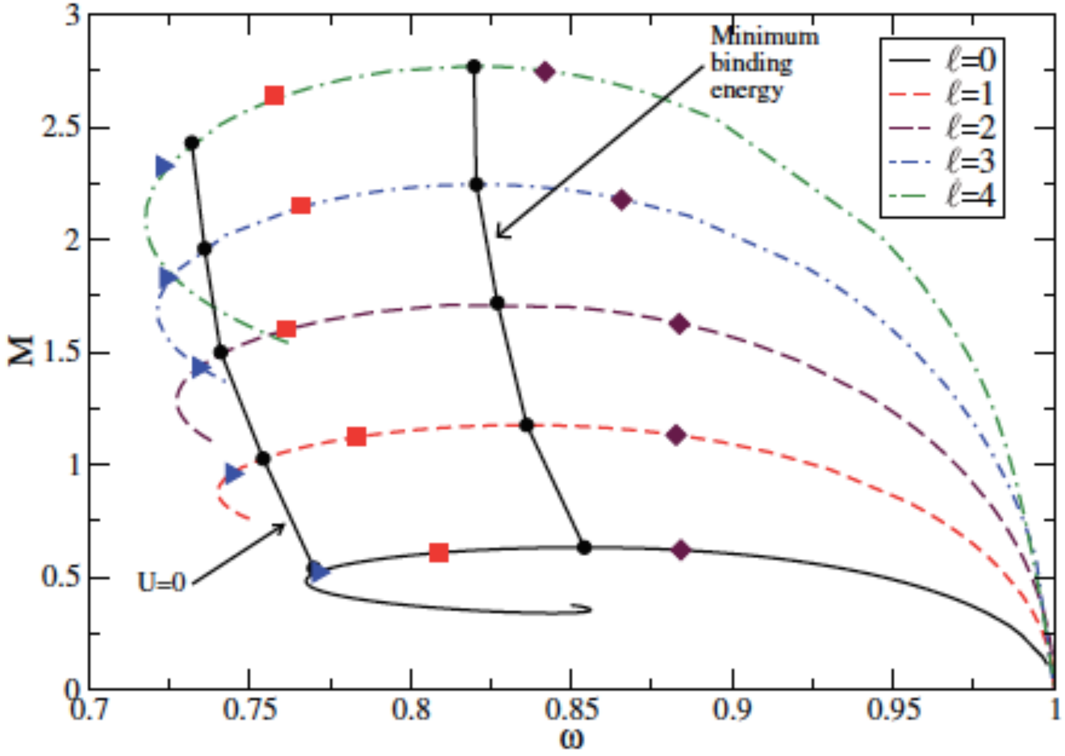


Stable if $\eta < 1$

Numerical perturbation analysis: $\ell-$Boson Star

ℓ	a_0	ω	Perturbation	M	N_B	U	ϵ/φ_R^{max}	s	r_0	End result
0	0.2	0.884 01	Type 0	0.6209	0.6391	−0.0182	—	—	—	Stable
0	0.2	0.884 01	Type I	0.6211	0.6394	−0.0183	+0.005	0	0.0	Stable
0	0.2	0.884 01	Type I	0.6207	0.6389	−0.0182	−0.005	0	0.0	Stable
0	0.2	0.884 01	Type II	0.6209	0.6391	−0.0182	+0.005	−1	0.0	Stable
0	0.2	0.884 01	Type II	0.6209	0.6391	−0.0182	−0.005	−1	0.0	Stable
0	0.2	0.884 01	Type III	0.6238	0.6412	−0.0174	+0.01	+1	20.0	Stable
0	0.2	0.884 01	Type III	0.6237	0.6372	−0.0135	+0.01	−1	20.0	Stable
0	0.4	0.808 66	Type 0	0.6088	0.6235	−0.0147	—	—	—	Black hole
0	0.4	0.808 66	Type I	0.6096	0.6246	−0.0150	+0.005	0	0.0	Black hole
0	0.4	0.808 66	Type I	0.6079	0.6225	−0.0146	−0.005	0	0.0	Migration to stable branch
0	0.4	0.808 66	Type II	0.6087	0.6235	−0.0148	+0.005	−1	0.0	Migration to stable branch
0	0.4	0.808 66	Type II	0.6088	0.6236	−0.0148	−0.005	−1	0.0	Black hole
0	0.4	0.808 66	Type III	0.6193	0.6305	−0.0112	+0.01	+1	20.0	Black hole
0	0.4	0.808 66	Type III	0.6193	0.6166	+0.0027	+0.01	−1	20.0	Black hole
0	0.6	0.771 34	Type 0	0.5248	0.5167	+0.0081	—	—	—	Black hole
0	0.6	0.771 34	Type I	0.5266	0.5190	+0.0075	+0.005	0	0.0	Black hole
0	0.6	0.771 34	Type I	0.5230	0.5144	+0.0086	−0.005	0	0.0	Explosion to infinity
0	0.6	0.771 34	Type II	0.5246	0.5165	+0.0081	+0.005	−1	0.0	Explosion to infinity
0	0.6	0.771 34	Type II	0.5250	0.5169	+0.0081	−0.005	−1	0.0	Black hole
0	0.6	0.771 34	Type III	0.5481	0.5314	+0.0167	+0.01	+1	20.0	Black hole
0	0.6	0.771 34	Type III	0.5480	0.5020	+0.0460	+0.01	−1	20.0	Black hole

Three fates of $\ell-$Boson Star



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Dynamical evolutions of ℓ-boson stars in spherical symmetry

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Linear stability analysis for $\ell-$Boson Star

Pertubre the system $\phi_\ell(t,r) = e^{i\omega t}\left[\psi_{\ell 1}(t,r) + i\psi_{\ell 2}(t,r)\right]$

$$\begin{aligned}
\delta\psi_{\ell 1}(t,r) &= \psi_{\ell 0}(r)\delta\varphi_{\ell 1}(t,r),\\
\delta\psi_{\ell 2}(t,r) &= \psi_{\ell 0}(r)\delta\varphi_{\ell 2}(t,r),\\
\delta\alpha(t,r) &= \frac{1}{2}\alpha_0(r)\delta\nu(t,r),\\
\delta\gamma(t,r) &= \frac{1}{2}\gamma_0(r)\delta\lambda(t,r),
\end{aligned}$$

$$\begin{aligned}
&\delta\varphi''_{\ell 1} + \left[\frac{2}{r} + \frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma}\right]\delta\varphi'_{\ell 1} + \frac{1}{\kappa_\ell r\psi_\ell^2}\delta\lambda' - \frac{\gamma^2}{\alpha^2}\delta\ddot{\varphi}_{\ell 1}\\
&+\left\{\frac{1-2r\frac{\gamma'}{\gamma}}{\kappa_\ell r^2\psi_\ell^2} + \frac{\psi'_\ell}{\psi_\ell}\left[\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma} + \frac{\psi'_\ell}{\psi_\ell} + \frac{1}{r}\right] - \gamma^2\left[\mu^2 + \frac{\ell(\ell+1)}{r^2} - \frac{\omega^2}{\alpha^2}\right]\right\}\delta\lambda\\
&-2\gamma^2\left\{\mu^2 + \frac{\ell(\ell+1)}{r^2} + \frac{\omega^2}{\alpha^2} + \frac{\psi'^2_\ell}{\gamma^2\psi_\ell^2} + \kappa_\ell r\left[\mu^2 + \frac{\ell(\ell+1)}{r^2}\right]\psi_\ell\psi'_\ell\right\}\delta\varphi_{\ell 1} = 0,
\end{aligned}$$

$$\begin{aligned}
&\delta\lambda'' + 3\left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma}\right)\delta\lambda' + 4\kappa_\ell\left\{2\psi_\ell\psi'_\ell - r\gamma^2\left[\mu^2 + \frac{\ell(\ell+1)}{r^2}\right]\psi_\ell^2\right\}\delta\varphi'_{\ell 1} - \frac{\gamma^2}{\alpha^2}\delta\ddot{\lambda}\\
&-2\left\{2\kappa_\ell\psi'^2_\ell + \frac{1}{r^2} + \left(\frac{\gamma'}{\gamma}\right)' - \left(\frac{\alpha'}{\alpha} - \frac{\gamma'}{\gamma}\right)^2 - \frac{1}{r}\left(2\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma}\right)\right\}\delta\lambda\\
&+4\kappa_\ell\left\{2\psi'^2_\ell - r\gamma^2\left[\mu^2 + \frac{\ell(\ell+1)}{r^2}\right]\psi_\ell^2\left[2\frac{\psi'_\ell}{\psi_\ell} + 2\frac{\alpha'}{\alpha} + \frac{\gamma'}{\gamma}\right] + \gamma^2\frac{\ell(\ell+1)}{r^2}\psi_\ell^2\right\}\delta\varphi_{\ell 1} = 0,
\end{aligned}$$

→ **Pulsation equations**

Pulsation equations can be rewriten:

Define $f_1 = \delta\varphi_{\ell 1}, \qquad f_2 = \frac{1}{\omega}\left[\frac{\delta\lambda}{2\kappa_\ell r \psi_\ell^2} - \frac{\psi_\ell'}{\psi_\ell}\delta\varphi_{\ell 1}\right].$

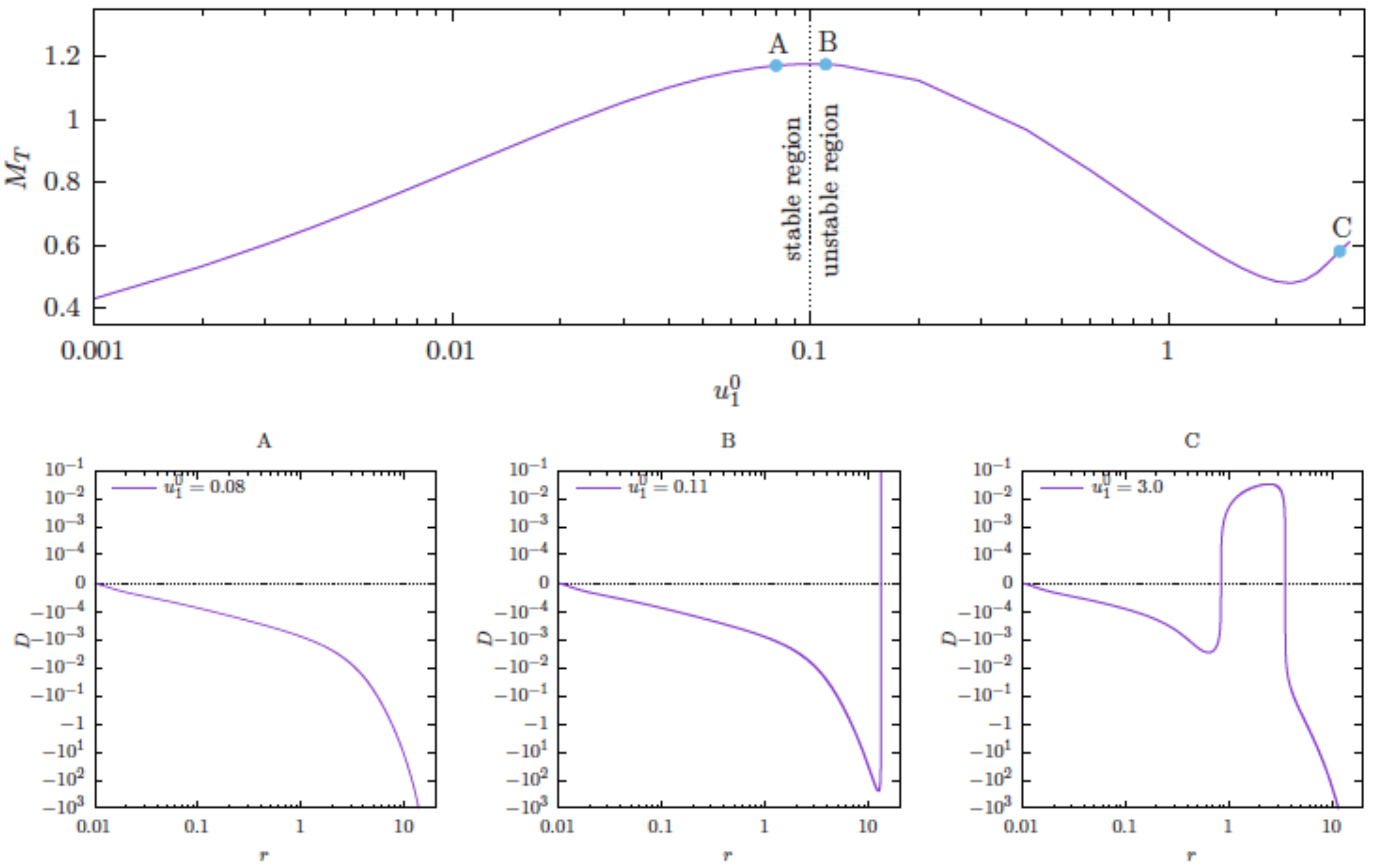
The system for $f := \begin{pmatrix} f_1 \\ f_2 \end{pmatrix}$ can be written in the form

$$\frac{\gamma^2}{\alpha^2} A\ddot{f} = \frac{d}{dr}\left(A\frac{df}{dr}\right) + \frac{d}{dr}(Bf) - B^T\frac{df}{dr} + Cf,$$

the form $\ddot{f} = -\mathcal{H}f$, with $\mathcal{H}$ the Schrödinger-type operator given by

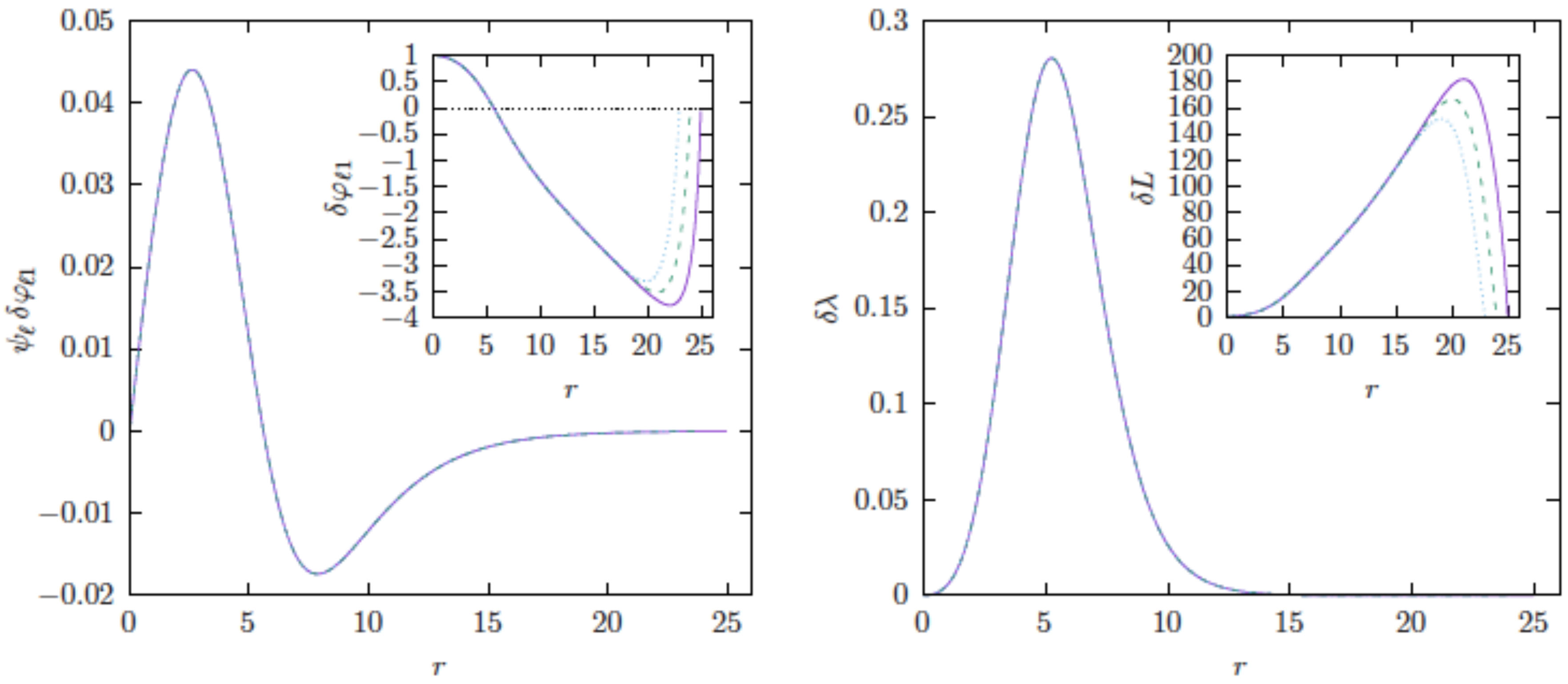
$$\mathcal{H} = \frac{\alpha^2}{\gamma^2}A^{-1}\left[-\frac{d}{dr}A\frac{d}{dr} - \frac{d}{dr}B + B^T\frac{d}{dr} - C\right].$$

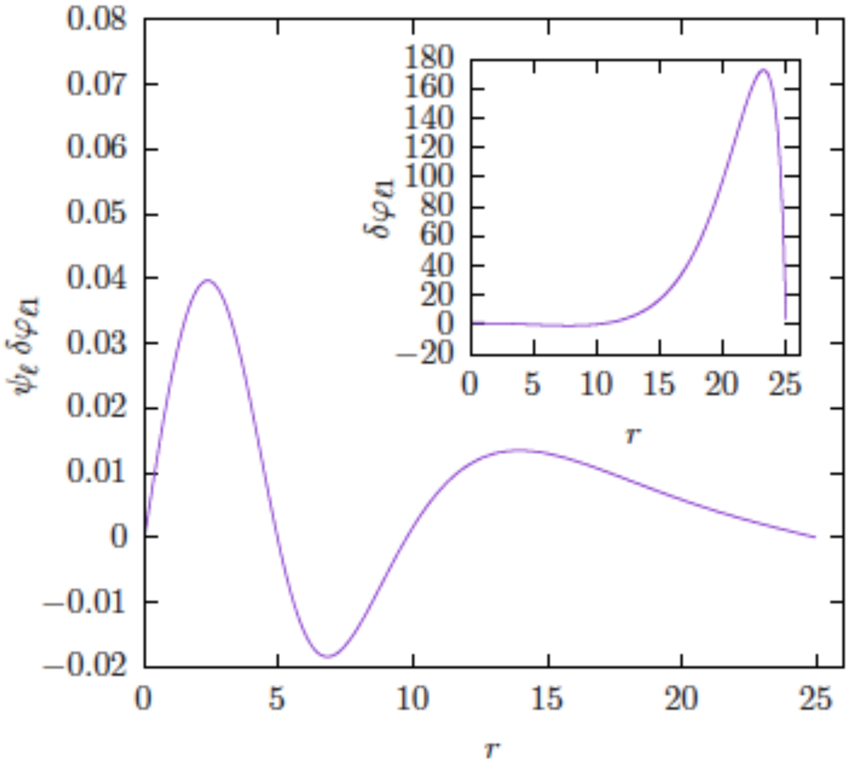
Thus it is posible to count the number of unstable modes by means of the Nodal Theorem

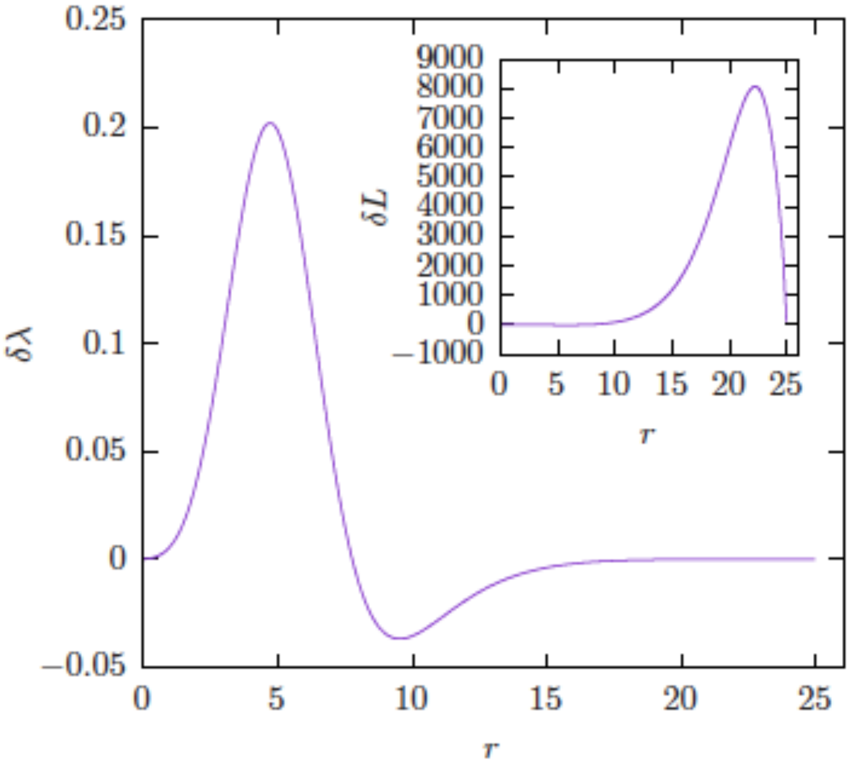


A
B
C
stable region
unstable region
M_T
u_1^0
A
B
C
$u_1^0 = 0.08$
$u_1^0 = 0.11$
$u_1^0 = 3.0$
D
r

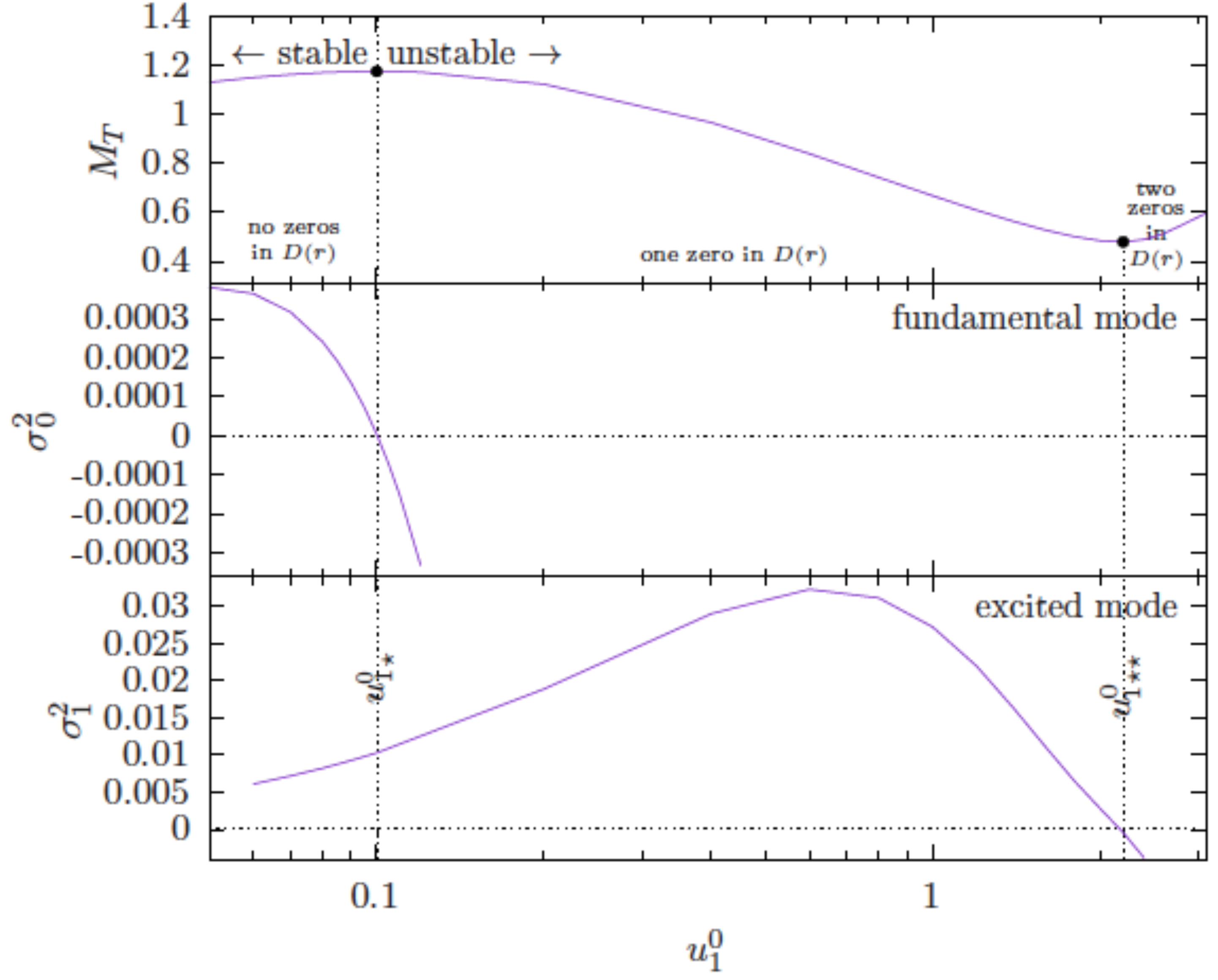
Solutions to the pulsation equations





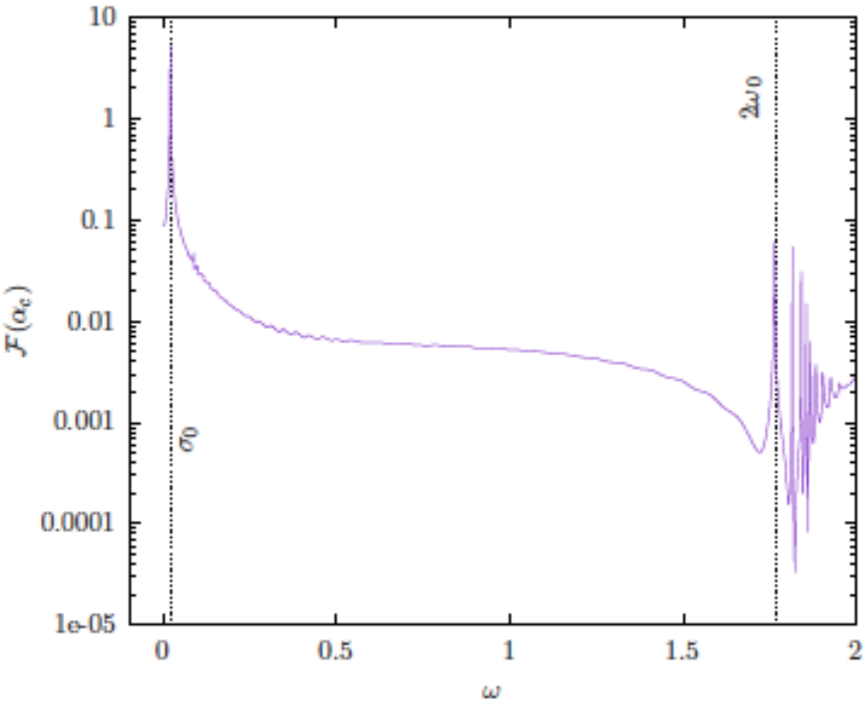


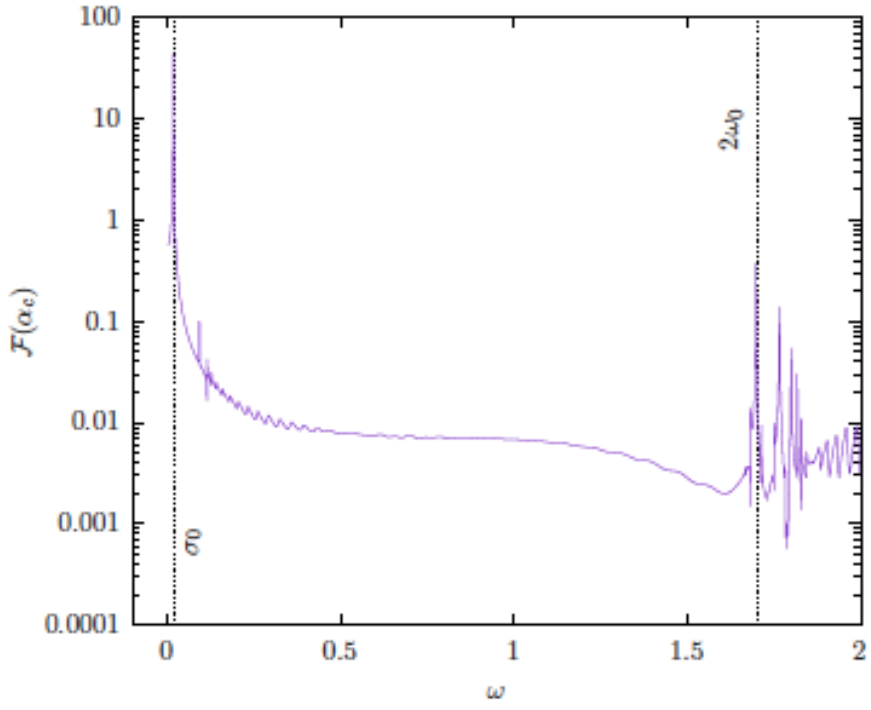
It is possible to find solutions with different nodes: energy levels



← stable unstable →
M_T
no zeros in $D(r)$
one zero in $D(r)$
two zeros in $D(r)$
fundamental mode
σ_0^2
excited mode
σ_1^2
u^0_{1*}
u^0_{1**}
u^0_1
1.4
1.2
1
0.8
0.6
0.4
0.0003
0.0002
0.0001
0
-0.0001
-0.0002
-0.0003
0.03
0.025
0.02
0.015
0.01
0.005
0
0.1
1

Complementarity between numerical perturbations and linear analysis





ℓ	$(2\ell+1)u_1^0$	ω	σ_0^2	c
1	0.050	8.832×10^{-1}	3.80×10^{-4}	-1.92×10^{-2}
1	0.080	8.519×10^{-1}	2.40×10^{-4}	-2.78×10^{-2}
1	0.085	8.475×10^{-1}	1.91×10^{-4}	-2.92×10^{-2}
1	0.090	8.4341×10^{-1}	1.35×10^{-4}	-3.06×10^{-2}
1	0.095	8.394×10^{-1}	7.27×10^{-5}	-3.20×10^{-2}
1	0.100	8.356×10^{-1}	3.95×10^{-6}	-3.33×10^{-2}
1	0.105	8.320×10^{-1}	-7.11×10^{-5}	-3.47×10^{-2}
1	0.110	8.285×10^{-1}	-1.53×10^{-4}	-3.60×10^{-2}

Dark matter halos