BLANDFORD-ZNAJEK POWER AS A STRONG GRAVITY SIGNATURE

FILIPPO CAMILLONI

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An IOP and SISSA journal
JCAP 01 (2024) , 047
ArXiv[2307.06878]

Blandford-Znajek jets in MOdified Gravity

Filippo Camilloni,[a,b] Troels Harmark,[c] Marta Orselli[a,c] and Maria J. Rodriguez[d,e]

Journal of Cosmology and Astroparticle Physics
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ArXiv[2201.11068]

Blandford-Znajek monopole expansion revisited: novel non-analytic contributions to the power emission

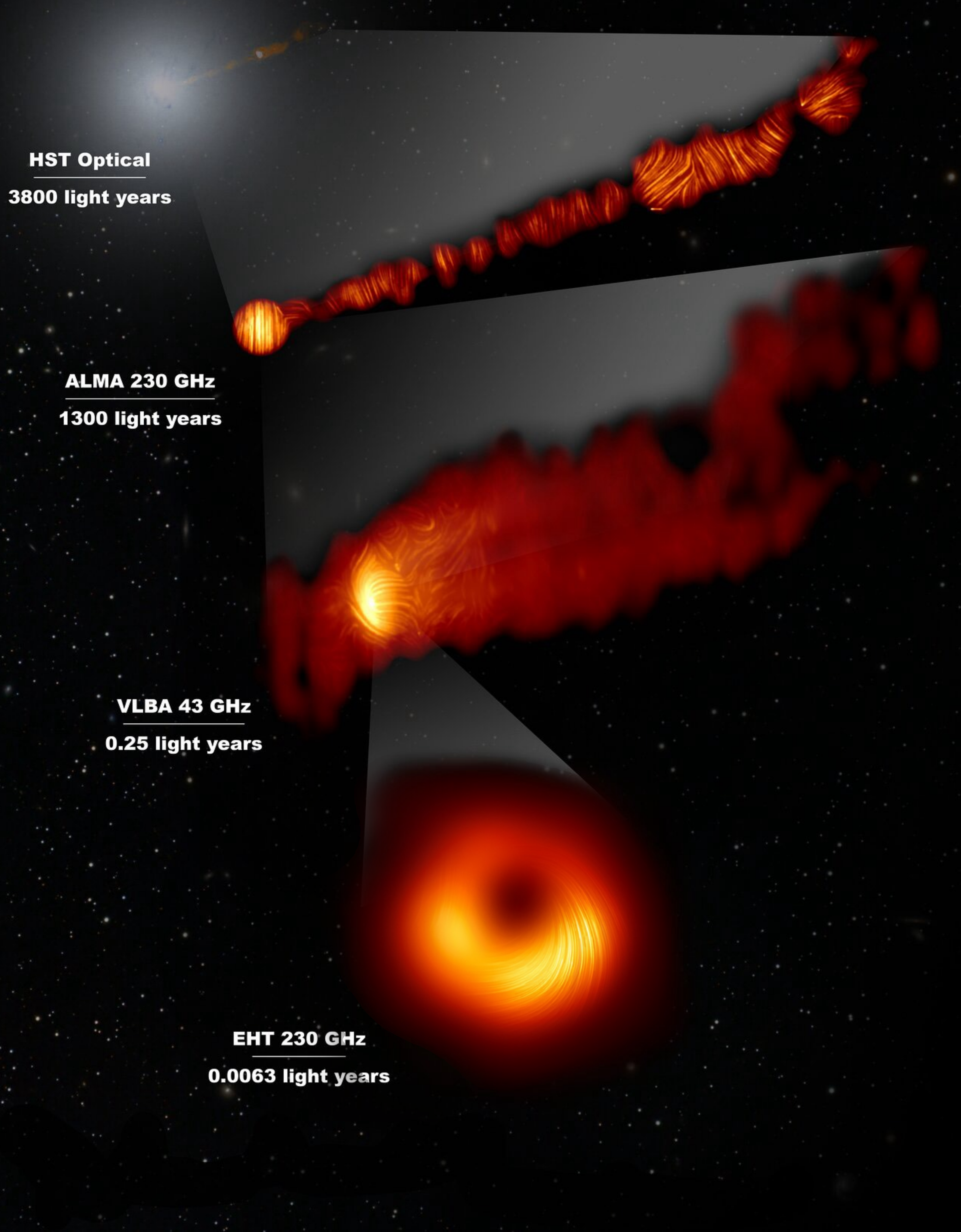
Filippo Camilloni,[a,c] Oscar J.C. Dias,[b] Gianluca Grignani,[a] Troels Harmark,[c] Roberto Oliveri,[d] Marta Orselli,[a,c] Andrea Placidi[a,c] and Jorge E. Santos[e]

Institute for Theoretical Physics

JetSet

GOETHE UNIVERSITÄT FRANKFURT AM MAIN

M87*


HST Optical
3800 light years
ALMA 230 GHz
1300 light years
VLBA 43 GHz
0.25 light years
EHT 230 GHz
0.0063 light years

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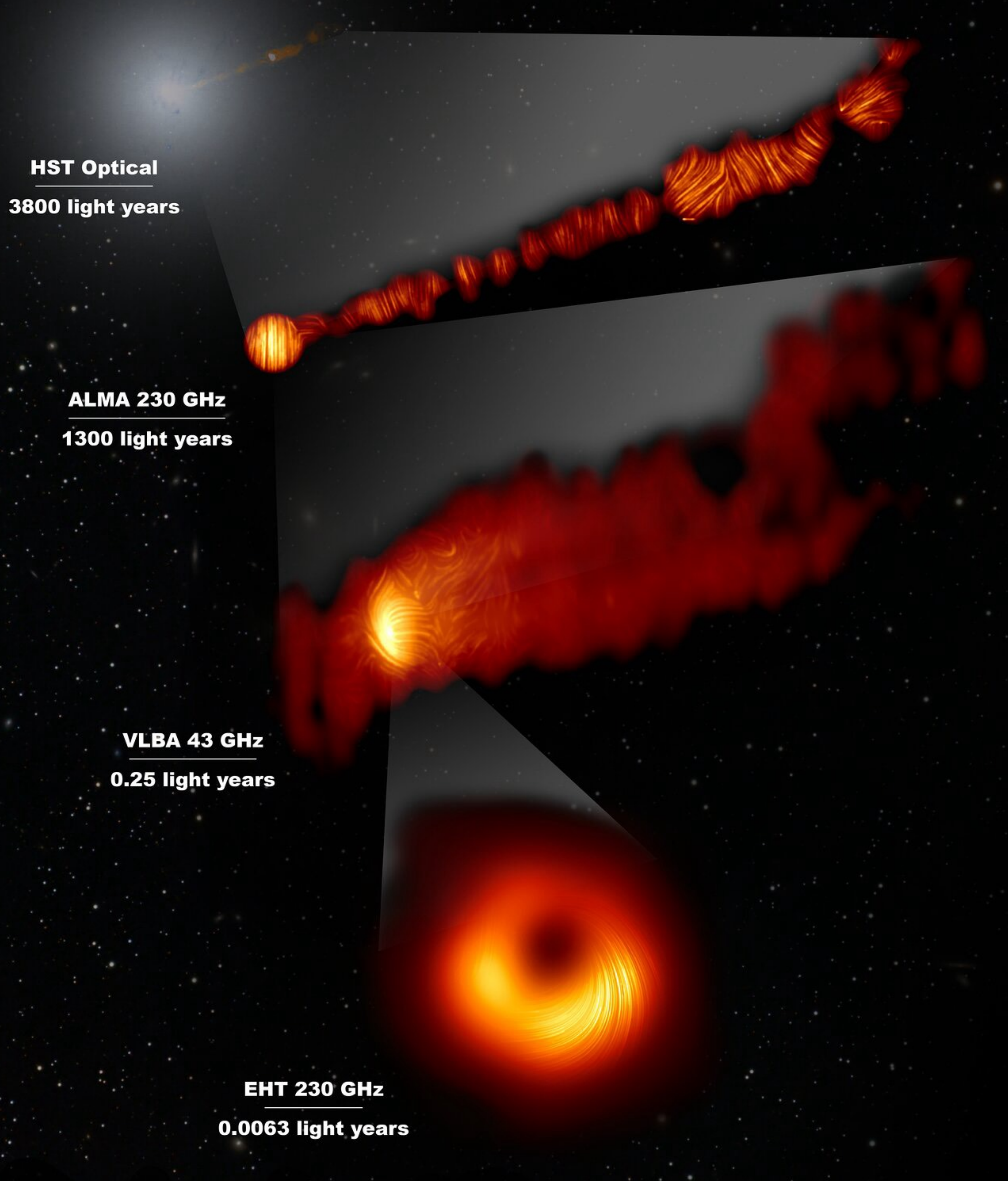
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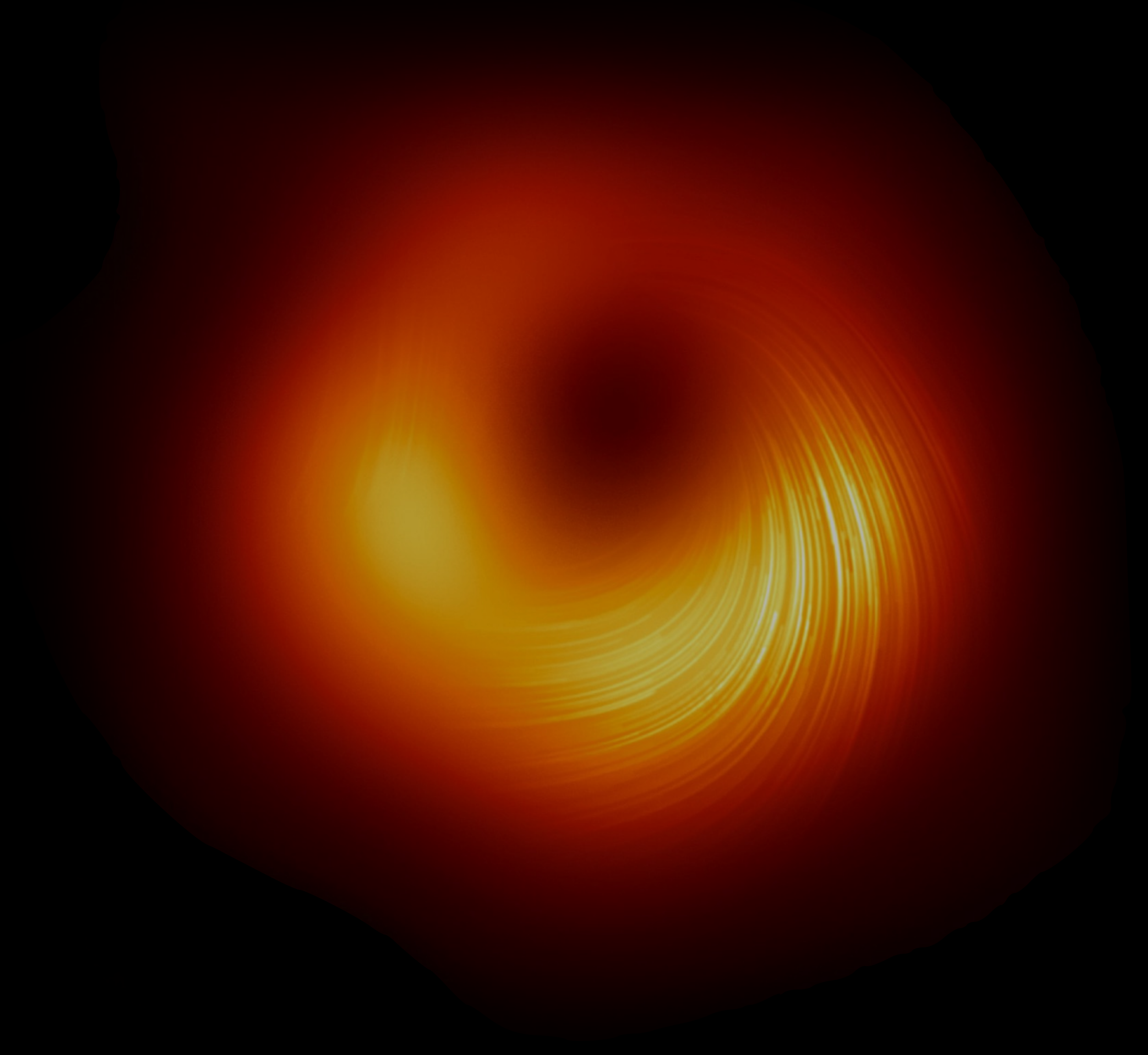

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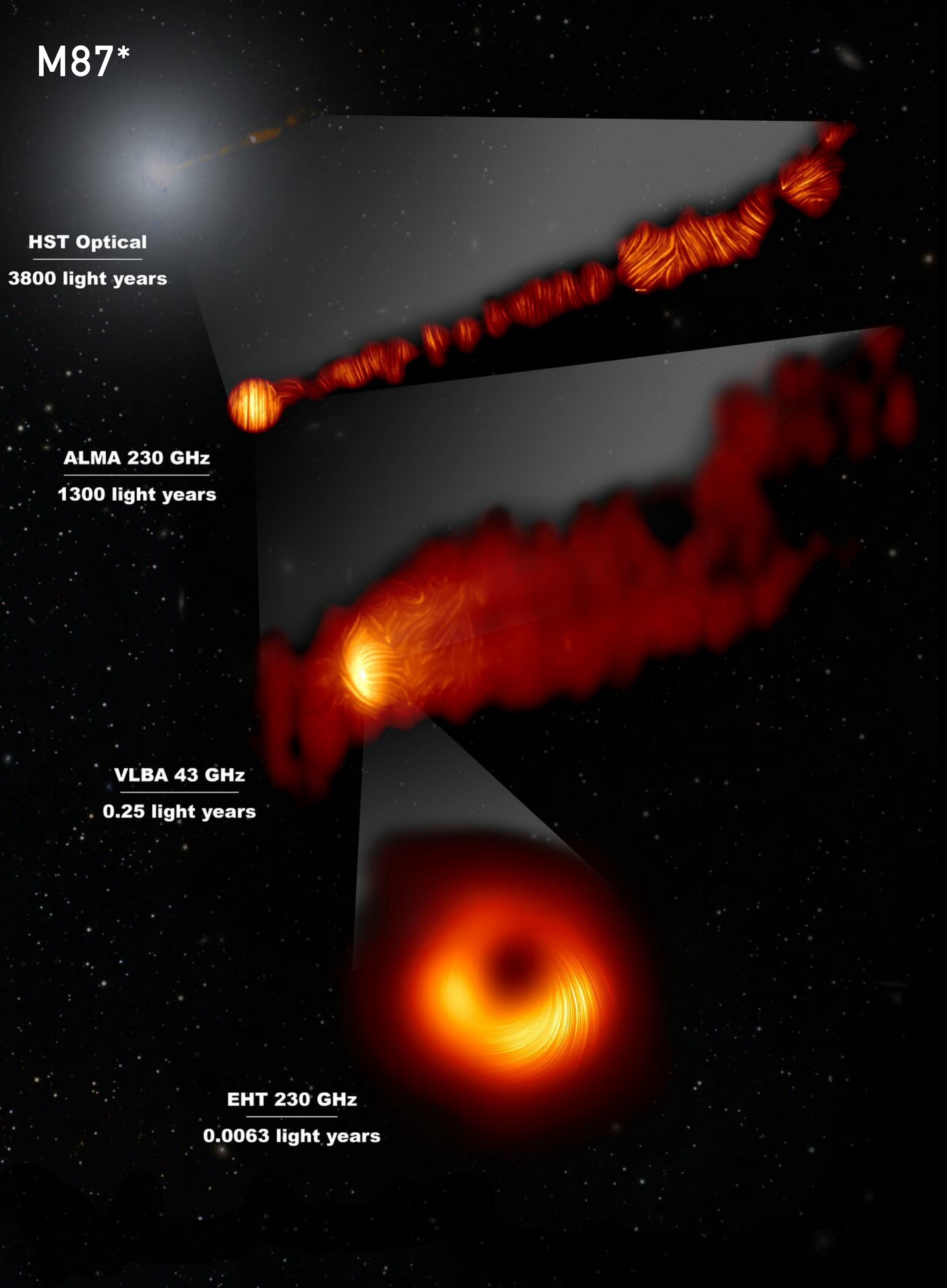

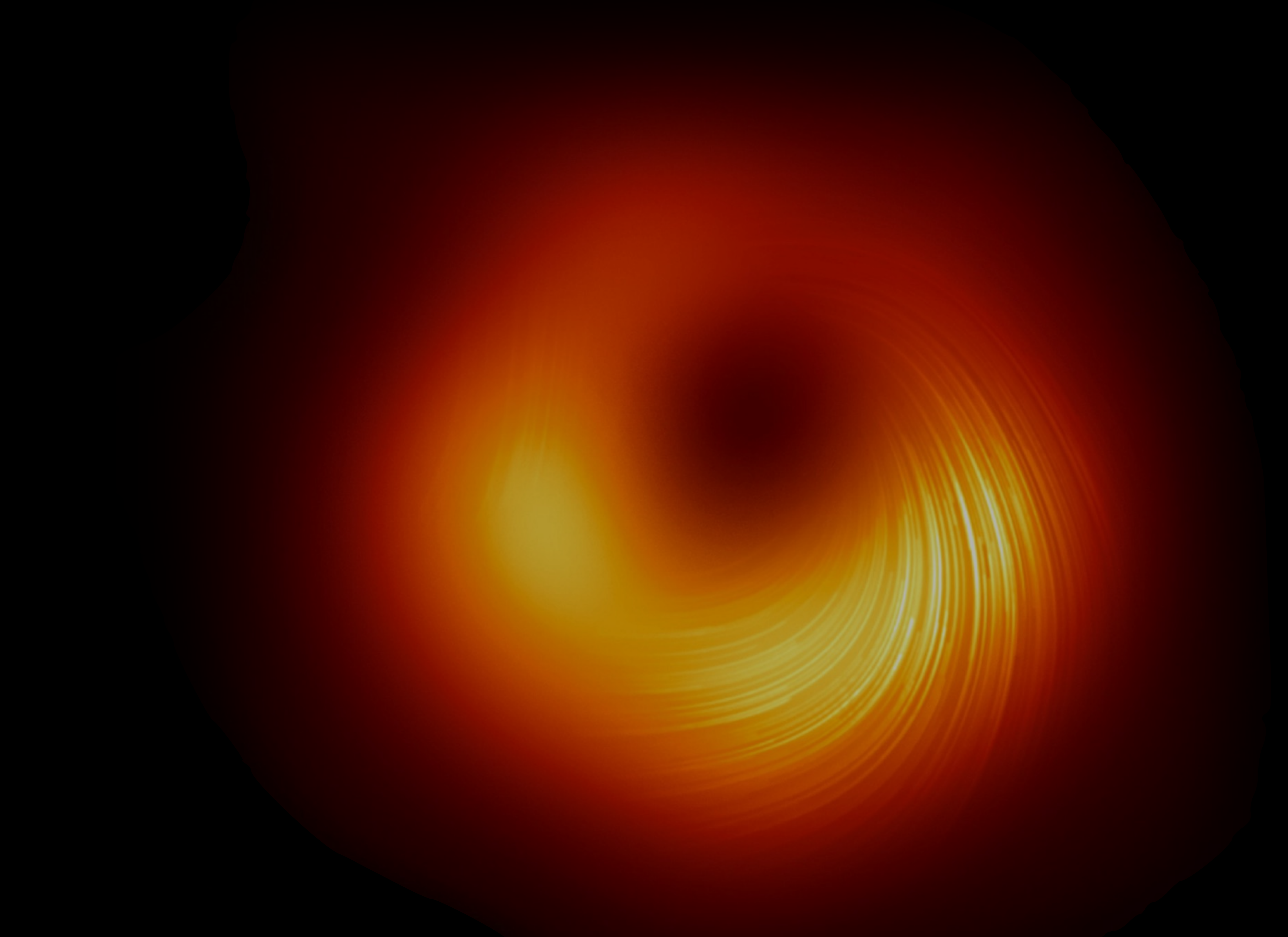

- Polarization is **magnetic field** signature!

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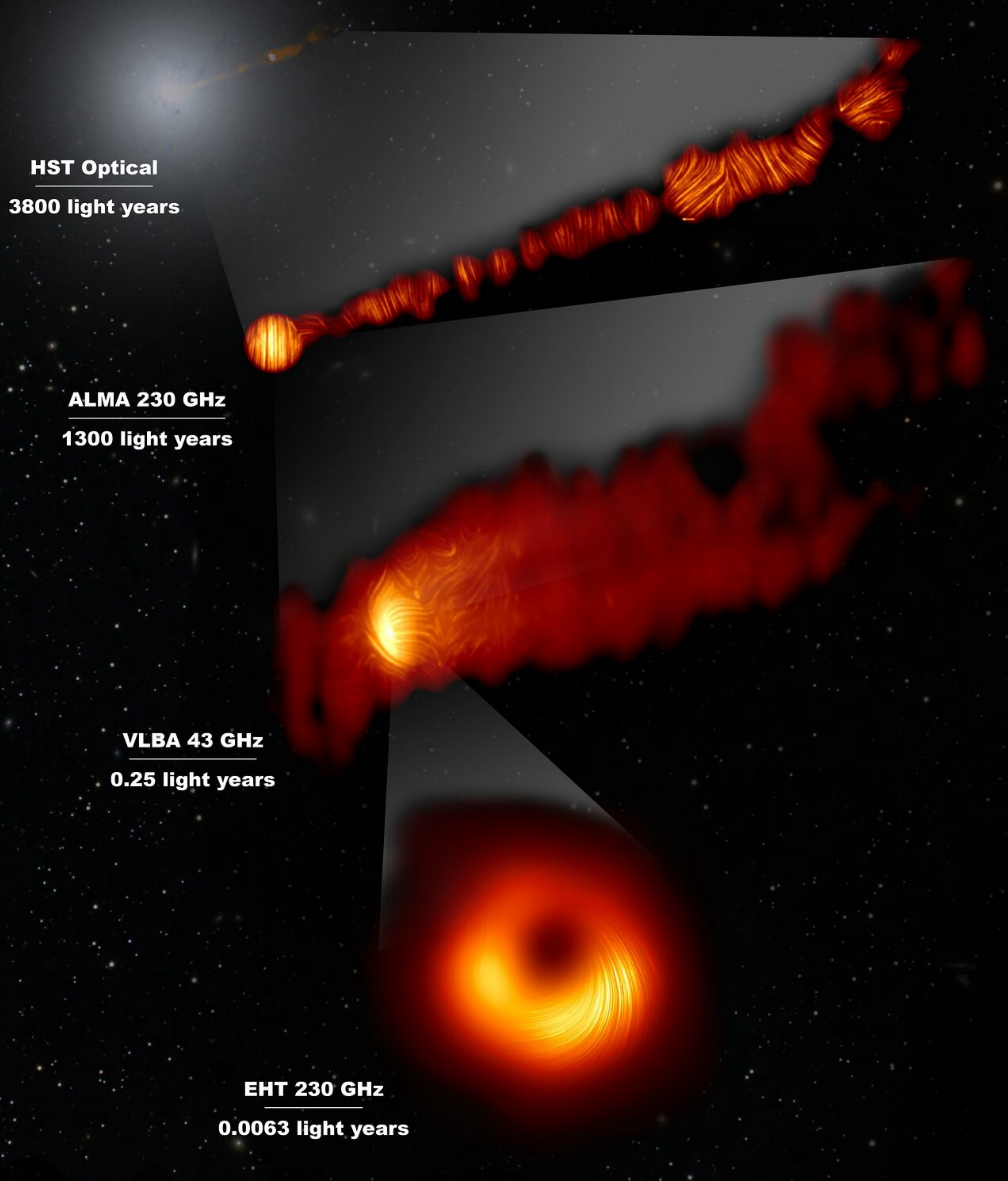

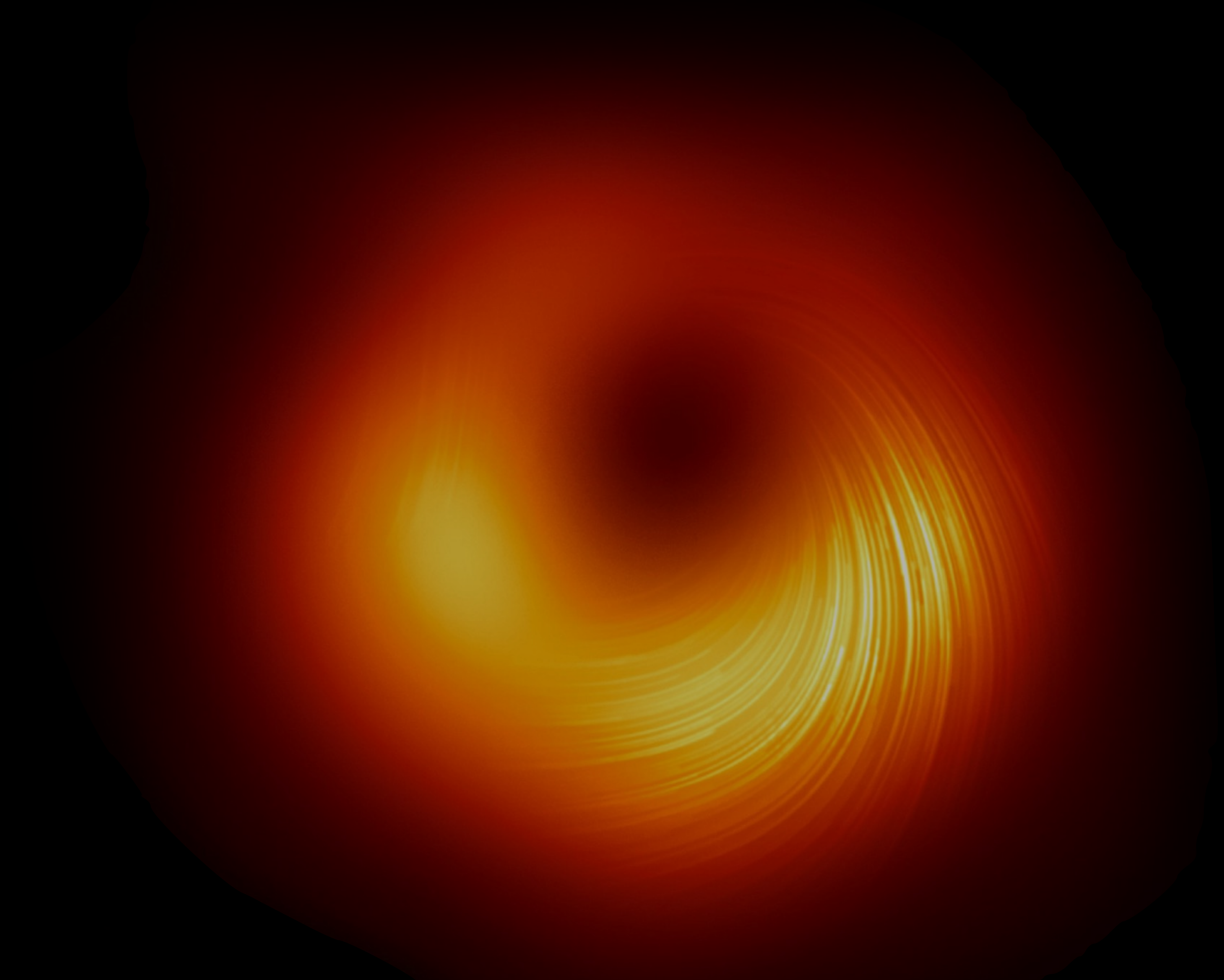

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- The observed Power emitted is consistent with the **Blandford–Znajek mechanism**!

EHT Collaboration (2019, 2021)
Blandford, Znajek (1977)

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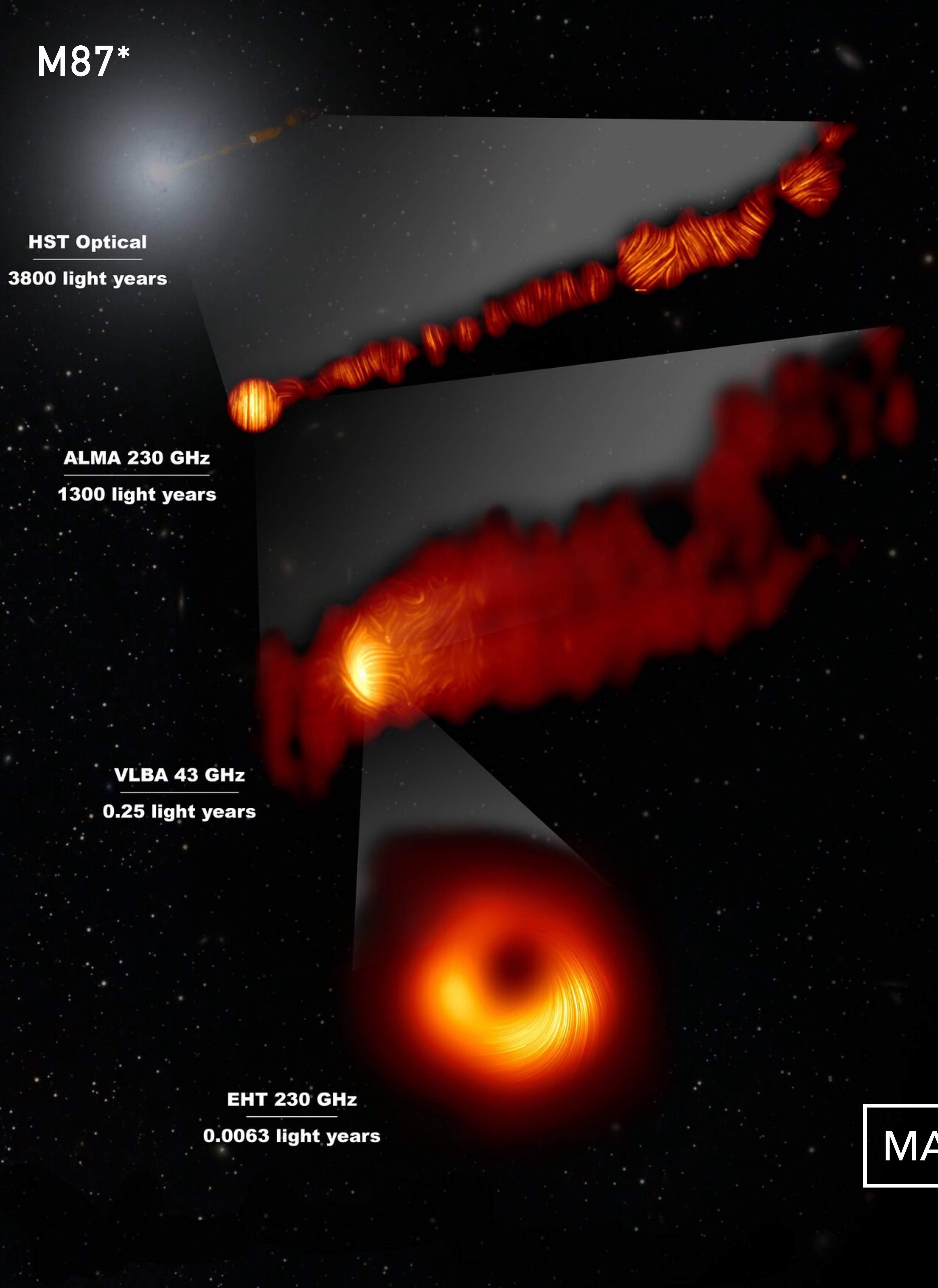

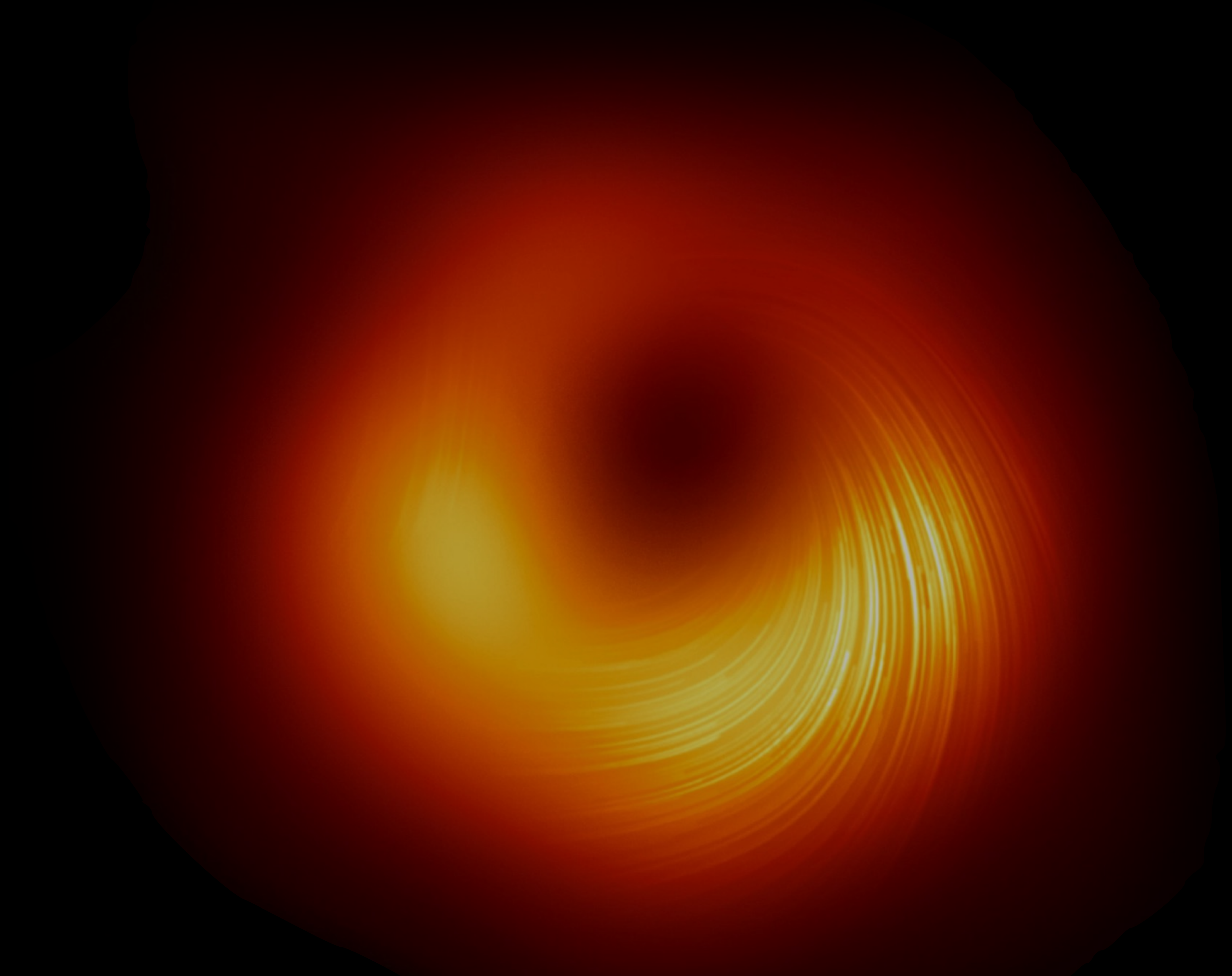

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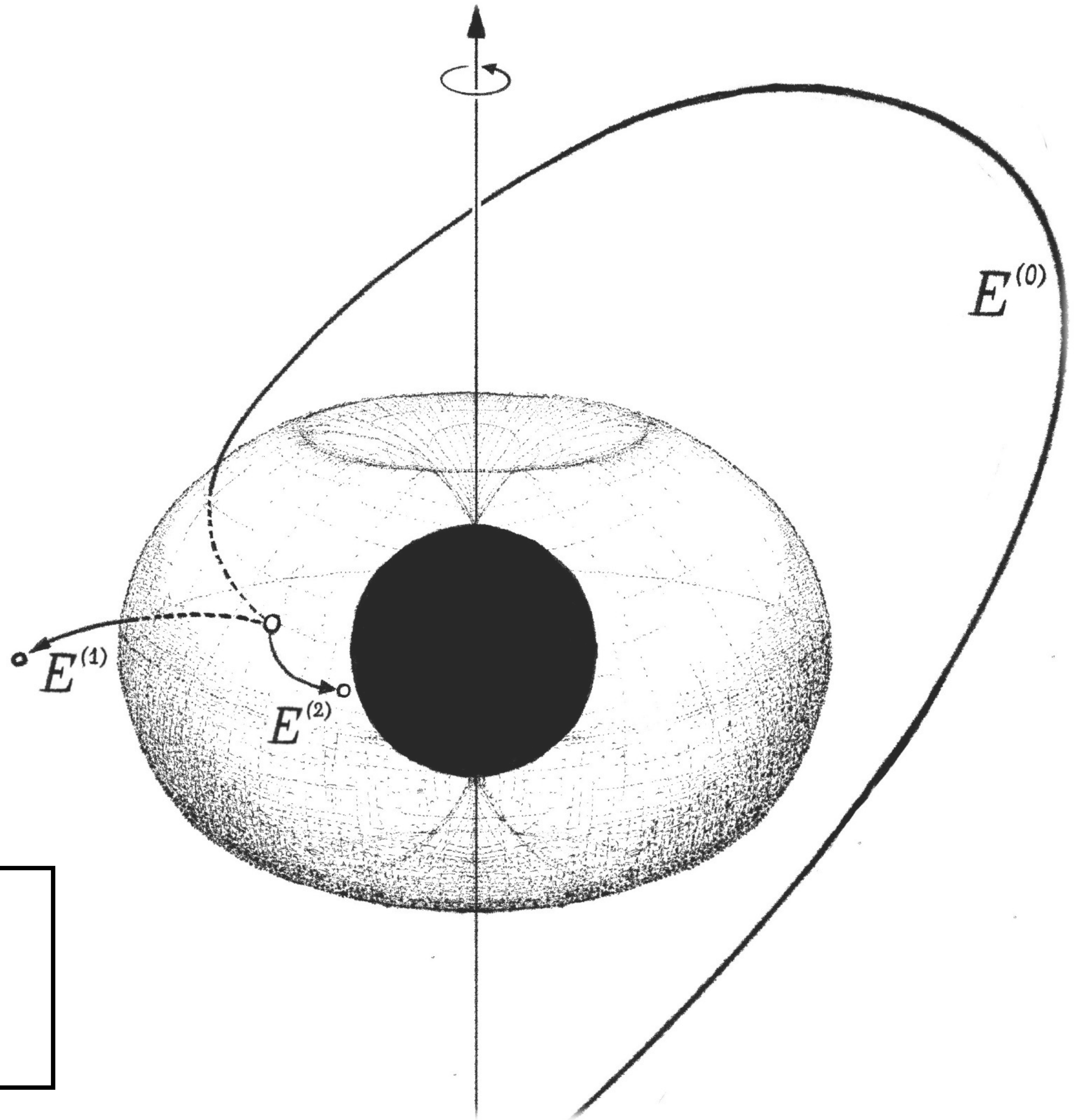
MAGNETOSPHERE + SPINNING BH = ENERGY AND ANGULAR MOMENTUM EXTRACTION

BLANDFORD–ZNAJEK MECHANISM



Spinning Black Hole (BH) geometries

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General Relativity ⟶ **KERR BH**

$$r_0 = \frac{2GM}{c^2} \quad , \quad \alpha = \frac{J}{M^2} \quad , \quad 1 \geq \alpha \geq 0 \quad , \quad \Omega_H = \frac{\alpha}{2Mr_+}$$

MASS **SPIN PARAMETER**

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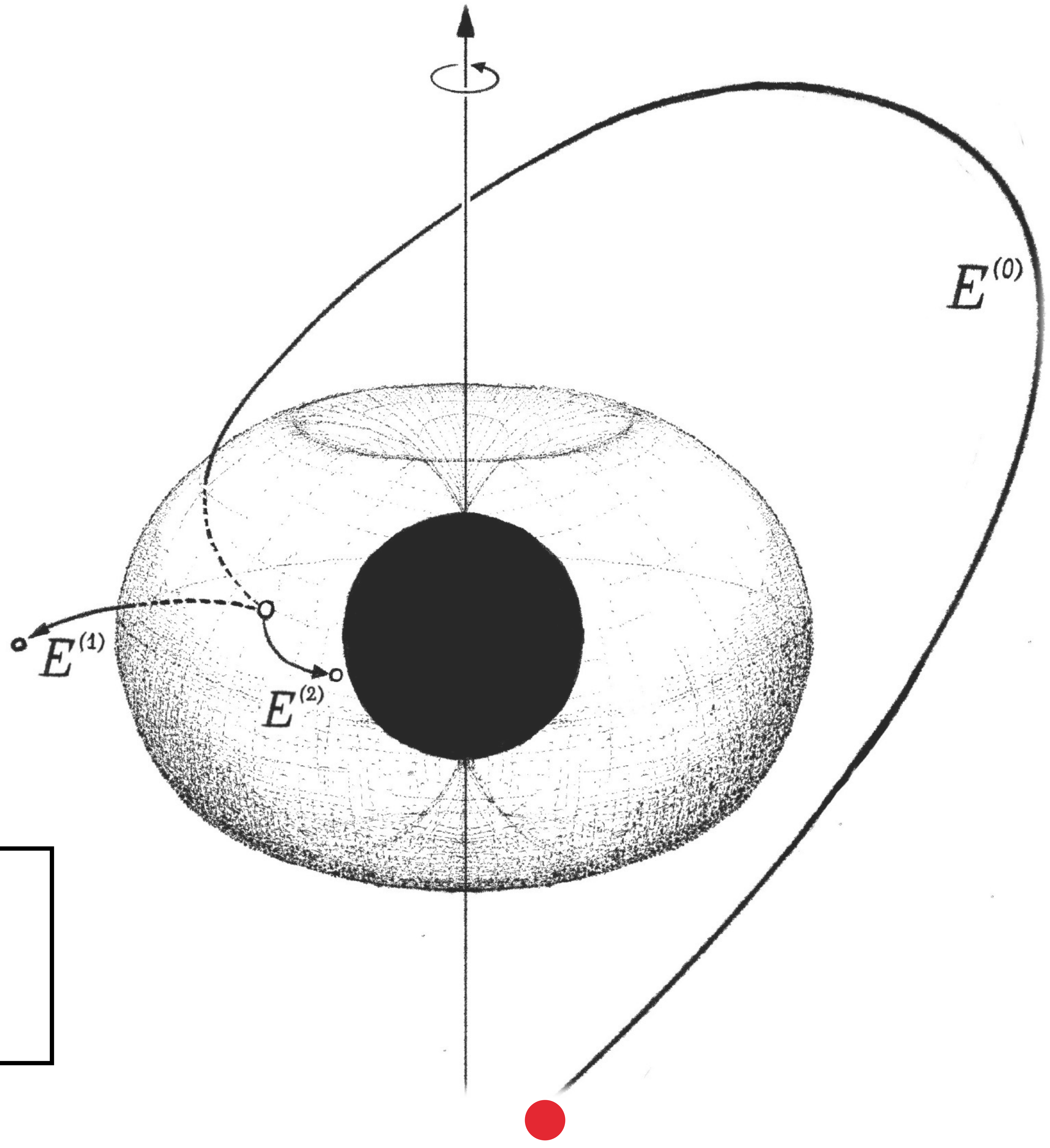

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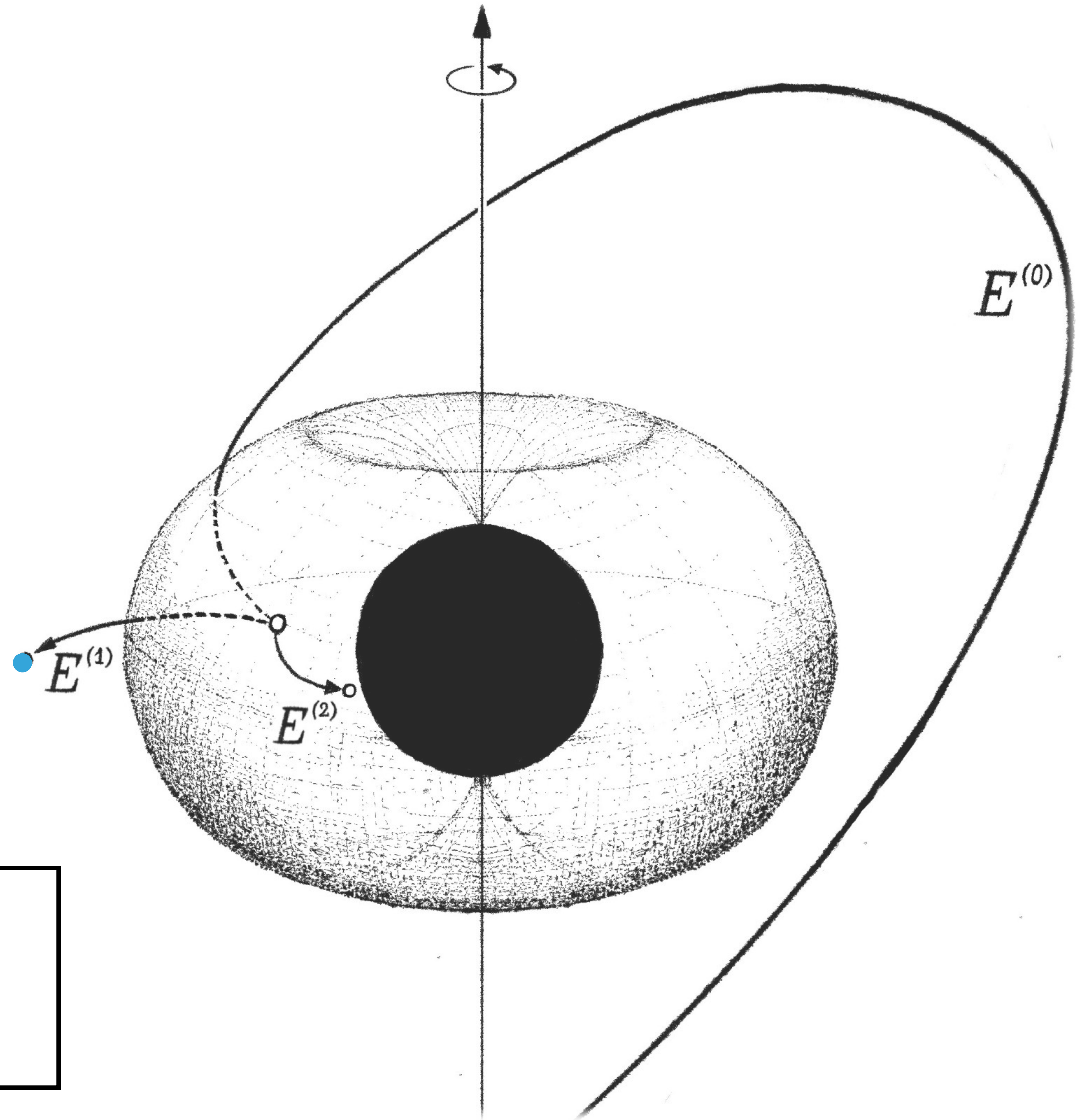

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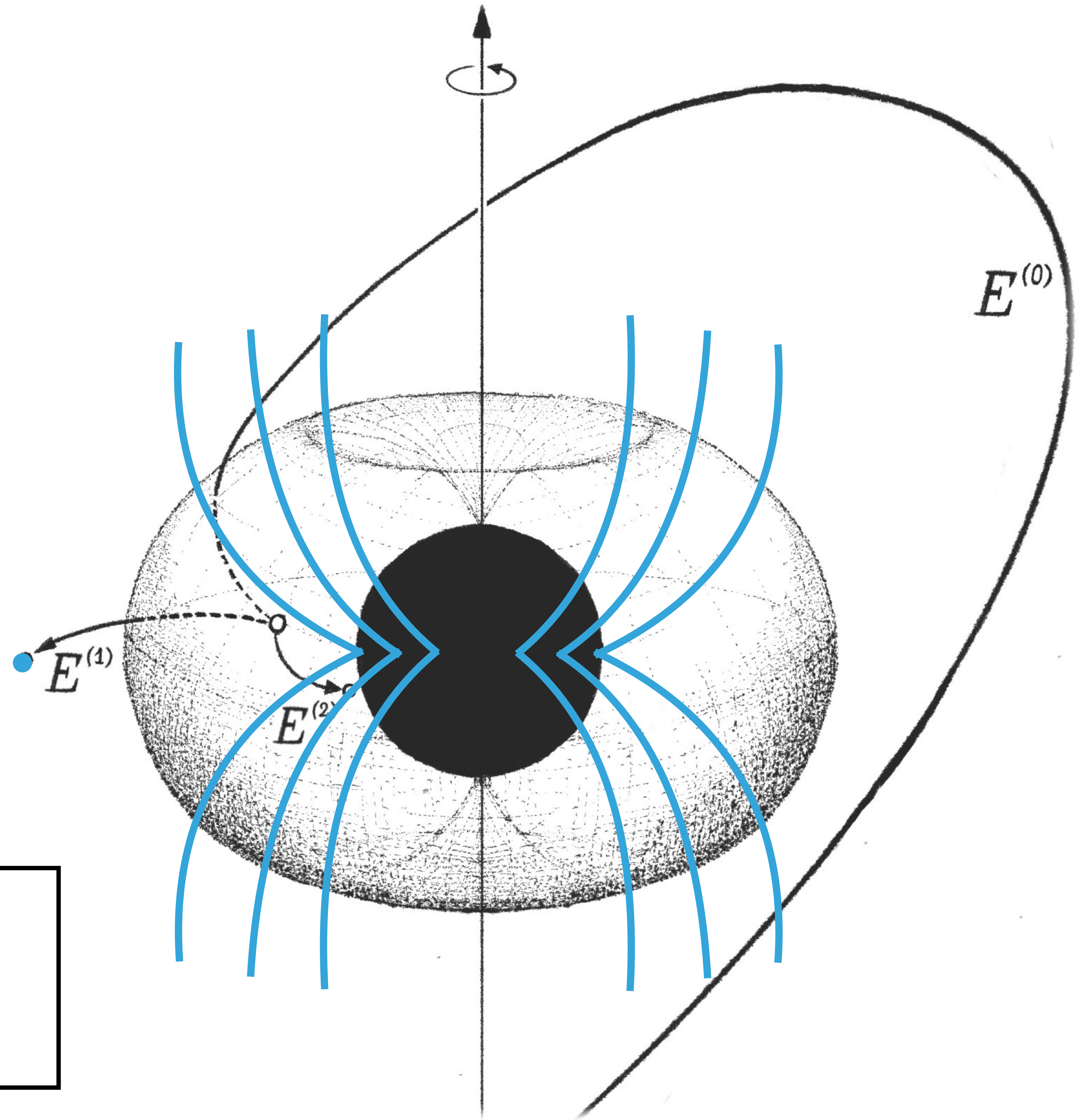

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Blandford, Znajek (1977)

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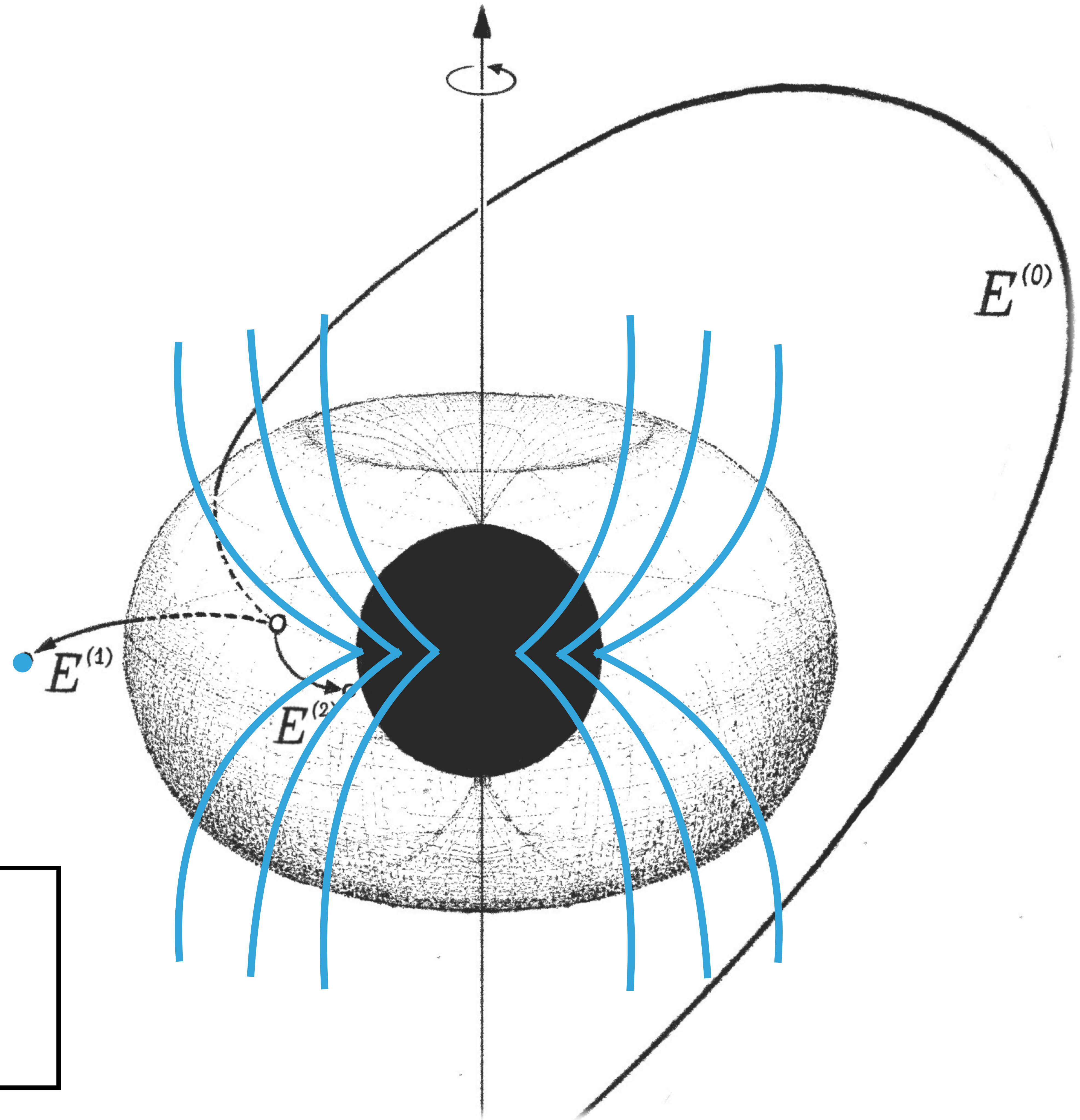

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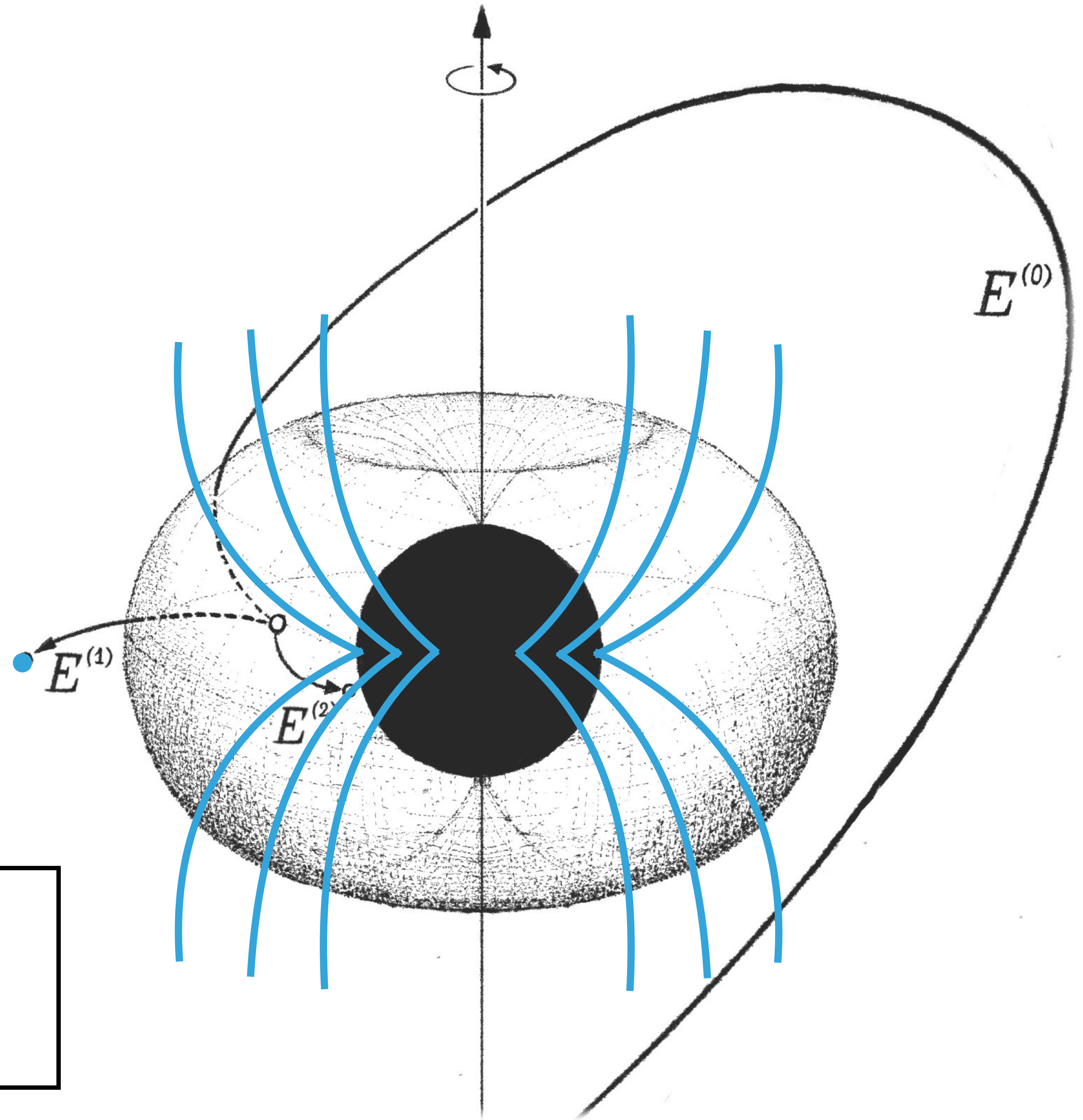

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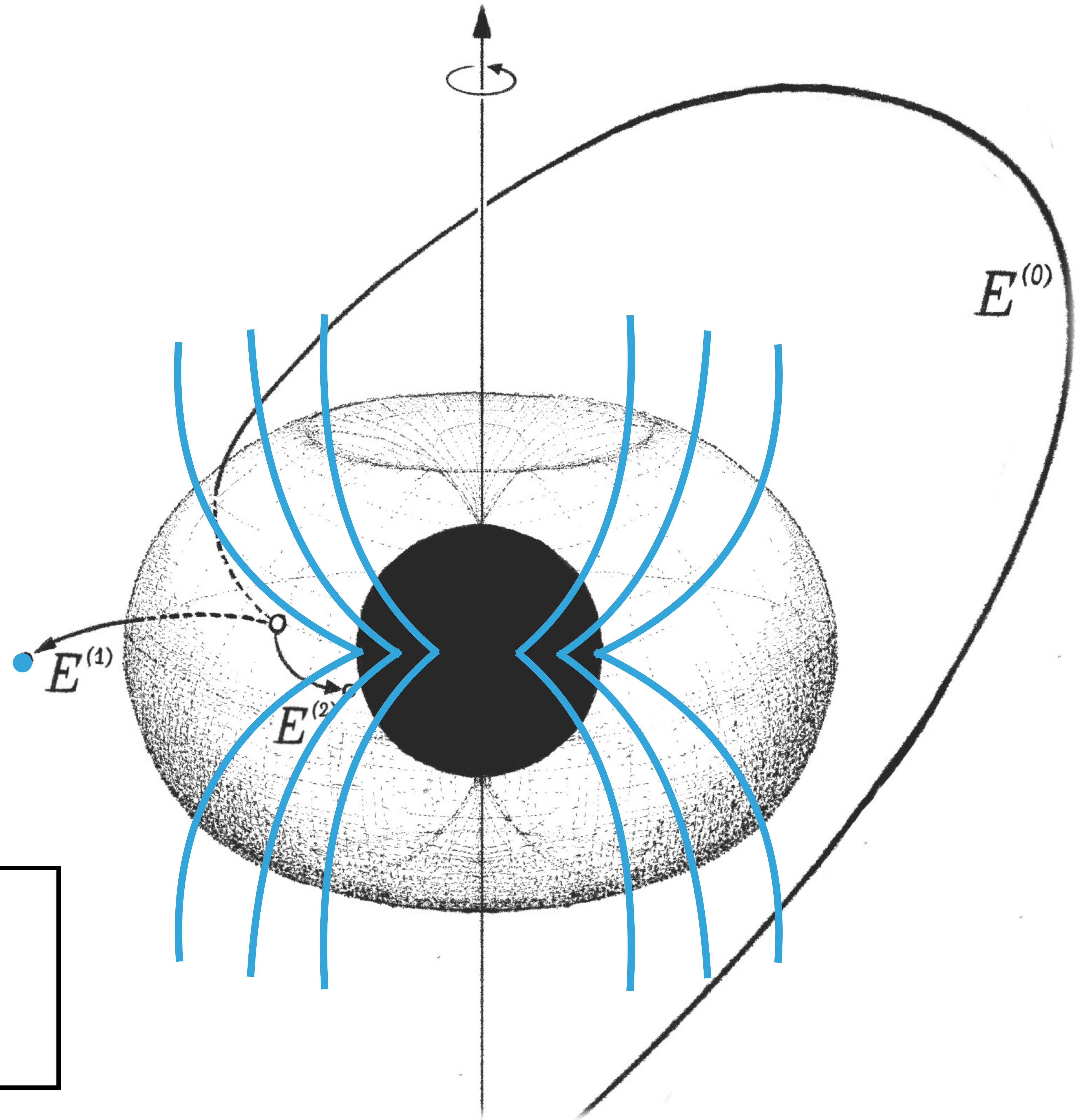

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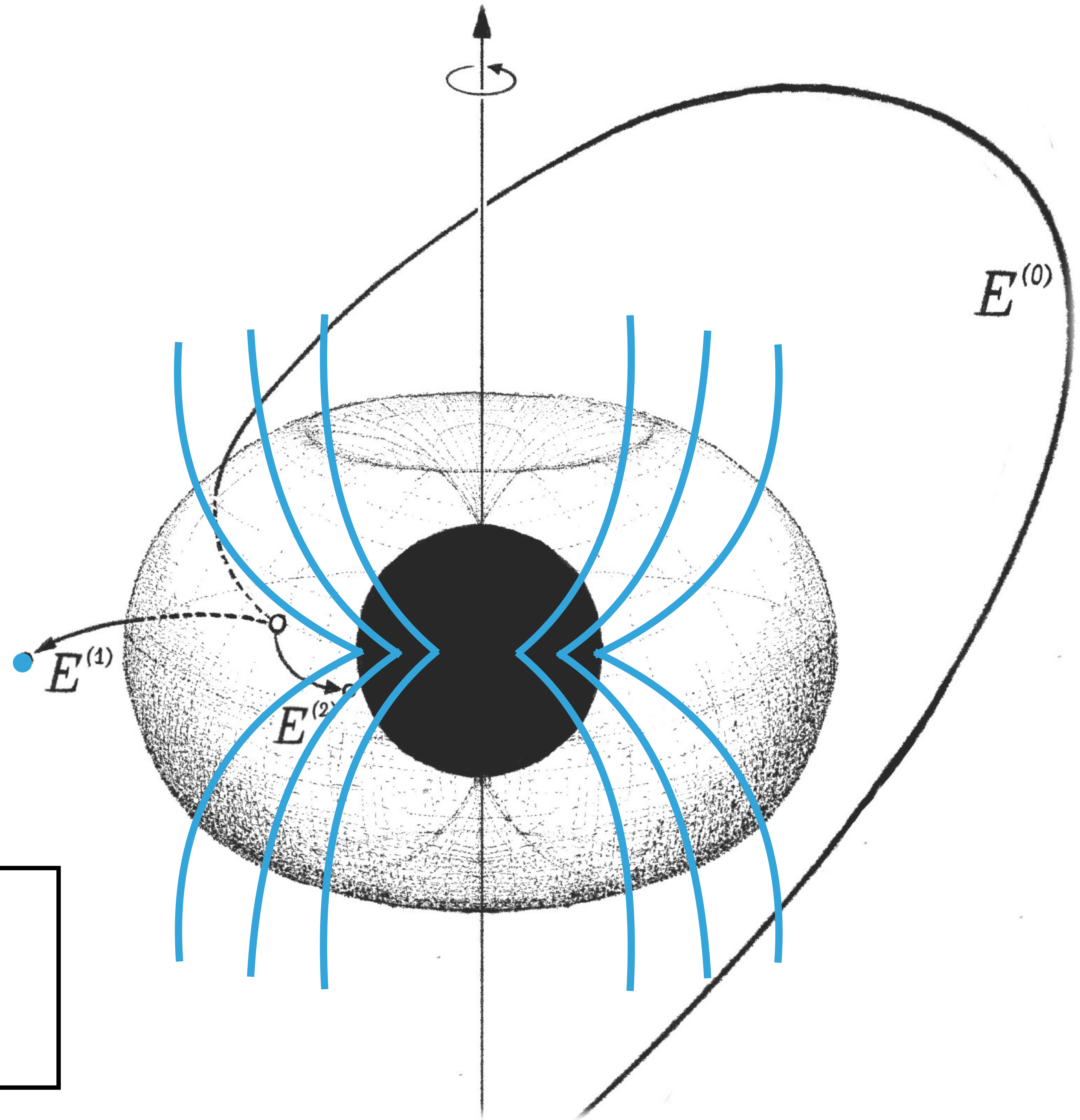

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High-Spin Factor , $f(\Omega_H) \neq 1$ for high-spin regime

$$f(\Omega_H) = 1 + c_2\ \Omega_H^2 + c_4\ \Omega_H^4 + \ldots$$

- **(Indications) Weak dependence on magnetic field** Tchekhovskoy, Narayan, McKinney (2010)

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Scalar-Tensor-Vector Gravity ⟶ **KERR–MOG BH**

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Moffat (2006), Moffat (2015)

$$q = \frac{G_\infty - G_N}{G_N}\ ,\quad 0 \leq q \leq \frac{1}{\alpha^2} - 1\ ,\quad M_q = (1+q)M$$

DEFORMATION PARAMETER

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- **Dependence on background metric, removes degeneracy !**

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BH MAGNETOSPHERES

$$T^{\mu\nu}_{em} \gg T^{\mu\nu}_{mat} \quad \Longrightarrow \quad \nabla_\mu T^{\mu\nu}_{em} = -F^{\nu}{}_{\sigma} j^{\sigma}_{mat} \approx 0$$

BH MAGNETOSPHERES

FORCE–FREE ELECTRODYNAMICS

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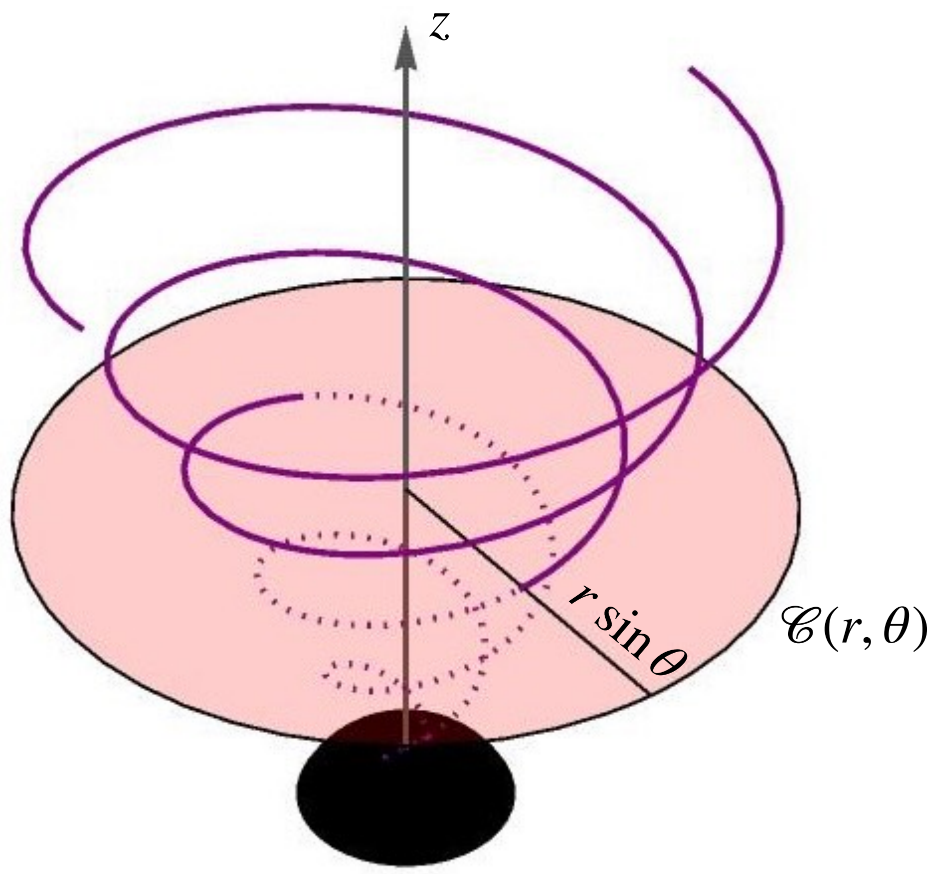

Stationary-Axisymmetric BH Magnetospheres

$$F = d\Psi \wedge \left(d\phi - \Omega(\Psi)dt\right) - I(\Psi)\frac{\Sigma}{\Delta \sin\theta}\, dr \wedge d\theta$$

- $\Psi(r,\theta)$ MAGNETIC FLUX
- $I(\Psi)$ POLOIDAL CURRENT
- $\Omega(\Psi)$ ANGULAR VELOCITY OF FIELD LINES

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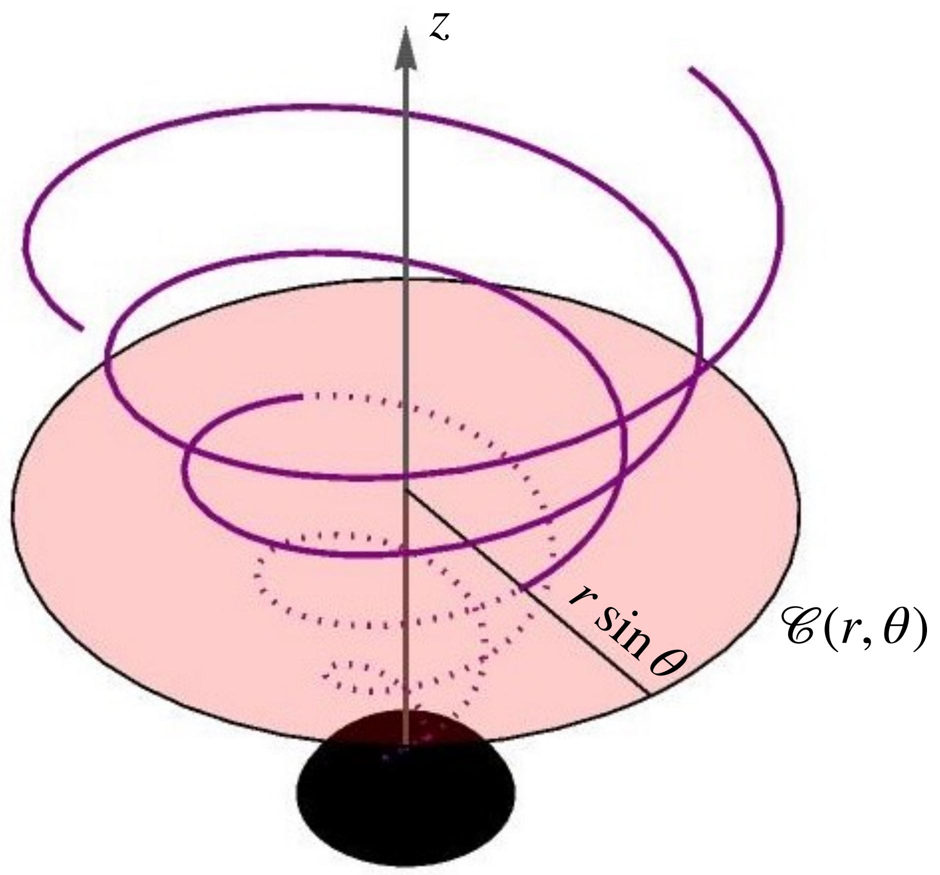

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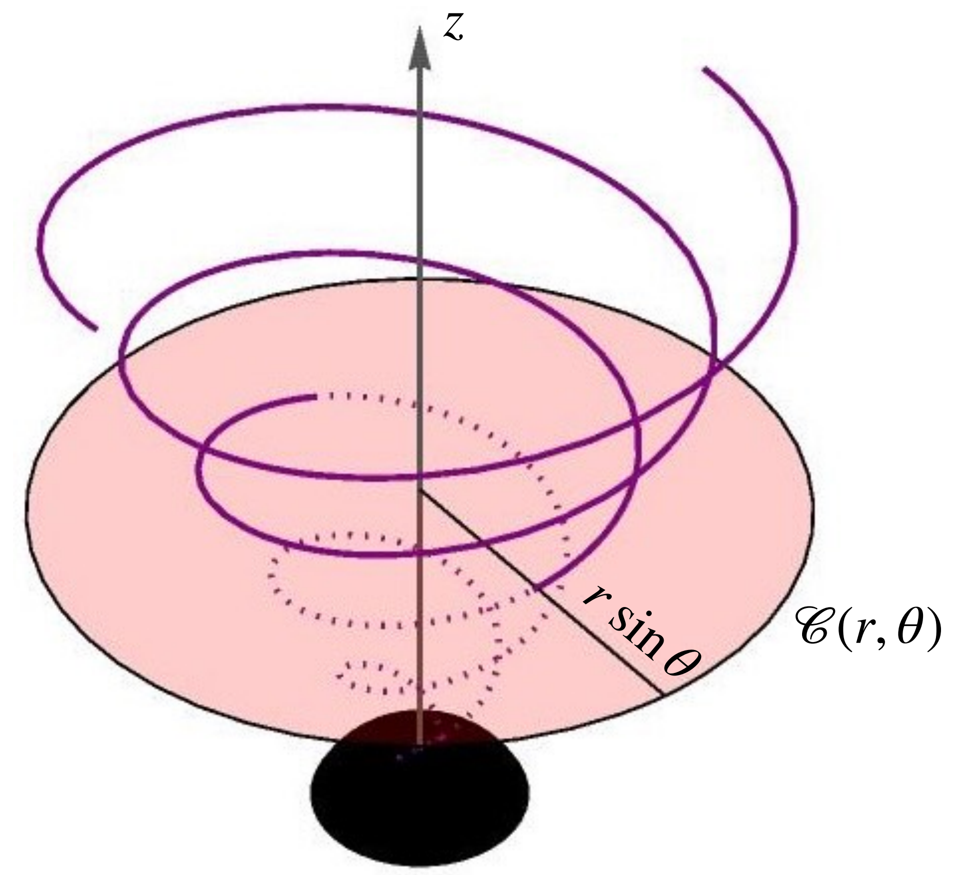

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GRAD-SHAFRANOV EQUATION in BH background

$$\eta_\mu \partial_r\left(\eta^\mu \Delta \sin\theta\, \partial_r \Psi\right) + \eta_\mu \partial_\theta\left(\eta^\mu \sin\theta\, \partial_\theta \Psi\right) + \frac{\Sigma}{\Delta \sin\theta} I \frac{dI}{d\Psi} = 0$$

$$\eta = d\phi - \Omega(\Psi)dt$$

FC, Dias et Al (2022)

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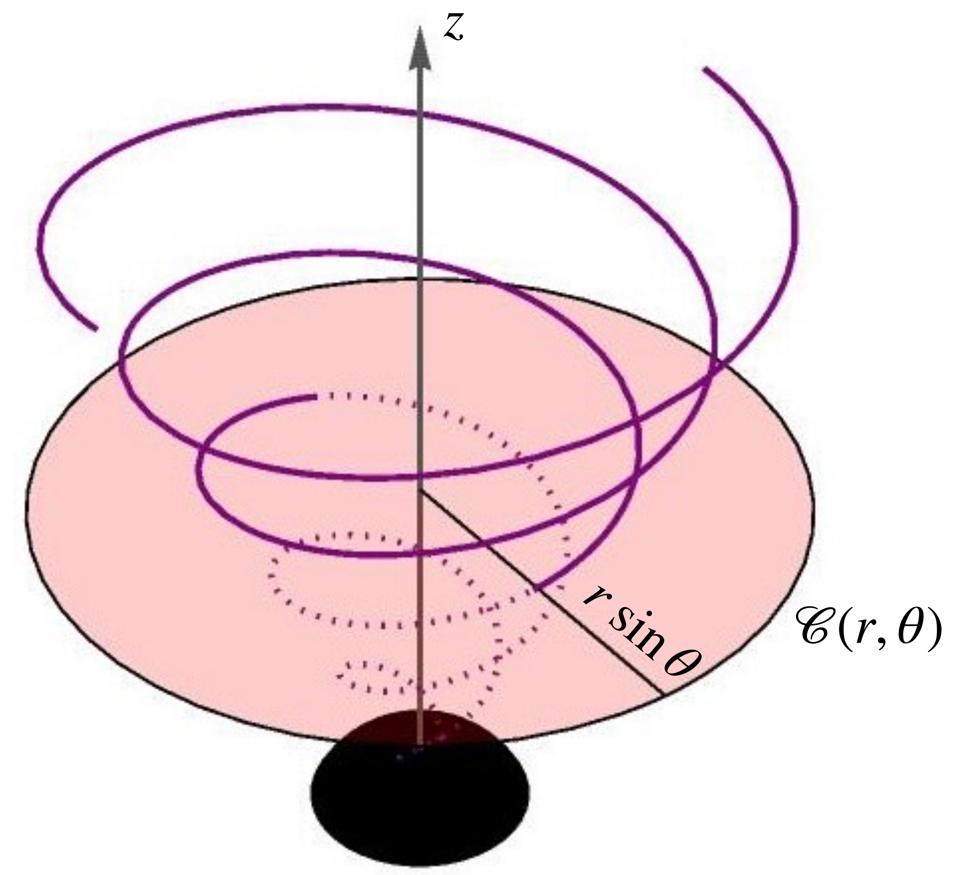

Stationary-Axisymmetric BH Magnetospheres

$$F = d\Psi \wedge \left(d\phi - \Omega(\Psi)dt\right) - I(\Psi)\frac{\Sigma}{\Delta \sin\theta} dr \wedge d\theta$$

- $\Psi(r,\theta)$ MAGNETIC FLUX
- $I(\Psi)$ POLOIDAL CURRENT
- $\Omega(\Psi)$ ANGULAR VELOCITY OF FIELD LINES

Power extracted $\dot{E}_+ = \int I(\Psi)\Omega(\Psi)d\Psi$ Gralla, Jacobson (2014)

GRAD-SHAFRANOV EQUATION in BH background

$$\eta_\mu \partial_r\left(\eta^\mu \Delta \sin\theta\, \partial_r \Psi\right) + \eta_\mu \partial_\theta\left(\eta^\mu \sin\theta\, \partial_\theta\Psi\right) + \frac{\Sigma}{\Delta \sin\theta} I \frac{dI}{d\Psi} = 0$$

$\eta = d\phi - \Omega(\Psi)dt$

FC, Dias et Al (2022)

REGULARITY CONDITIONS

Znajek Conditions at the **HORIZON**, $r \to r_+$, and at **INFINITY**, $r \to \infty$

$$I(r_+,\theta) = \left[\left(\frac{r_0 r_+}{\Sigma}\sin\theta\right)\left(\Omega_H - \Omega\right)\partial_\theta\Psi\right]\Bigg|_{r_+}$$

$$I^\infty(\theta) = \sin\theta\, \Omega^\infty(\theta)(\partial_\theta\Psi)^\infty$$

At the **LIGHT SURFACES**, $\eta^\mu\eta_\mu = 0$ (ILS/OLS)

$$\Delta\, \eta_\mu \partial_r \eta^\mu \partial_r \Psi + \eta_\mu \partial_\theta \eta^\mu \partial_\theta \Psi + \frac{\Sigma}{\Delta \sin^2\theta} I \frac{dI}{d\Psi} = 0$$

PERTURBATIVE APPROACH

SPIN PARAMETER

$$ds^2 = -\frac{\Delta}{r^2}dt^2 + \frac{r^2}{\Delta}dr^2 + r^2 d\Omega^2 + \mathcal{O}(\alpha) \qquad \alpha \ll 1$$

Blandford, Znajek (1977)

- Start from Static BH and **turn-on a small rotation**!

$$F \sim \mathcal{O}(\alpha^0) \;\;,\;\; j \sim \mathcal{O}(\alpha) \;\;\Longrightarrow\;\; \Psi(r,\theta) = \psi_0(r,\theta) + \mathcal{O}(\alpha)\,,\;\; I(\Psi) = \mathcal{O}(\alpha)\,,\;\; \Omega(\Psi) = \mathcal{O}(\alpha)$$

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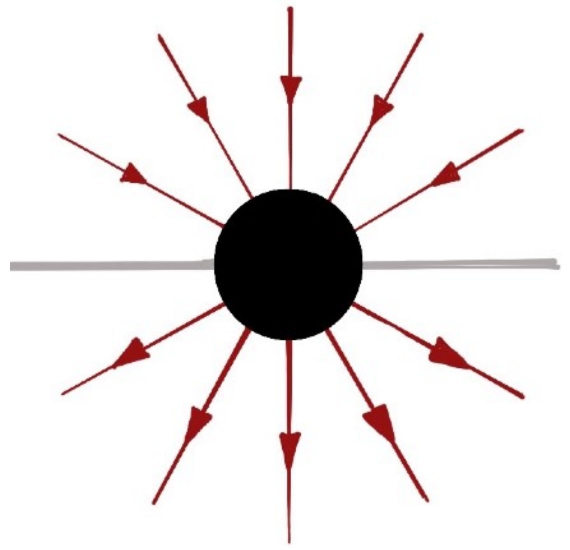
SPLIT MONOPOLE

$$\psi_0 = 1 - \cos\theta$$



$$\psi|_{\theta=0} = \partial_\theta \psi|_{\theta=0} = 0,$$

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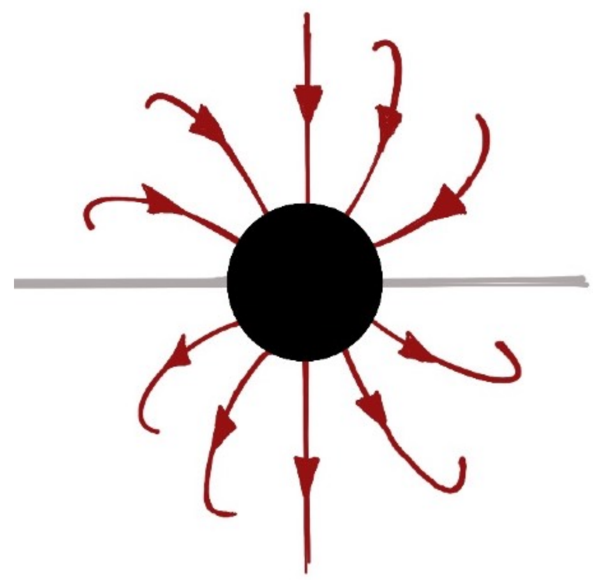

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Grad-Shafranov operator

$$\mathcal{L} = \frac{1}{\sin\theta}\partial_r\left(\frac{\Delta}{r^2}\partial_r\right) + \frac{1}{r^2}\partial_\theta\left(\frac{1}{\sin\theta}\partial_\theta\right)$$

STATIONARY FORCE-FREE solutions

$$\psi_n(r,\theta) = R_\ell^{(n)}(r)\Theta_\ell(\theta)$$

Superposition of STATIC–VACUUM SOLUTIONS + SOURCE TERMS

SPLIT MONOPOLE

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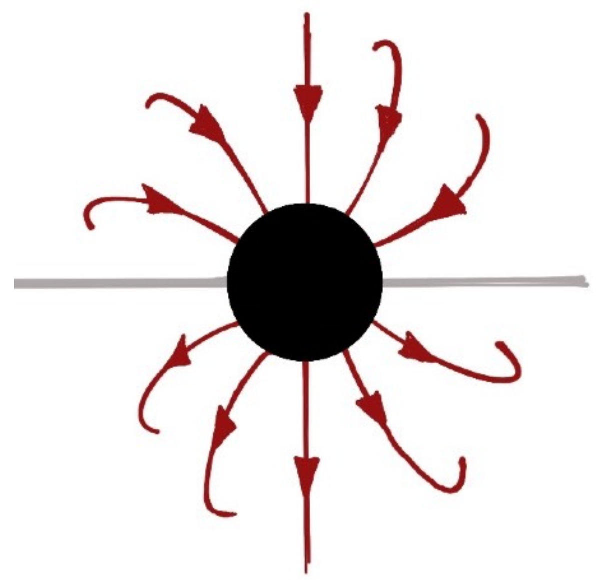

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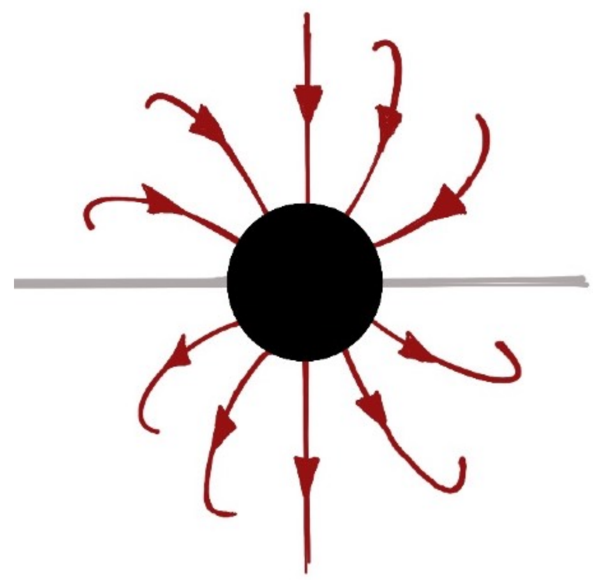

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KERR $\Delta = r(r-2M) + \mathcal{O}(\alpha)^2 \implies$ Hypergeometric Functions

KERR-MOG $\Delta = r^2 - \frac{2M_q r}{\sqrt{1+q}} - \frac{q}{1+q}M_q^2 + \mathcal{O}(\alpha)^2 \implies$ **Heun Polynomials**

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FC, Dias et Al (2022)

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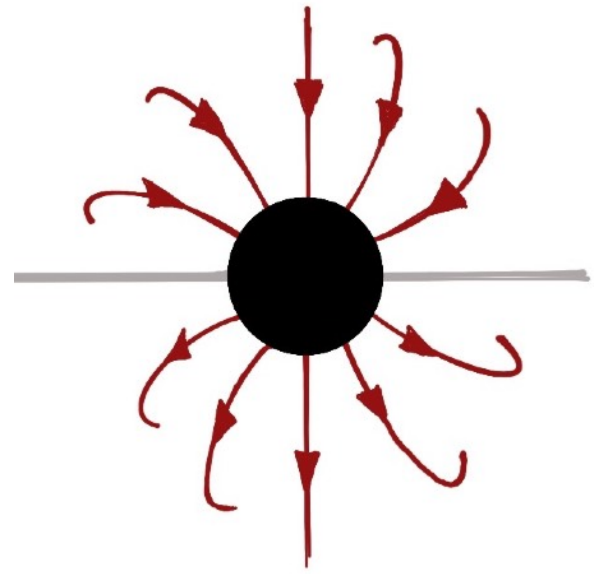

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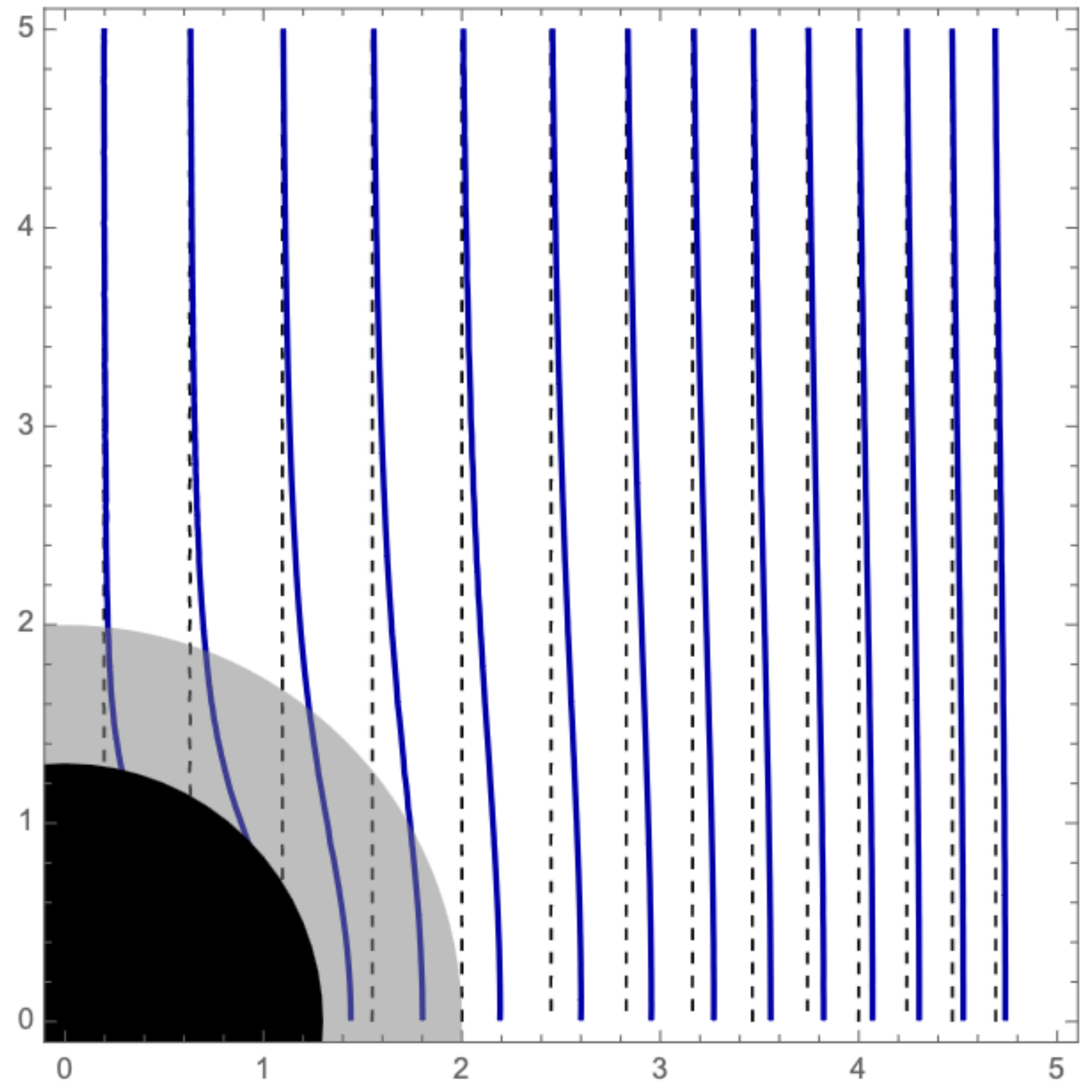

$$\Psi(r,\theta) = \frac{r^2(1+q) - qM_q^2}{2M_q^2\left(1+\sqrt{1+q}\right)}\sin^2\theta$$

At the Outer Light Surface the **perturbation theory breaks down** due to a **NON-PERTURBATIVE SCALING**

$$\frac{r_+}{r_0} = \frac{r_{\rm ILS}}{r_0} = 1 + \mathcal{O}(\alpha^2) \quad , \quad \boxed{\frac{r_{\rm OLS}}{r_0} \sim \frac{1}{\alpha} + \mathcal{O}(\alpha^0)}$$

→ **MATCHED ASYMPTOTIC EXPANSION** to resolve the OLS $\bar{r} = \alpha r$

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HORIZON

INNER LIGHT SURFACE

OUTER LIGHT SURFACE

INFINITY

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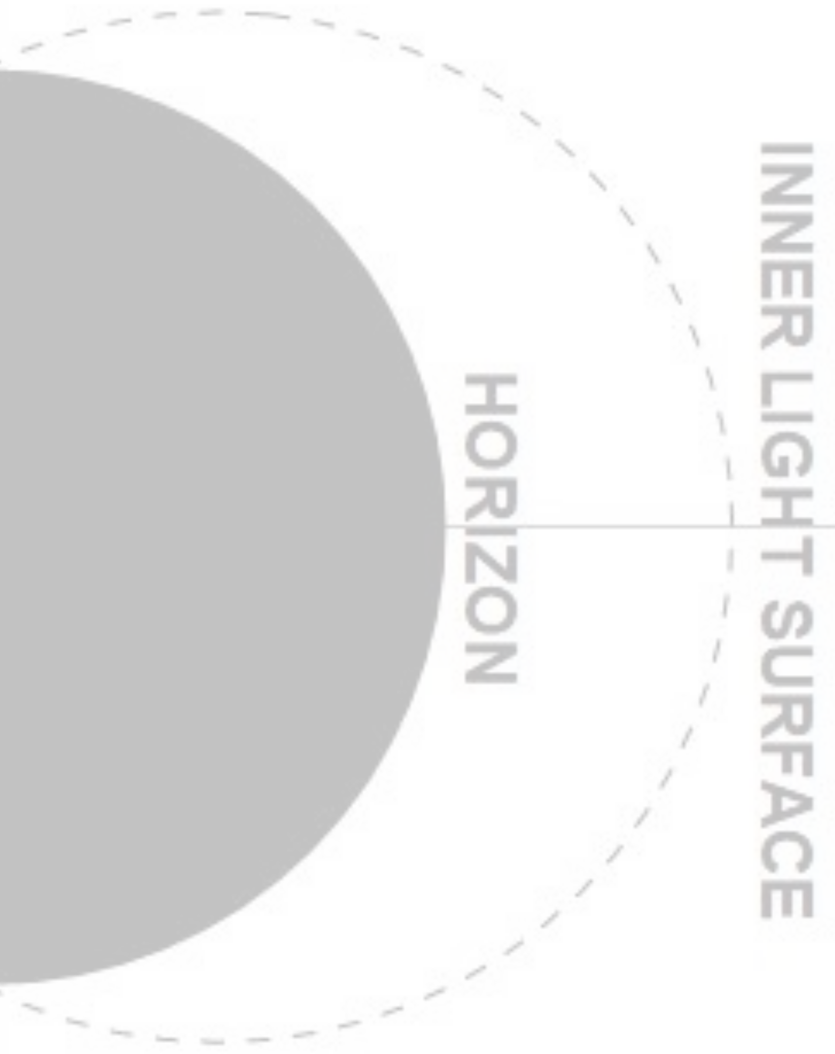

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$$\bar{\psi}_4(\bar{r},\theta) \xrightarrow{\bar{r}\to 0} -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}\Theta_2(\theta) + \dots$$

$$\log\alpha\ \bar{\psi}_{4L}(\bar{r},\theta) \xrightarrow{\bar{r}\to 0} \frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\alpha\,\Theta_2(\theta) + \dots$$

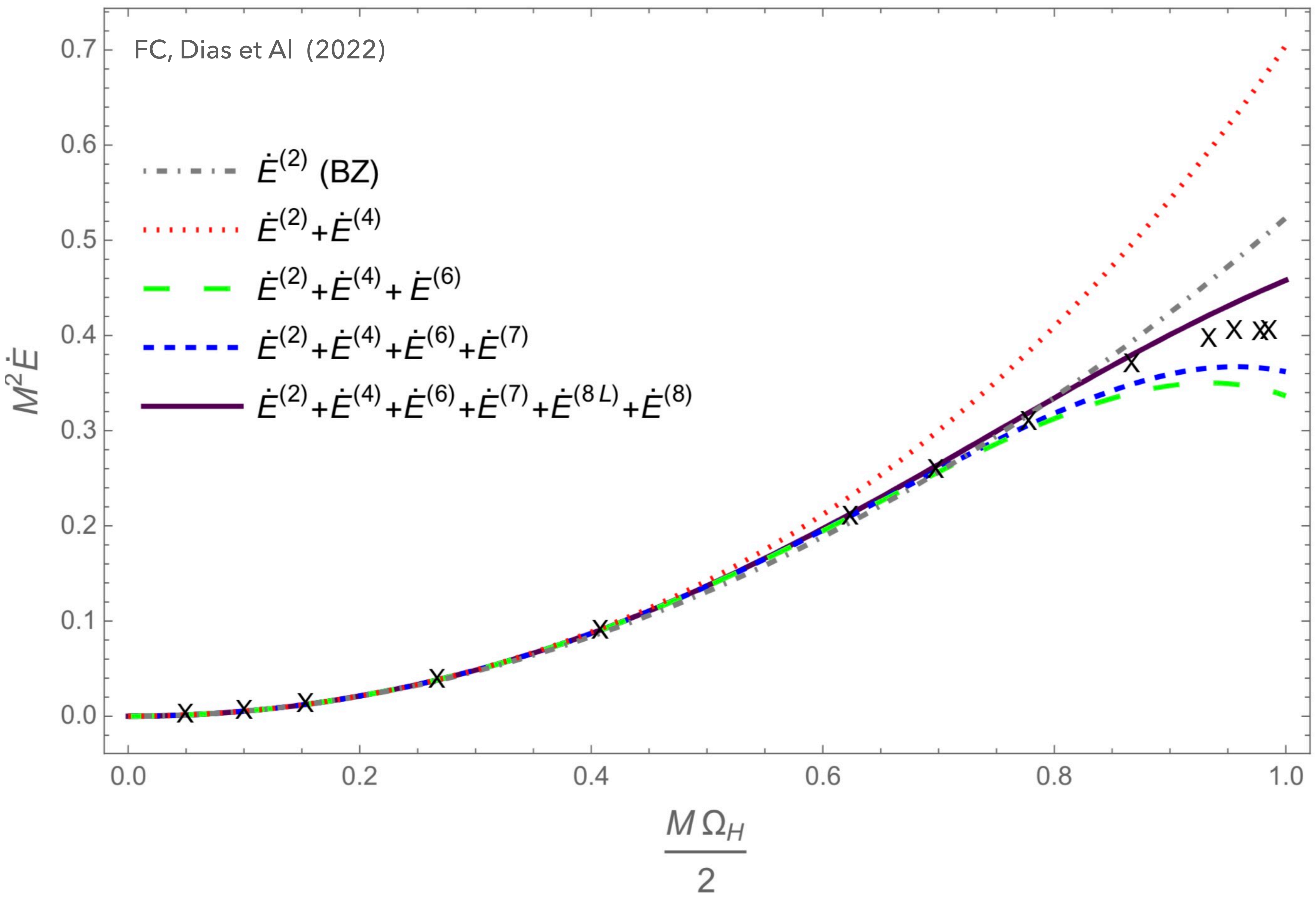
$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2 \underbrace{\left[1 + 0.3459\ r_0^2\Omega_H^2 - 0.7031\ r_0^4\Omega_H^4 + 0.0483\ r_0^5|\Omega_H|^5 + \left[0.1837 - 0.0027\log(r_0|\Omega_H|)\right] r_0^6\Omega_H^6 + \cdots\right]}_{f(\Omega_H)}$$

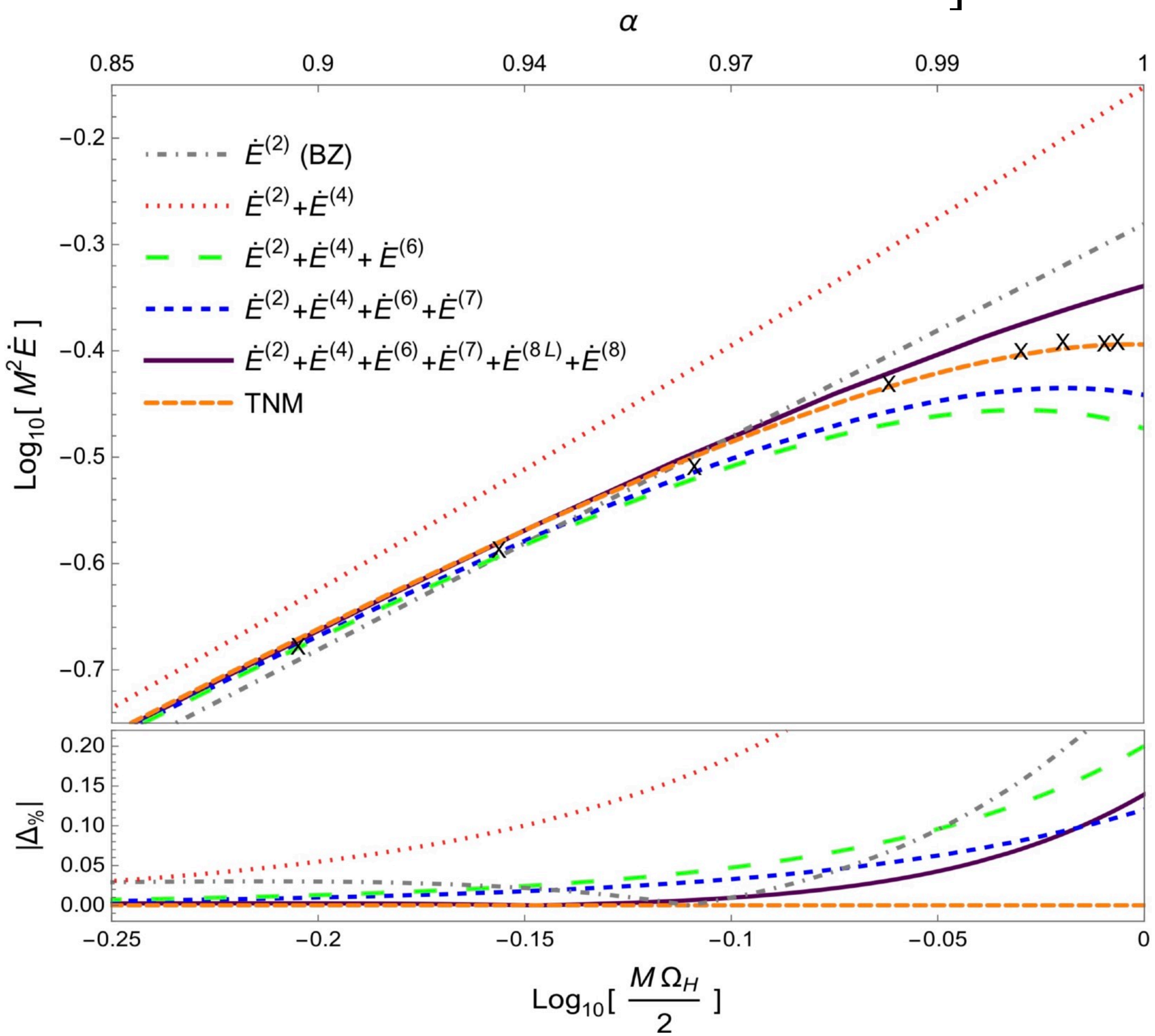
FC, Dias et Al (2022)

HIGH-SPIN FACTOR IN THE KERR SPACETIME

The power emitted in GR (Kerr) :

$$\dot{E}_+ = \frac{2\pi}{3}\Omega_H^2\Big[1 + 0.3459\ r_0^2\Omega_H^2 - 0.7031\ r_0^4\Omega_H^4 + 0.0483\ r_0^5|\Omega_H|^5 + \big[0.1837 - 0.0027\log(r_0|\Omega_H|)\big] r_0^6\Omega_H^6 + \cdots\Big]$$





$|\Delta_\%| \approx 7.5\,\%$ for $\alpha_T = 0.998$ **Thorne Limit** Thorne (1974)

Tchekhovskoy, Narayan, McKinney (2010) $\dot{E}_{(\mathrm{TNM})} = \frac{2\pi}{3}\Omega_H^2\left[1 + 0.3459\ r_0^2\Omega_H^2 - 0.575\ r_0^4\Omega_H^4\right]$

Non trivial given this was obtained assuming $\alpha \ll 1$

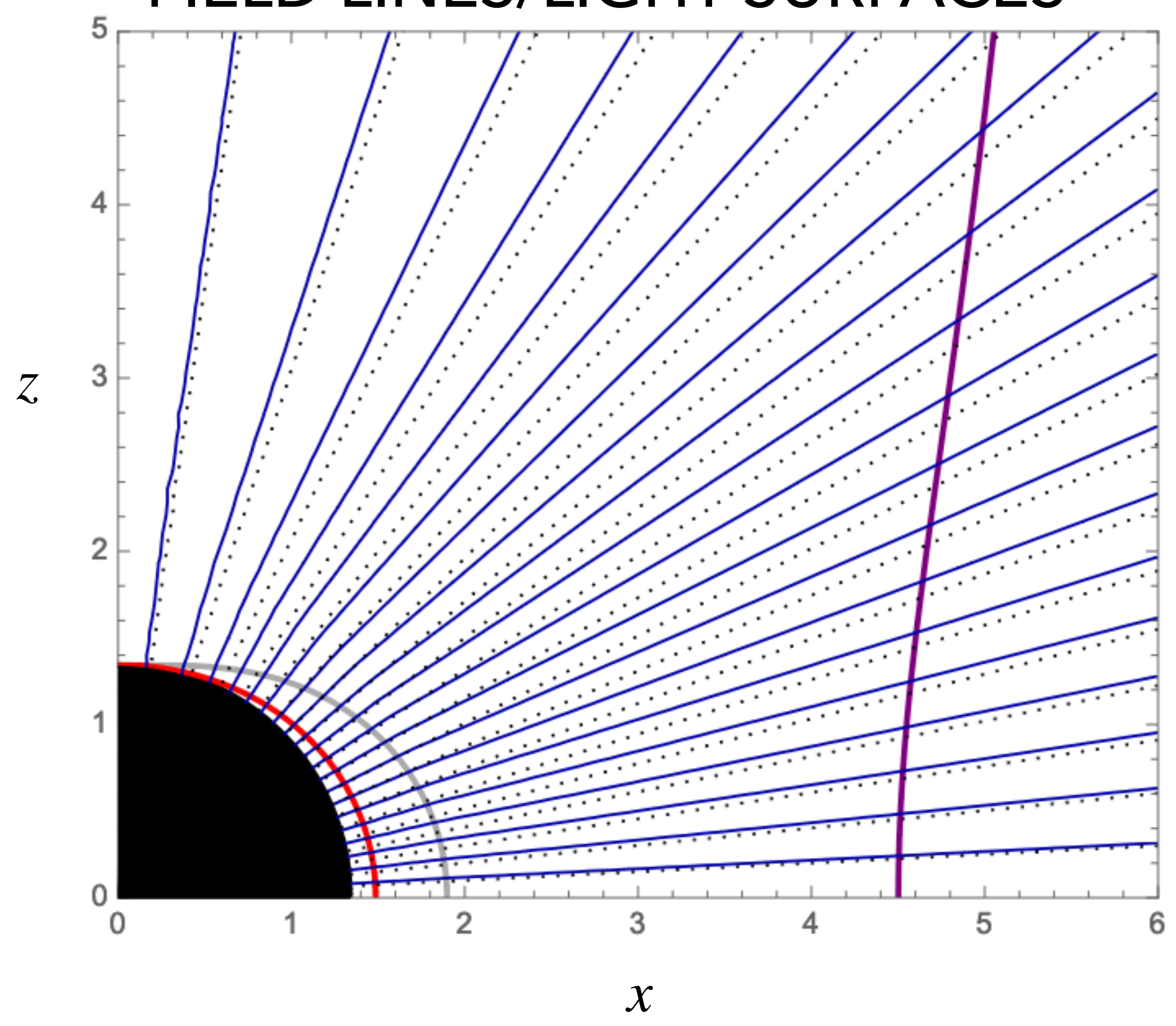
BZ MECHANISM IN MODIFIED GRAVITY

KERR–MOG BACKGROUND

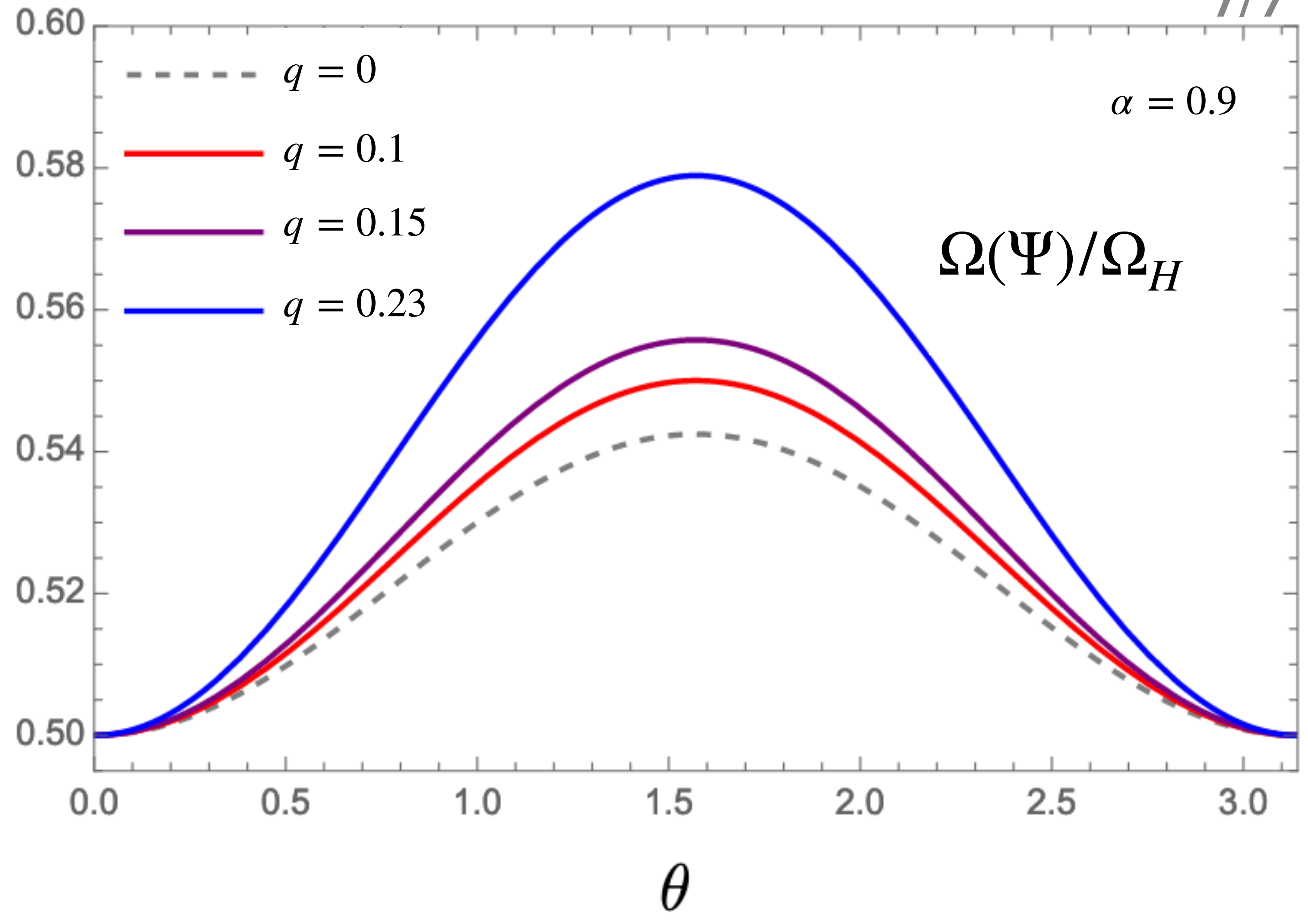
$$q = \frac{G_\infty - G_N}{G_N} \quad , \quad 0 \le q \le \frac{1}{\alpha^2} - 1 \quad , \quad M_q = (1+q)M$$

$$\Omega_H(\alpha, q) = \frac{\alpha}{2r_+ - \frac{q}{1+q}M_q}$$

FIELD LINES/LIGHT SURFACES



FIELD LINES ANGULAR VELOCITY



$$\dot{E}_+ = \kappa |2\pi\Psi_H|^2 \Omega_H^2 \; f(\Omega_H)$$

BZ MECHANISM IN MODIFIED GRAVITY

KERR–MOG BACKGROUND

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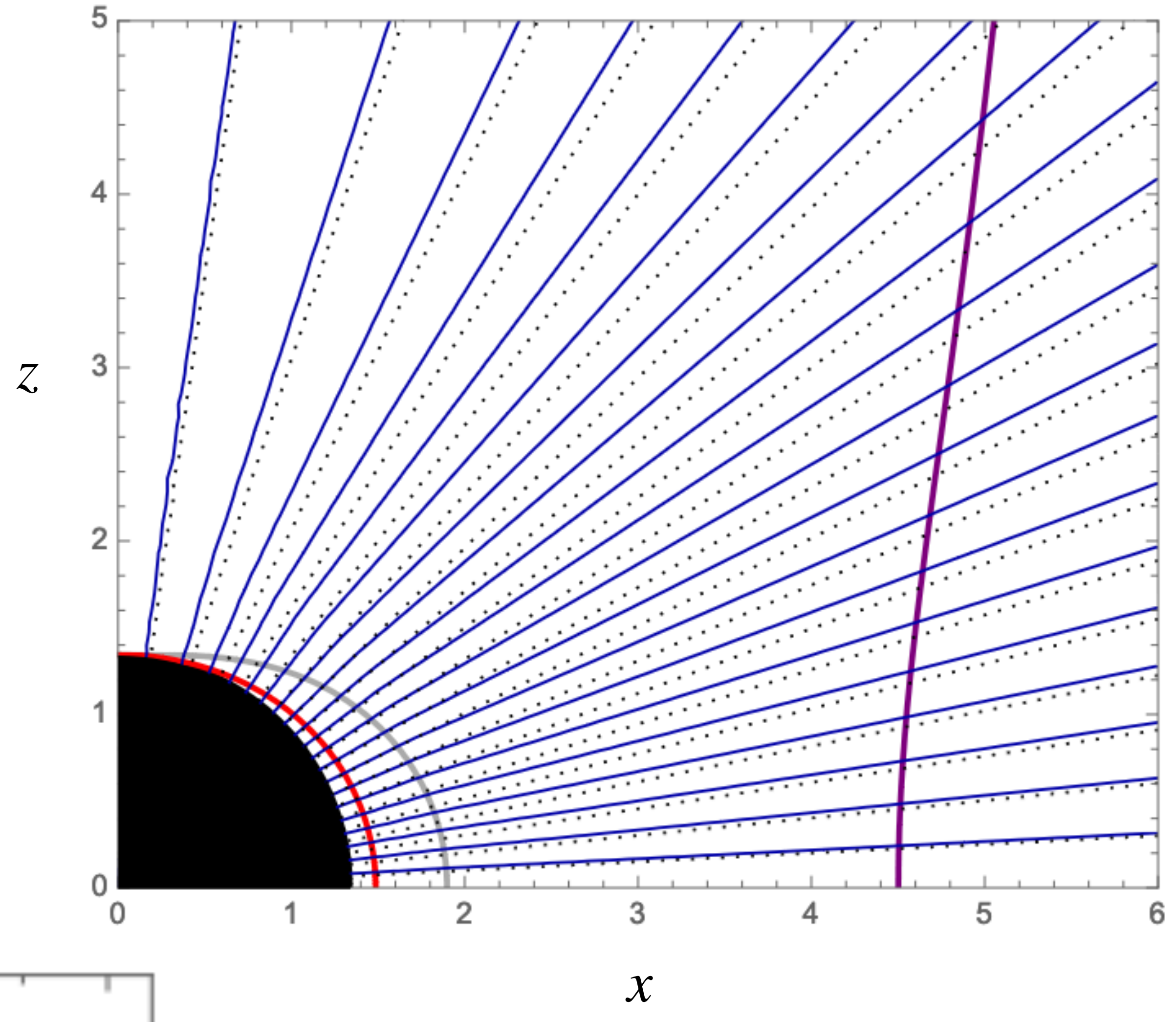
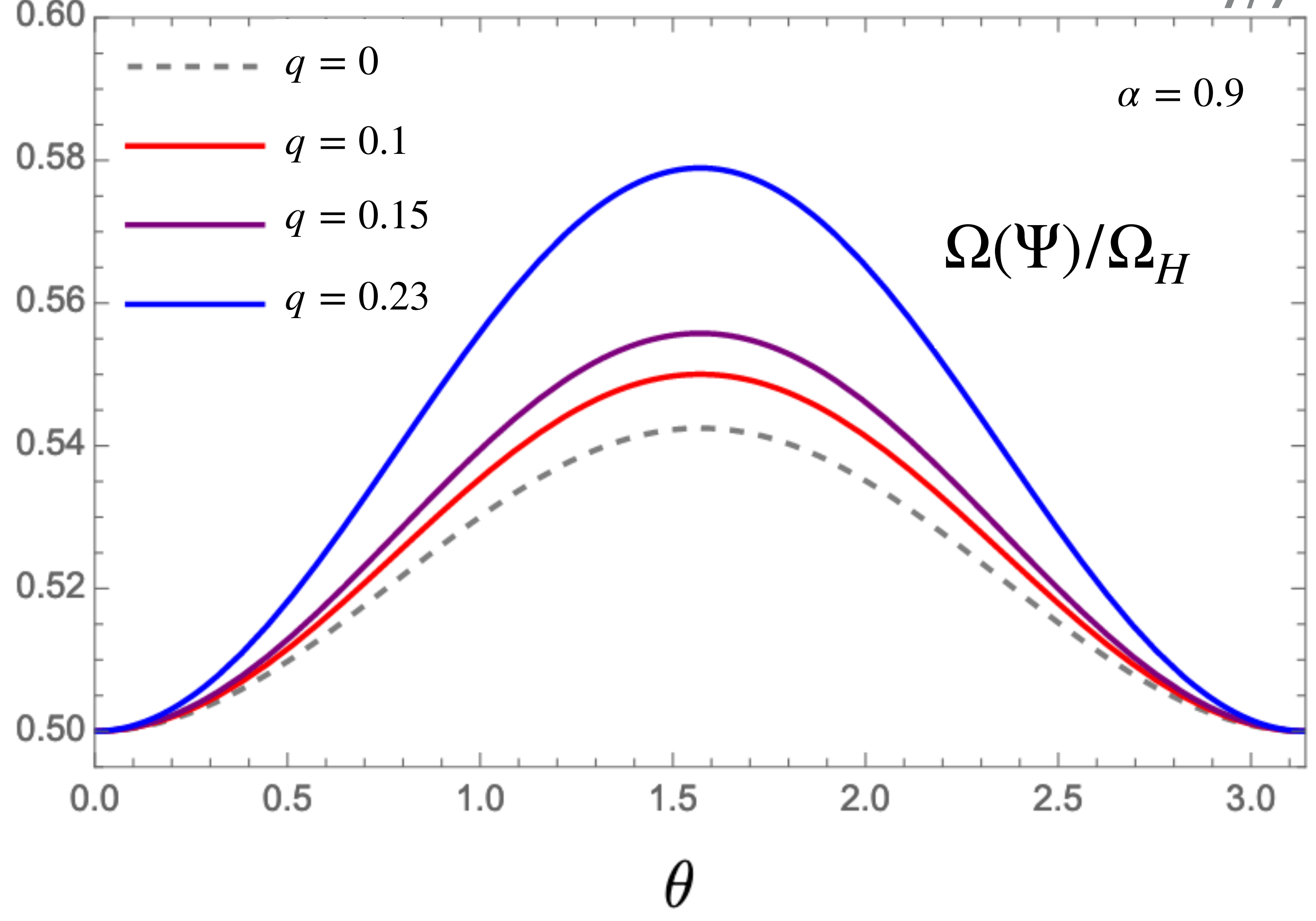
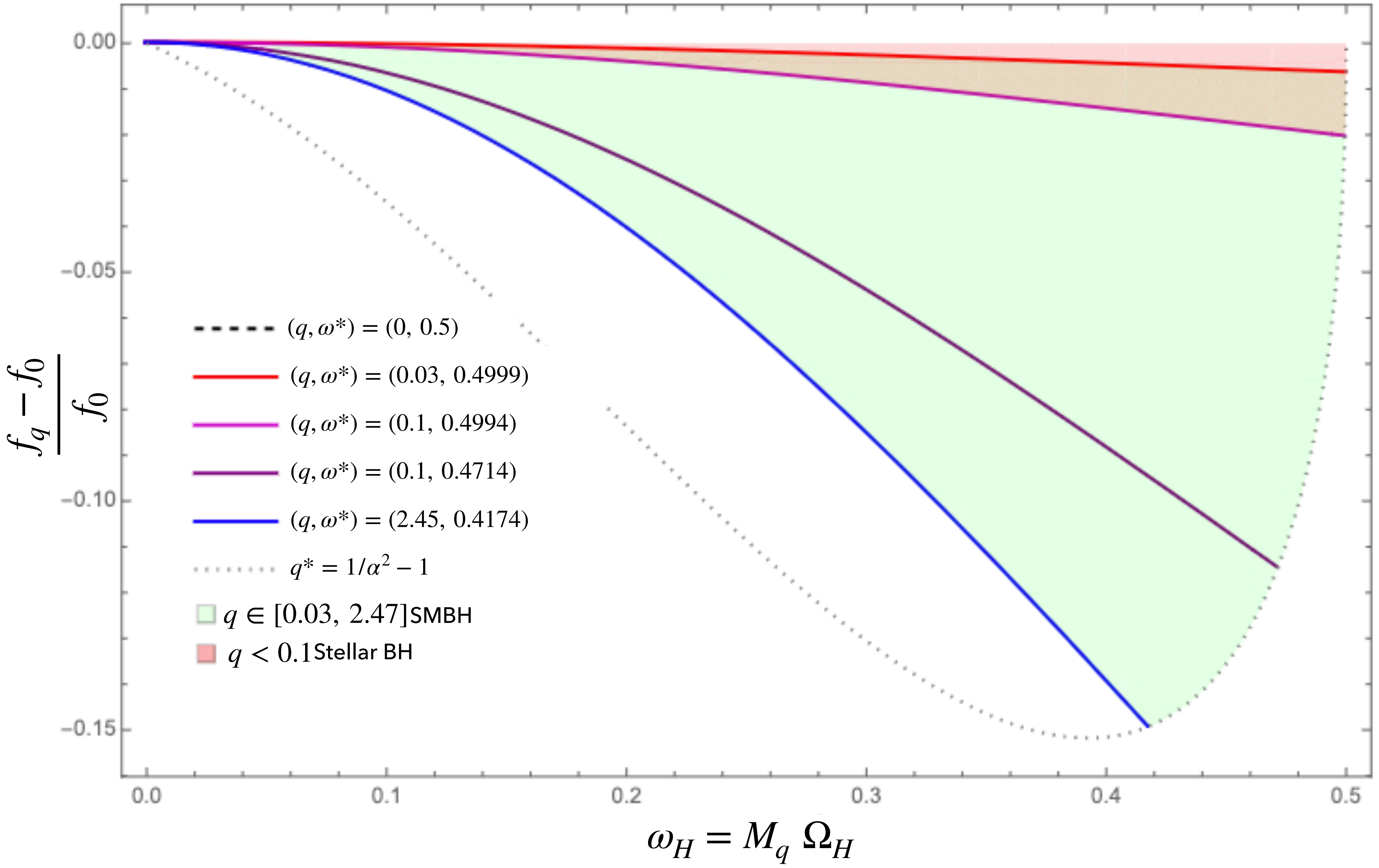
FIELD LINES/LIGHT SURFACES



FIELD LINES ANGULAR VELOCITY



FRACTIONAL DEVIATION FROM GR



MAXIMUM DEVIATION ~ $-15\,\%$ at $\omega_H \approx 0.42$

$$\dot{E}_+ = \kappa \,|\, 2\pi\Psi_H |^2\, \Omega_H^2\, f(\Omega_H)$$

FC, Harmark et Al (2024)

$$f_q(\Omega_H) = 1 + \frac{4}{5} M_q^2 \Omega_H^2 \frac{\left(1+\sqrt{1+q}\right)^2}{1+q}\left[1 - \frac{\left(1+\sqrt{1+q}\right)^2}{1+q} R_2^H(q)\right] + \mathcal{O}(\Omega_H^4)$$

$$R_2^H(q) = \frac{(1+w_q)^2}{2(1-w_q)w_q}\left[\frac{w_q(3\pi^2 - 47 + 2w_q)}{18} - \frac{1}{2} + Li_2(w_q) - \frac{1-w_q^2}{2w_q}\log(1-w_q)\right]$$

$$w_q = \frac{1+q-\sqrt{1+q}}{1+q+\sqrt{1+q}}$$

CONCLUSION AND OUTLOOKS

$$\dot{E}_{+} = \kappa\ (2\pi\Psi_H)^2\ \Omega_H^2\ f(\Omega_H)$$

The Ω_H^2 scaling is **not always** representative of the BZ mechanism and **cannot distinguish** GR and non-GR case

The **High-Spin Factor** $f(\Omega_H)$ contains **crucial** and **non-degenerate information on strong gravity regime!**

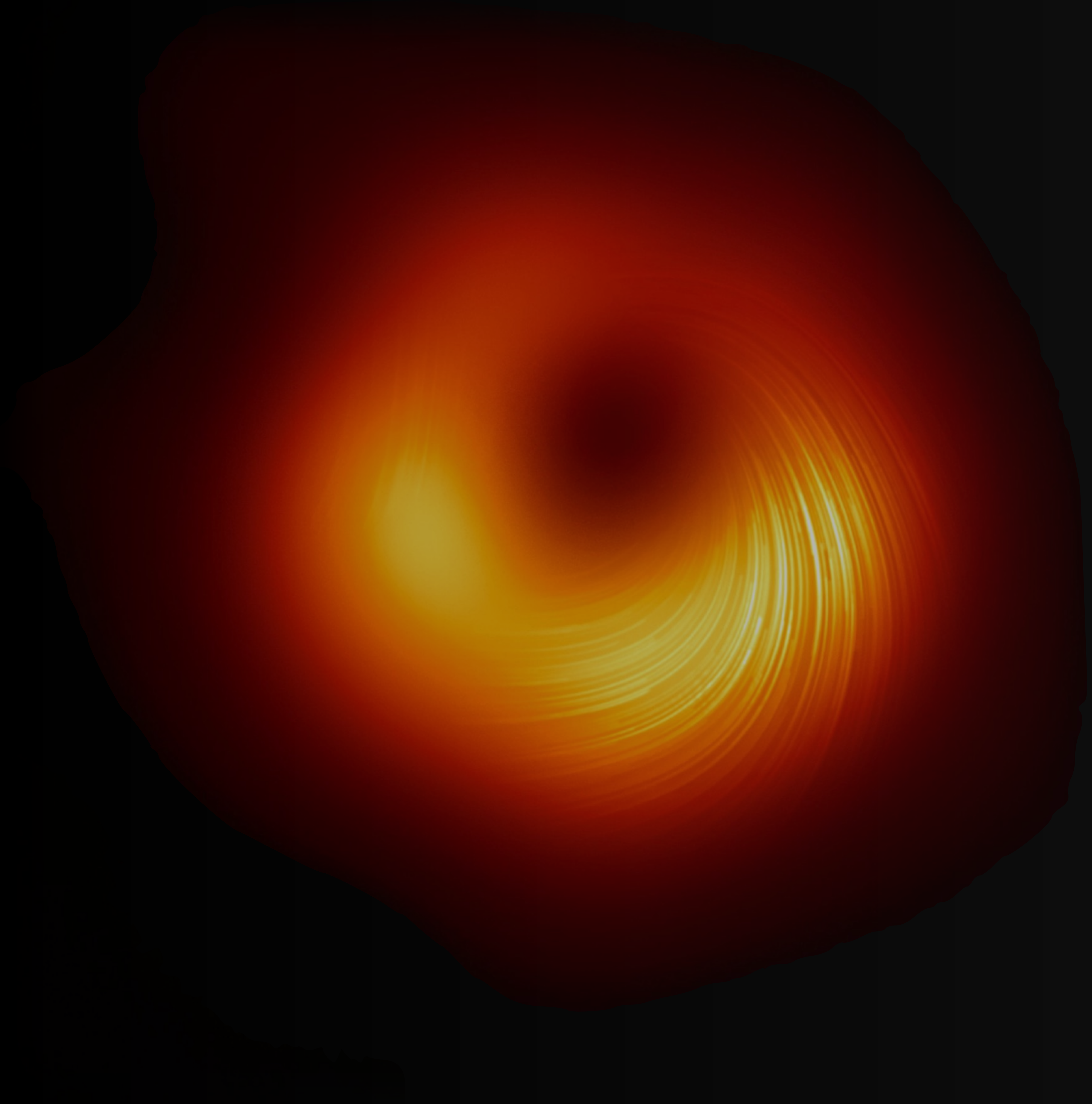
- **Can be computed perturbatively at high orders** **(Matched Asymptotic Expansion Scheme)**

FC, Dias et Al (2022)

- **Logarithmic terms necessary** **(Good agreement with simulations for** $\alpha \approx 0.99$**)**

- Contains **insights** about the **underlying theory of gravity** **(test the Kerr–Hypothesis)** FC, Harmark et Al (2024)

Extend the knowledge of $f(\Omega_H)$ to **THEORY–AGNOSTIC FRAMEWORKS** FC, Rezzolla (in preparation)



THANKS!

MATCHED ASYMPTOTIC EXPANSION

r-REGION ANSATZ

$$\psi(r,\theta) = \psi_0(\theta) + \alpha^2\psi_2(r,\theta) + \alpha^4\psi_4(r,\theta) + \alpha^5\boldsymbol{\psi_5(r,\theta)} + \mathcal{O}(\alpha^6\log\alpha),$$

$$r_0 I(\psi) = \alpha\, i_1(\psi_0) + \alpha^3 i_3(\psi_2) + \alpha^4 i_4(\psi_0) + \alpha^5\left[\boldsymbol{I_5(r,\theta) + \log\alpha\, I_{5L}(r,\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha),$$

$$r_0\Omega(\psi) = \alpha\,\omega_1(\psi_0) + \alpha^3\omega_3(\psi_0) + \alpha^4\omega_4(\psi_0) + \alpha^5\left[\boldsymbol{\Omega_5(r,\theta) + \log\alpha\,\Omega_{5L}(r,\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha)\,.$$

$\bar{r}$-REGION ANSATZ

$$\psi(\bar{r},\theta) = \psi_0(\theta) + \alpha^3\bar{\psi}_3(\bar{r},\theta) + \alpha^4\left[\boldsymbol{\bar{\psi}_4(\bar{r},\theta) + \log\alpha\,\bar{\psi}_{4L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^5\log\alpha),$$

$$r_0 I(\psi) = \alpha\, i_1(\psi_0) + \alpha^3\,\bar{i}_3(\psi_0) + \alpha^4\,\bar{i}_4(\bar{\psi}_3) + \alpha^5\left[\boldsymbol{\bar{I}_5(\bar{r},\theta) + \log\alpha\,\bar{I}_{5L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha),$$

$$r_0\Omega(\psi) = \alpha\,\omega_1(\psi_0) + \alpha^3\omega_3(\psi_0) + \alpha^4\omega_4(\psi_0) + \alpha^5\left[\boldsymbol{\bar{\Omega}_5(\bar{r},\theta) + \log\alpha\,\bar{\Omega}_{5L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha)\,.$$

- $r \to \infty$ in the r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_4(r,\theta) = \left[\frac{1}{224}\frac{r}{r_0} + \frac{227}{100800} + \frac{1}{1680}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \left[\frac{9}{8960}\frac{r}{r_0} + \frac{363}{896000} + \frac{3}{22400}\log\frac{r}{r_0}\right]\Theta_4(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

$$\psi_5(r,\theta) = \frac{r^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{r}{r_0}\right)$$

- $\bar{r} \to 0$ in the $\bar{r}$-region

$$\bar{\psi}_3(\bar{r},\theta) = \frac{1}{8}\frac{r_0}{\bar{r}}\Theta_2(\theta) + \frac{\bar{r}}{r_0}\left[\frac{1}{224}\Theta_2(\theta) + \frac{9}{8960}\Theta_4(\theta)\right] + \frac{\bar{r}^2}{r_0^2}\mathcal{F}(\theta) + \mathcal{O}\left(\frac{\bar{r}^3}{r_0^3}\right)$$

$$\bar{\psi}_4(\bar{r},\theta) = -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}\,\Theta_2(\theta) + \frac{227}{100800}\Theta_2(\theta) + \frac{363}{896000}\Theta_4(\theta) + \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]\log\frac{\bar{r}}{r_0} + \mathcal{O}\left(\frac{\bar{r}}{r_0}\right)$$

$$\bar{\psi}_{4L}(\bar{r},\theta) = -\frac{1}{40}\frac{r_0^2}{\bar{r}^2}\,\Theta_2(\theta) - \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]$$

MATCHED ASYMPTOTIC EXPANSION

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$$\psi(r,\theta) = \psi_0(\theta) + \alpha^2\psi_2(r,\theta) + \alpha^4\psi_4(r,\theta) + \alpha^5\boldsymbol{\psi_5(r,\theta)} + \mathcal{O}(\alpha^6\log\alpha),$$

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$$r_0 I(\psi) = \alpha\, i_1(\psi_0) + \alpha^3\,\bar{i}_3(\psi_0) + \alpha^4\bar{i}_4(\bar{\psi}_3) + \alpha^5\left[\boldsymbol{\bar{I}_5(\bar{r},\theta) + \log\alpha\,\bar{I}_{5L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha),$$

$$r_0\Omega(\psi) = \alpha\,\omega_1(\psi_0) + \alpha^3\omega_3(\psi_0) + \alpha^4\omega_4(\psi_0) + \alpha^5\left[\boldsymbol{\bar{\Omega}_5(\bar{r},\theta) + \log\alpha\,\bar{\Omega}_{5L}(\bar{r},\theta)}\right] + \mathcal{O}(\alpha^6\log\alpha)\,.$$

$$\log(r/r_0) \xrightarrow{r\to\alpha^{-1}\bar{r}} \log(\bar{r}/r_0) - \log\alpha$$

▸ $r \to \infty$ in the r-region

$$\psi_2(r,\theta) = \left[\frac{1}{8}\frac{r_0}{r} - \frac{11}{800}\frac{r_0^2}{r^2} + \frac{1}{40}\frac{r_0^2}{r^2}\log\frac{r}{r_0}\right]\Theta_2(\theta) + \mathcal{O}\left(\frac{r_0^3}{r^3}\log\frac{r}{r_0}\right)$$

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$$\bar{\psi}_4(\bar{r},\theta) = -\frac{11}{800}\frac{r_0^2}{\bar{r}^2}\Theta_2(\theta) + \frac{1}{40}\frac{r_0^2}{\bar{r}^2}\log\frac{\bar{r}}{r_0}\Theta_2(\theta) + \frac{227}{100800}\Theta_2(\theta) + \frac{363}{896000}\Theta_4(\theta) + \left[\frac{1}{1680}\Theta_2(\theta) + \frac{3}{22400}\Theta_4(\theta)\right]\log\frac{\bar{r}}{r_0} + \mathcal{O}\left(\frac{\bar{r}}{r_0}\right)$$

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NON-PERTURBATIVE STRUCTURE

Kerr geometry is analytic in α, we therefore expect that also the magnetosphere is analytic!

EXAMPLE

For finite α the function is analytic

$$\frac{1}{\alpha^2+1} + \alpha^2 \overbrace{\sqrt{\left(\alpha^2\right)^{\alpha^4} - \frac{1}{\alpha^2+1}}} = 1 - \alpha^2 + |\alpha|^3 + \alpha^4 \underbrace{- \frac{1}{2}|\alpha|^5 + |\alpha|^5 \log|\alpha|} + \mathcal{O}(\alpha^6)$$

The limit $\alpha \to 0$ leads to non-analytic terms