Martin Kološ (Silesian University, Opava, CZ)

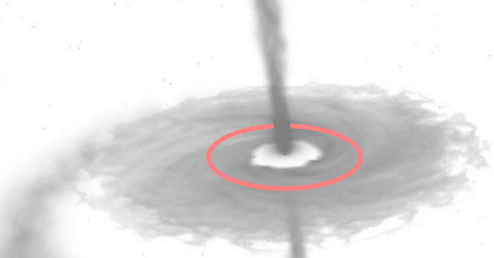
Vibrating ring around black hole

A thin circular structure vibrating in the central plane of a black hole will be investigated. This circular ring (string loop) can be considered as a simplified model for thin magnetic flux tubes (in plasma physics), and connections to accreting fluid structures around the black hole will be demonstrated. The stability of the string loop and the frequencies of its vibrational modes will be provided and compared with the vibrational modes of thick toroidal fluid structure around black holes, which is the standard analytical model for the temporal properties of accretion flow.

in collaboration with: Arman Tursunov, José Natário, Maria Churilova, Zdeněk Stuchlík

Marcel Grossmann Meeting in Pescara 2024

Motivation: magnetic field influence on accretion structures
we focuss on thin circular structure vibrating around black hole



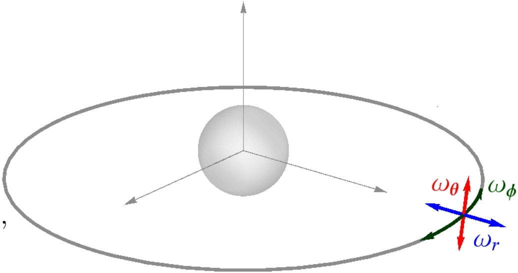
- axially symmetric thin structure (ring,string)
- particle motion vs. linear axially symmetric string vs. fluid accretion tori
- oscillating thick tori gets diluted and could be reinforced by magnetic field
- vibration frequency can provide basic time scale on astrophysical processes

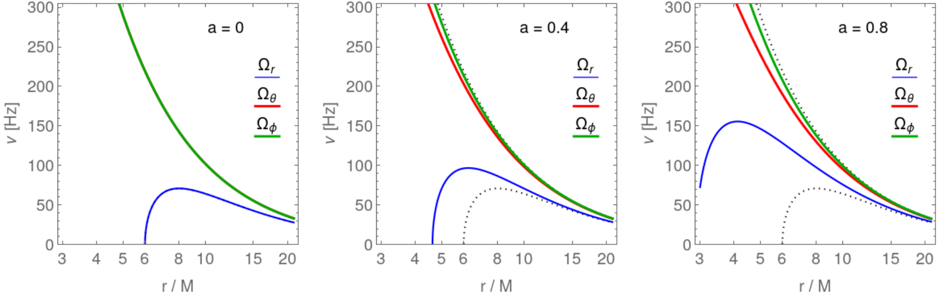
"*Magnetic tension is important in plasma physics - controls dynamics of some systems and the shape of magnetic structures; in a homogeneous magnetic field (no gravity), magnetic tension is the sole driver of linear Alfvén waves.*"
• J.C.Vial, O.Engvold: Solar Prominences, Springer (2015).

thin Keplerian disk = test particles on circular orbits

Perturbation of circular orbit, located at minima of $V_{\rm eff}(r,\theta)$, leads to particle oscillations with radial $\omega_{\rm r}$, vertical ω_θ, Keplerian ω_ϕ frequencies:

$$\omega_{\rm r}^2 \sim \frac{\partial^2 V_{\rm eff}}{\partial r^2}, \quad \omega_\theta^2 \sim \frac{\partial^2 V_{\rm eff}}{\partial \theta^2}, \quad \omega_\phi = \frac{{\rm d}\phi}{{\rm d}\tau},$$





thin Kelerian ring and **THICK** accretion disk around BH

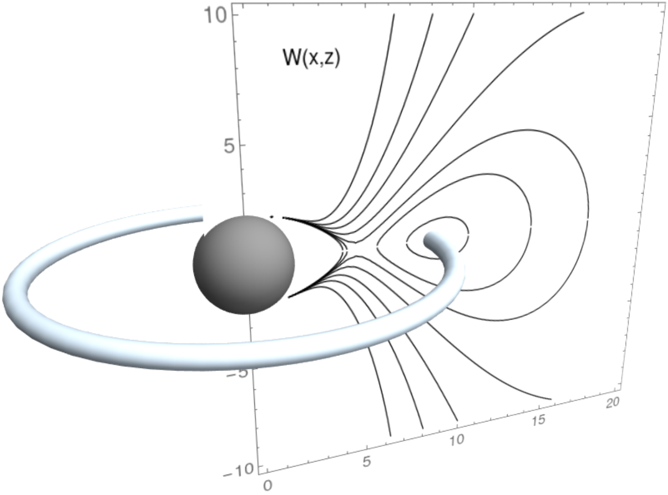
ideal GRMHD:

$$
\begin{aligned}
(\rho u^{\mu})_{;\mu} &= 0, \\
(T^{\mu}{}_{\nu})_{;\mu} &= 0, \\
(u^{\nu} b^{\mu} - u^{\nu} b^{\nu})_{;\mu} &= 0,
\end{aligned}
$$

+ eq.of state + angular momentum profile → equipressure surfaces: analytic $W(x,z) =$ int.GRMHD simulation

torus thickens ∼ pressure inside

no thickness, no pressure = particle circular orbit



• M.Kozlowski, M.Jaroszynski, M.Abramowicz: The analytic theory of fluid disks orbiting the Kerr black hole, Astronomy and Astrophysics, 63, 1-2, 209-220 (1978)

Thick accretion tori vibration frequencies

thin linear structure:

- radial mode
- vertical mode

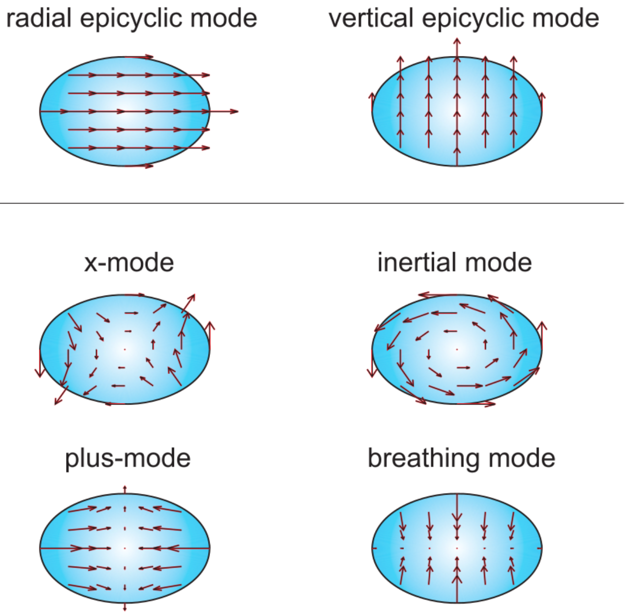
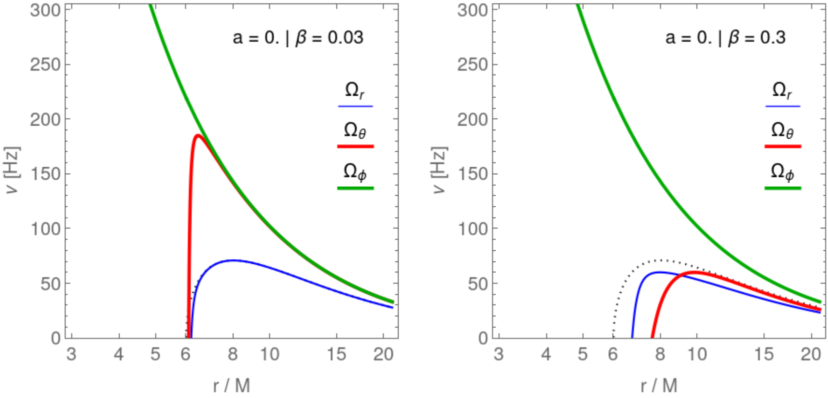


figure from:
• Eva Šrámková: Oscillations of disc structures around compact objects, PhD dissertation (2010)

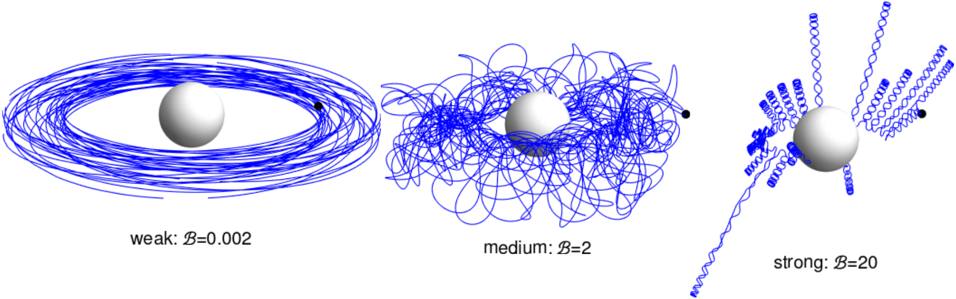
Thick tori frequencies || torus pressure decrees the frequencies

- torus thickness $\beta \leftrightarrow$ pressure in torus center
- frequencies are decreasing with pressure inside torus (torus thickness)



• O.Straub, E.Šrámková: Epicyclic oscillations of non-slender fluid tori around Kerr BH, CQG, 26,5,055011 (2009) • A.Kotrlová, E.Šrámková+, Astronomy & Astrophysics 643, A31 (2020)

Magnetic field: single charged particle dynamic



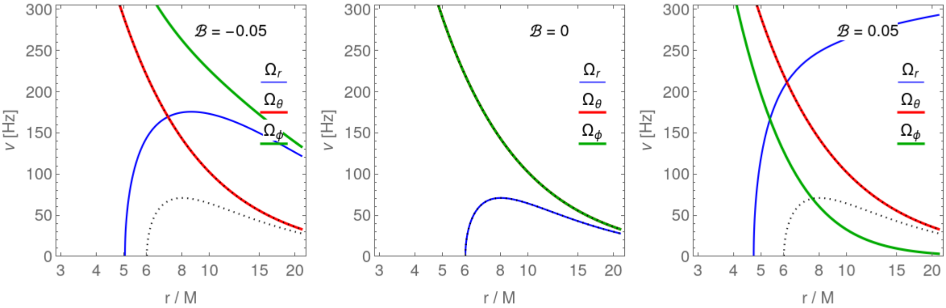
- astrophysically relevant:
  weak $\mathcal{B} \ll 1$ case - small oscillations
  strong $\mathcal{B} \gg 1$ case - motion along magnetic field lines

- Lorentz force: $\mathcal{B} < 0$ attractive || $\mathcal{B} > 0$ repulsive
- $\mathcal{B} \sim 1$ Lorentz force is comparable to gravity - the richest case

Charged particle in uniform magnetic field

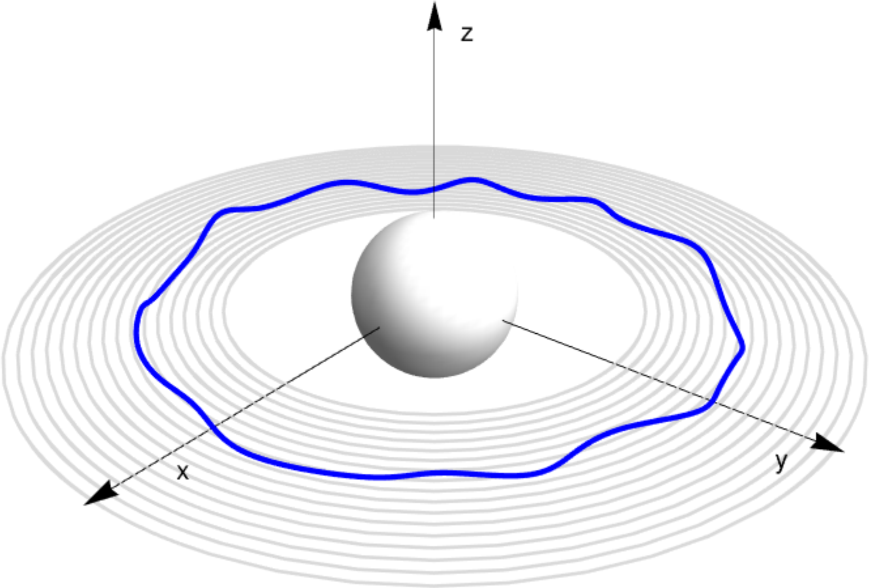
Lorentz force:

$\mathcal{B} < 0$ attractive $\qquad\qquad$ $\mathcal{B} > 0$ repulsive

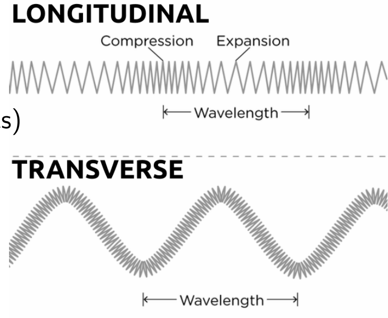


• M.Kološ, Z.Stuchlík, A.Tursunov: *Quasi-harmonic oscillatory motion of charged particles around a Schw.BH immersed in uniform mag. field*, CQG 165009 (2015) [arXiv:1506.06799]

Linear structure vibrating around black hole - string loop



How to describe 1D objects?



particle (0D object) is specified by its rest mass m;
all infinitesimally thin structures (=strings, 1D objects)
are described by only two numbers:
energy density U and **tension** T per unit length
string equation of state $F(U,T)=0$; $\boxed{p=-T}$

$$S=\int \mathcal{L}\,\mathrm{d}\sigma \mathrm{d}\tau \quad \rightarrow \quad s_{\mathrm{T}}^2=\frac{T}{U}, \quad c_{\mathrm{L}}^2=-\frac{\mathrm{d}T}{\mathrm{d}U}$$

.

perturbation velocity: transverse (normal) $s=s_{\mathrm{T}}$ / logitunidal (along) $c=c_{\mathrm{L}}$

"dust" string	Nambu–Goto string	Current-Carrying String	more models
non-interacting test particles	simplest case for string with tension $T=\mu$	scalar field φ added on the string $\rightarrow$ currents $\varphi_{\|a}$ CCS $\in$ rigid strings - the most stiff strings	...
$T=0$	$T=U$		
$s^2=c^2=0$	$s^2=1,\ c^2=-1$	$s^2=1,\ c^2\in[0,1)$	

1D object around black hole (Nambu–Goto string)

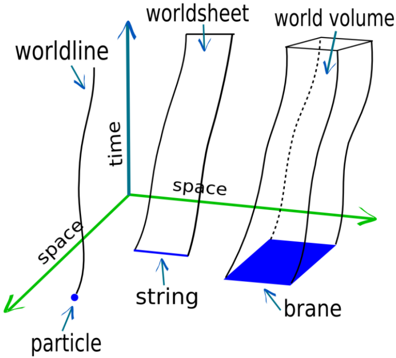
1D string immersion into 4D spacetime

$$X^\alpha(\tau,\sigma) = (t, r, \theta, \phi);$$

worldsheet: $a, b \in \{\tau, \sigma\}$ / $\alpha \in \{t, r, \theta, \phi\}$
τ - string evolution / σ - along string

worldsheet should be min. (Nambu–Goto)

$$S = -\mu \int_w \mathrm{d}w = -\mu \int \sqrt{-h}\, \mathrm{d}\sigma \mathrm{d}\tau$$



$$h_{ab} = g_{\alpha\beta} X^\alpha_{|a} X^\beta_{|b}, \qquad \Sigma^{ab} = -\mu\sqrt{-h}\, h^{ab}$$

induced metric on the worldsheet h_{ab} / worldsheet stress-energy tensor Σ^{ab}

take $\delta S = 0$ and (after some algebra) NG string equation of motion are obtain

$$(\Sigma^{ab} g_{\mu\lambda} X^\mu_{,a})_{,b} - {}^1\!/_2 \Sigma^{ab} g_{\mu\nu,\lambda} X^\mu_{,a} X^\nu_{,b} = 0$$

1D object around black hole (Nambu–Goto string)

1D string immersion into 4D spacetime

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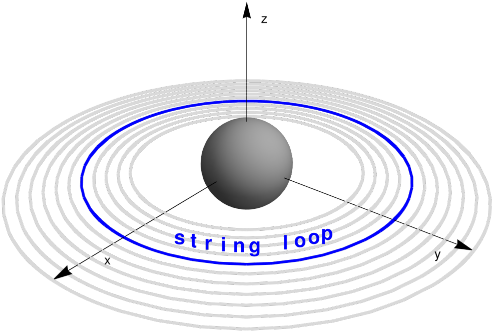

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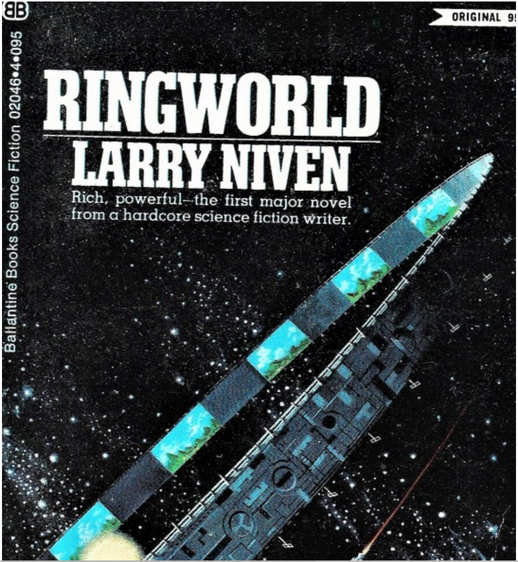
induced metric on the worldsheet h_{ab} / worldsheet stress-energy tensor Σ^{ab}

take $\delta S = 0$ and (after some algebra) NG string equation of motion are obtain

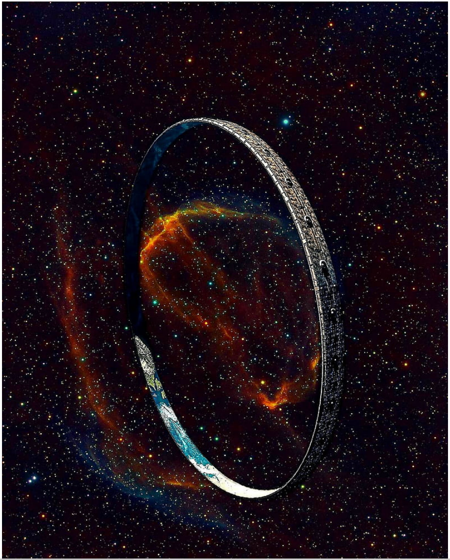
$$-\frac{\partial}{\partial\tau} X^\alpha_{|\tau} \quad - \Gamma^\alpha_{\beta\gamma} X^\beta_{|\tau} X^\gamma_{|\tau} \quad + \frac{\partial}{\partial\sigma} X^\alpha_{|\sigma} \quad + \Gamma^\alpha_{\beta\gamma} X^\beta_{|\sigma} X^\gamma_{|\sigma} \quad = 0$$

Ringworld - circular megastructure rotating around star

sci-fi novel written by Larry Niven (1970)

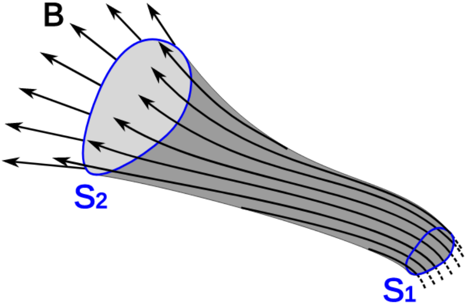


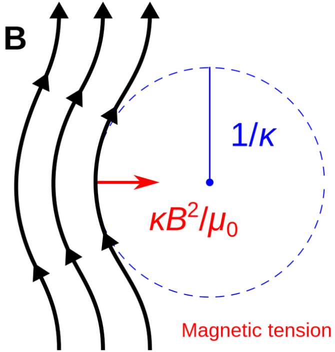
Halo (2001-) computer game



Magnetic flux tubes & Magnetic tension

A flux tube is tube-like (cylindrical) region of space containing a magnetic field $\vec{B}$.
en.wikipedia.org/Magnetic_tension





Magnetic tension is a restoring force that acts to straighten bent magnetic field lines

$$f_{\mathrm{T}} = \frac{1}{\mu_0}\frac{1}{r_0}B^2$$

where μ_0 is the vacuum permeability.

Motivation: magnetic flux tube = relativistic string (**tension**)

$$\text{GRMHD}: \quad \nabla_\alpha T^{\alpha\beta} = 0, \quad \nabla_\alpha \rho u^\alpha = 0, \quad \nabla_\alpha (b^\alpha u^\beta - b^\beta u^\alpha) = 0 \qquad \boxed{x^\alpha_\tau = \frac{u^\alpha}{q}, \; x^\alpha_\sigma = \frac{b^\alpha}{\rho}}$$

$$T^{\alpha\beta} = Q u^\alpha u^\beta - P g^{\alpha\beta} - 1/(4\pi) b^\alpha b^\beta \quad P = p - \frac{1}{8\pi} b^\alpha b_\alpha, \quad Q = p + \epsilon - \frac{1}{4\pi} b^\alpha b_\alpha$$

$$\begin{array}{lcll}
-\frac{\partial}{\partial\tau} X^\alpha_{|\tau} - \Gamma^\alpha_{\beta\gamma} X^\beta_{|\tau} X^\gamma_{|\tau} + \frac{\partial}{\partial\sigma} X^\alpha_{|\sigma} + \Gamma^\alpha_{\beta\gamma} X^\beta_{|\sigma} X^\gamma_{|\sigma} & = & 0 & (\text{string}) \\
-\frac{\partial}{\partial\tau}\left(\frac{Qq}{\rho} x^\alpha_{|\tau}\right) - \frac{Qq}{\rho}\Gamma^\alpha_{\beta\gamma} x^\beta_{|\tau} x^\gamma_{|\tau} + \frac{\partial}{\partial\sigma}\left(\frac{\rho}{4\pi q} x^\alpha_{|\sigma}\right) + \frac{\rho}{4\pi q}\Gamma^\alpha_{\beta\gamma} x^\beta_{|\sigma} x^\gamma_{|\sigma} & \simeq & 0 & (\text{plasma})
\end{array}$$

string with tension μ vs. magnetic flux tube (plasma) - "string along mag. filed-lines"

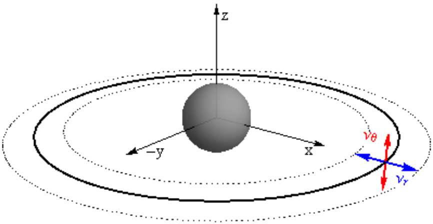
- H.C.Spruit: *Equations for thin flux tubes in ideal MHD*, Astn.&Astp., **102**, 1 (1981)
- V.S.Semenov and L.V.Bernikov: *Magnetic flux tubes - nonlinear strings in relativistic magnetohydrodynamics*, Astro. and Space Sci. **184**, 157-166 (1990)
- S.A.Dyadechkin, V.S.Semenov, H.K.Biernat, T. Penz: *Comparison of magnetic flux tube and cosmic string behavior in Kerr metric* AiSR, **42**, 3, (2008)

Axisymmetric perturbation, frequencies of oscillations in Schw.

- instability? - loop linear perturbation
  $r = r_0 + \delta r(\tau), \quad \theta = \pi/2 + \delta\theta(\tau)$
- frequencies $(x \in (r, \theta))$
  stable: $\Omega^2 > 0$ vs. unstable: $\Omega^2 < 0$

$$\ddot{x} + \Omega^2 x = 0$$

- vertical oscillations have Keplerian frequency in all cases $\sim r^{-3/2}$



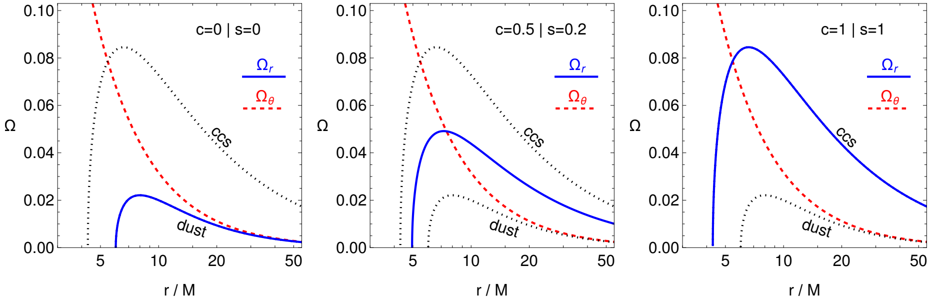
$$\Omega_\theta^2 = \frac{1}{r^3}, \qquad \Omega_\mathrm{r}^2 = \Omega_\mathrm{r}^2(r; c, s) \qquad \Omega_\mathrm{r}^2 \text{ is quite long, but special cases are:}$$

$$\text{"dust" string (particle)} \quad \Omega_\mathrm{r}^2 = \frac{r-6}{r^4}, \qquad \text{c. c. string} \quad \Omega_\mathrm{r}^2 = \frac{r^2 - 5r + 3}{r^4}$$

with the help of coefficients from:

• J.Natário, L.Queimada, R.Vicente: Rotating elastic string loops in flat and black hole spacetimes: stability, cosmic censorship and the Penrose process, CQG 35(7) (2018)

Axisymmetric perturbation, frequencies of oscillations in Schw.



$$\Omega_\theta^2 = \frac{1}{r^3}, \qquad \Omega_r^2 = \Omega_r^2(r; c, s) \qquad \Omega_r^2 \text{ is quite long, but special cases are:}$$

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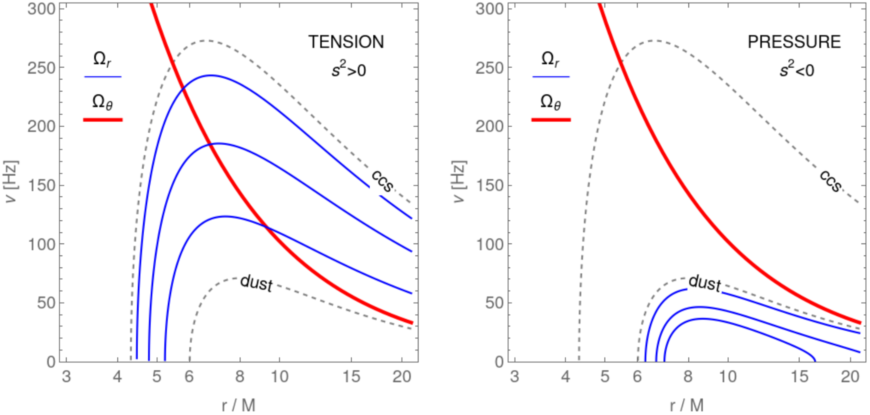
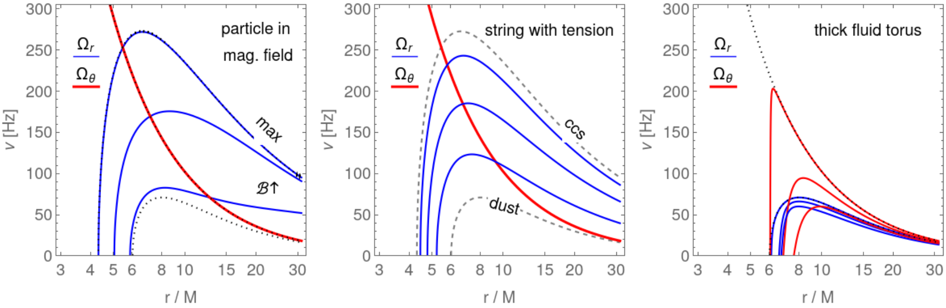
Influence of - tension: string $p < 0$ vs. pressure: ring $p > 0$



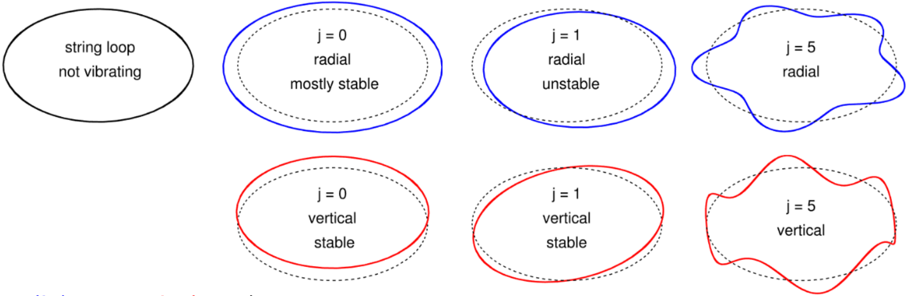
- string frequencies Ω_r, Ω_θ are functions of velocities $s^2 = \frac{T}{U}$ and $c^2 = -\frac{\mathrm{d}T}{\mathrm{d}U}$
- equation of state - relation between energy density U and tension T
- tension is negative pressure $\boxed{p = -T}$ Can we have 1D structures with positive pressure (negative tension)? $p > 0 \Leftrightarrow (U > 0) \wedge (T < 0) \Leftrightarrow s^2 < 0$ (unstable!)

particle on perturbed circular orbit ∥ vibrating string loop ∥ oscillating fluid torus



- string tension: frequencies $\uparrow$ vs. fluid torus pressure: frequencies $\downarrow$
- charged particle in mag. field (attractive Lorentz) = rigid CCS string (maximum)

So far axisymmetric loop only (0th mode), but there can be also higher vibrational modes on string loop



- radial vs. vertical modes
- previous slides are for 0th ($j = 0$) modes only
- how to find higher mode instability? - loop linear perturbation - frequencies!

$$\ddot{x} + \Omega^2 x = 0, \qquad \mathrm{stable}: \Omega^2 > 0 \;||\; \mathrm{unstable}: \Omega^2 < 0$$

- testing higher modes + solving string EOM numerically(?) (PDEs!)

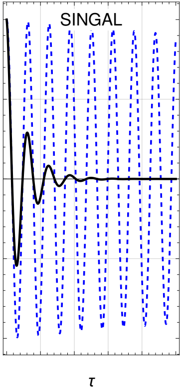
Instabilities: RingWorld (string) & Papaloizou-Pringle (torus)

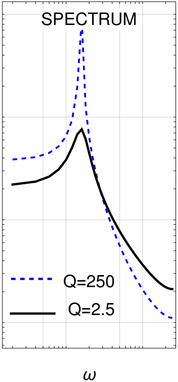
- RingWorld instability - radial $j = 1$ mode - gravitational attraction of the near-side is greater than that of the far-side
- Papaloizou-Pringle instability - fluid rings are dynamically unstable against global, nonaxisymmetric perturbations
- Dyson sphere is stable (loop instability has geometric origin)

The frequencies $\Omega \in \mathbb{C}$ $\quad (P_4(\Omega) = 0$ roots)
$\mathrm{Re}(\Omega)$ - harmonic oscillations
$\mathrm{Im}(\Omega)$ - mode instability
- exponentially grow or decay $\mathrm{sign}(\mathrm{Im}(\Omega))$

$\rightarrow$ **!** $\leftarrow$ you will see oscillations even for unstable modes, if $\mathrm{Im}(\Omega)$ is small; quality factor:

$$Q \equiv \frac{\mathrm{Re}(\Omega)}{2\,\mathrm{Im}(\Omega)}.$$





Summary & Future work

- relativistic elasticity
- magnetic filed lines (flux tubes) behave like relativistic strings
- string tension (frequencies $\uparrow$) vs. fluid torus pressure (frequencies $\downarrow$)

- solving EQM numerically: charged particle ✓ (code in Mathematica, link) $\|$ relativistic string - now NG only $\|$ thick torus - GRMHD simulation (HARM)
- torus frequencies with toroidal magnetic field (analytic, Komissarov model)
- extension to Kerr black hole (partially done)

Thank you for your attention.

codes and more info: https://github.com/XyhwX martin.kolos@physics.slu.cz

• M.Kološ, Z.Stuchlík, A.Tursunov: *Quasi-harmonic oscillatory motion of charged particles around a Schw.BH immersed in uniform mag. field*, CQG 165009 (2015) [arXiv:1506.06799]
• M Churilova, M Kološ, Z Stuchlík: *String loop vibration around Schwarzschild black hole*, The European Physical Journal C 84 (1), 1-13 (2024) [arXiv:2310.07540]