Initial Conditions and Evolution for Spherically Symmetric Collapse

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Content

Context

Context

There are more than 127 exact solutions for an isolated, static, spherically symmetric and perfect fluid solutions of Einstein's equations.

[M.S.R. Delgaty and Kayll Lake1] 1998 → *"how many could represent the field associated with an isolated static spherically symmetric perfect fluid source?"*

Only 9 of these passed all the tests.

Context

[Quevedo (private communication)] $\rightarrow$ The GTD theory says that the other 9 solutions are not thermodynamically correct

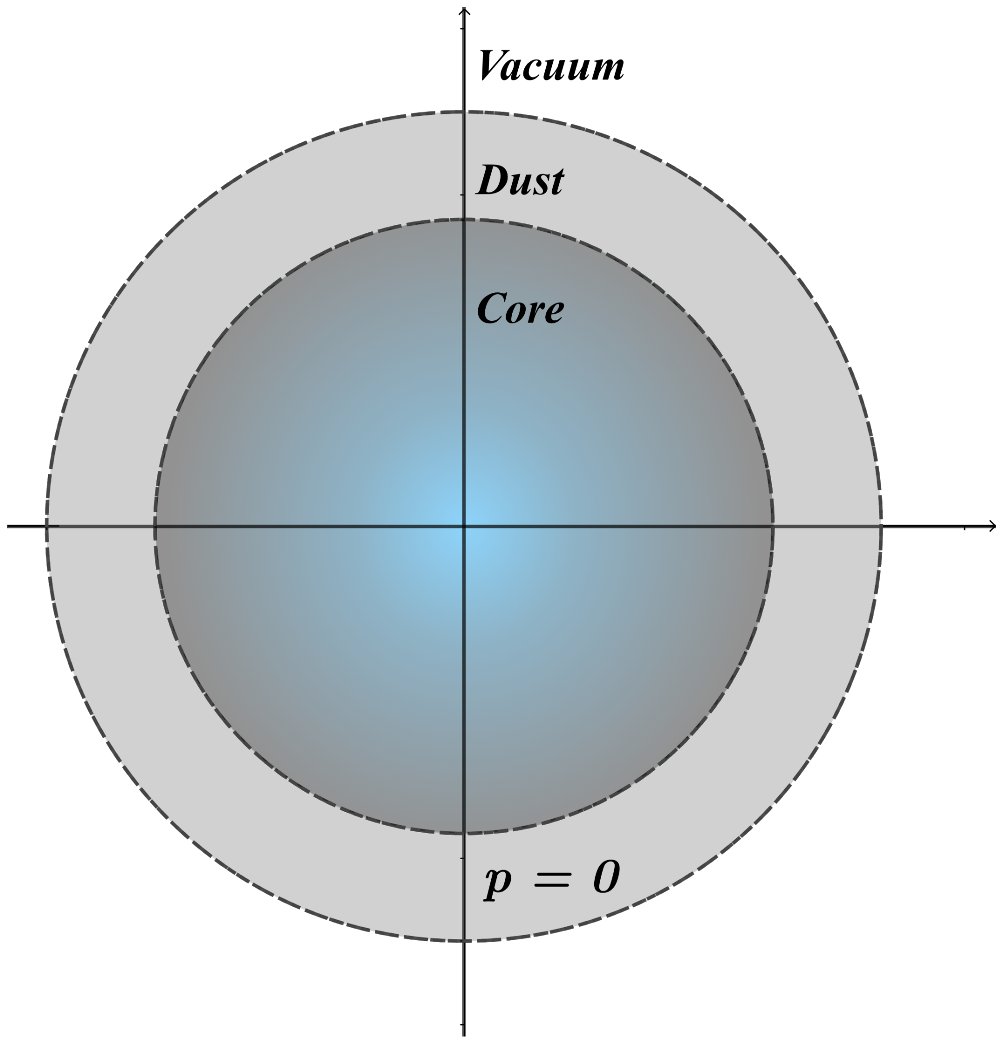
On the other hand,

[Gutiérrez & Quevedo] $\rightarrow$ "The C^3 procedure does not allow to match spherically symmetric perfect fluid spacetimes, whose density and pressure are different from zero on the matching surface conditions"

We also can simplify the problem!

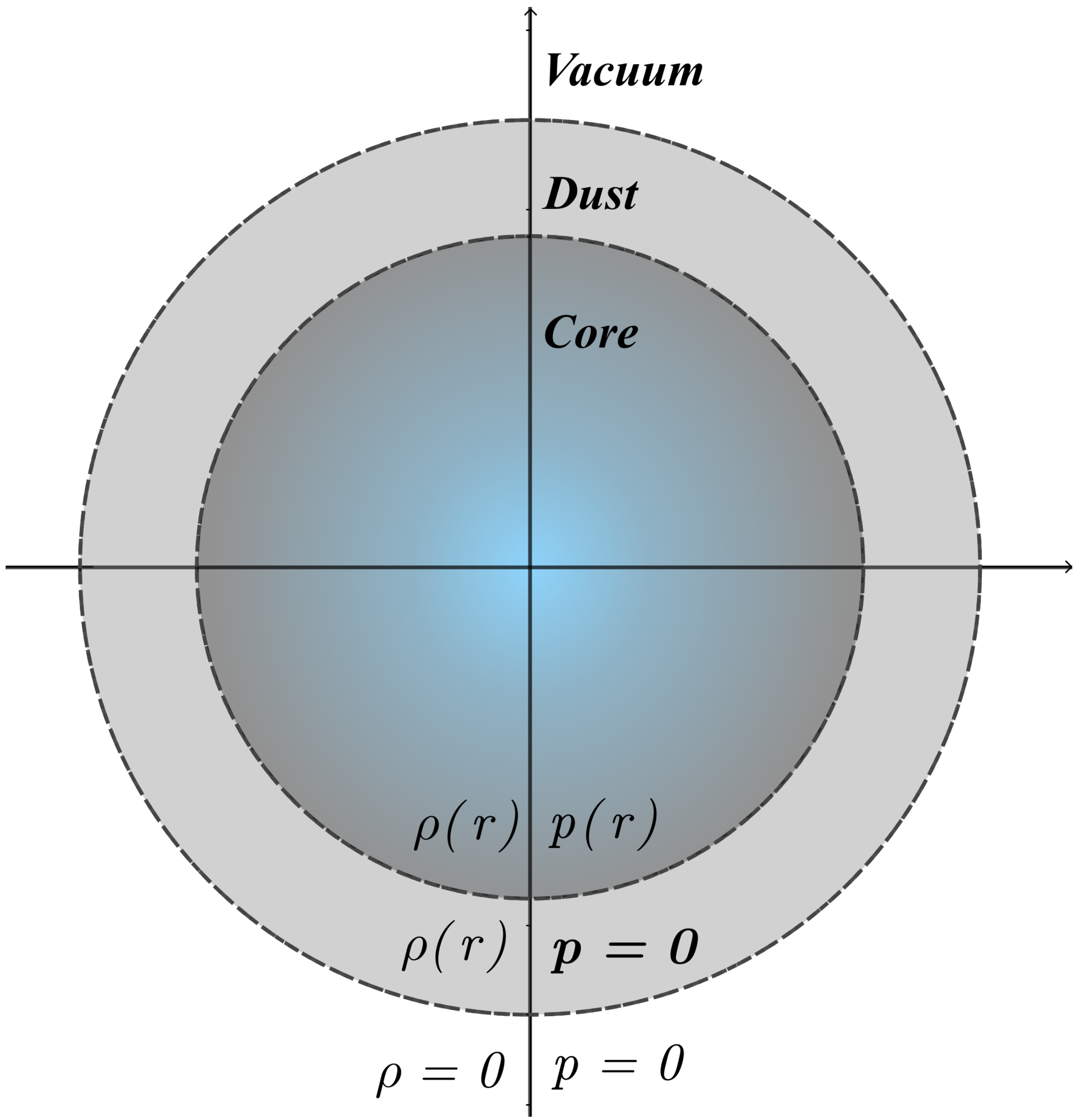
Main Idea

What happens if... $\rho = c_0 - c_2 r^2 - c_4 r^4$



Main Idea

You get a structure like this:



Initial Problem

Initial Problem

The energy stress tensor is given by

$$T_{\mu\nu} = (\rho + p)u_\mu u_\nu + p g_{\mu\nu}$$

A static and spherically symmetric spacetime is

$$ds^2 = -e^{2\phi}dt^2 + e^{2\psi}dr^2 + r^2 d\Omega^2$$

Where,

$$d\Omega^2 = d\theta^2 + \sin^2(\theta)d\phi^2$$
$$u^\mu = e^\phi(\partial_t)^\mu$$

Initial Problem

The Einstein equations lead to,

$$e^{2\psi} = \left(1 - \frac{2m(r)}{r}\right)^{-1} \qquad m(r) = 4\pi \int_0^r \rho(r')r'^2 dr'$$

$$\frac{d\phi}{dr} = \frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))}$$

$$\boxed{\frac{dp}{dr} = -\left(p(r) + \rho(r)\right)\frac{m(r) + 4\pi r^3 p(r)}{r(r - 2m(r))},} \tag{1}$$

Tolman–Oppenheimer–Volkoff (TOV)

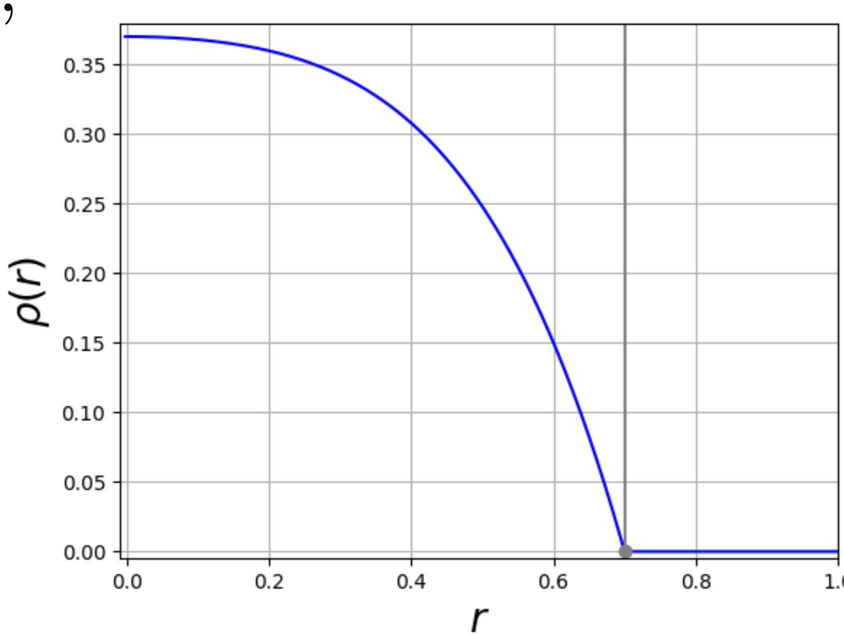
Initial Problem

Let the energy density,

$$\rho(r) = \begin{cases} \rho_c - c_2 r^2 - c_4 r^4, & r < R \\ 0, & r > R \; . \end{cases} \quad (2)$$

And by using the C^3 matching conditions:
$\boxed{\rho(R) = 0,}\; p(R) = 0$ we obtain,

$$(3) \quad \rho_c = c_4 R^4 + c_2 R^2 \; .$$



Initial Problem

So, the metric functions can be written as,

$$m(r) = \begin{cases} 4\pi r^3 \left[\frac{\rho_c}{3} - \frac{c_2}{5} r^2 - \frac{c_4}{7} r^4\right] \ , & r > R \ , \\ M = 4\pi R^3 \left[\frac{\rho_c}{3} - \frac{c_2}{5} R^2 - \frac{c_4}{7} R^4\right] \ , & r > R \ . \end{cases}$$

$$A(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} = \begin{cases} \left(1 - 8\pi r^2 \left[\frac{\rho_c}{3} - \frac{c_2 r^2}{5} - \frac{c_4 r^4}{7}\right]\right)^{-1}, & r < R \\ \left(1 - \frac{2M}{r}\right)^{-1}, & r > R \end{cases}$$

Thermodynamic Information

We assume,

- Local thermodynamic equilibrium.
- Isoentropic fluid.

$$\underline{de} = -pd\frac{1}{\rho_0} = \frac{p}{\rho_0^2}d\underline{\rho_0}\ .$$

Then, we have two new variables, the specific internal energy e, and rest energy density ρ_0.

$$\rho(r) = \rho_0(r)(1 + e(r))\ .$$

$$\frac{d\rho_0}{dr} = \frac{\rho_0}{[p(r)+\rho(r)]}\frac{d\rho}{dr}\ , \qquad (4)$$

Physical Conditions

1. By imposing the Buchdahl limit we ensure that the metric function $A(r)$ does not diverge, i.e.,

$$\frac{2m(r)}{r} < \frac{8}{9} \qquad \Longrightarrow \qquad A(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} < 9 \tag{5}$$

2. The speed of sound at the center must be lower than 1 (in GU);

$$v_s^2\Big|_{r=0} = \frac{dP}{d\rho}\Big|_S = \frac{2\pi}{C_2}(p_c + \rho_c)\left(p_c + \frac{\rho_c}{3}\right) < 1 \tag{6}$$

3. Specific internal energy must be positive

Numerical Solutions

Numerical Solutions

Let's look at two cases,

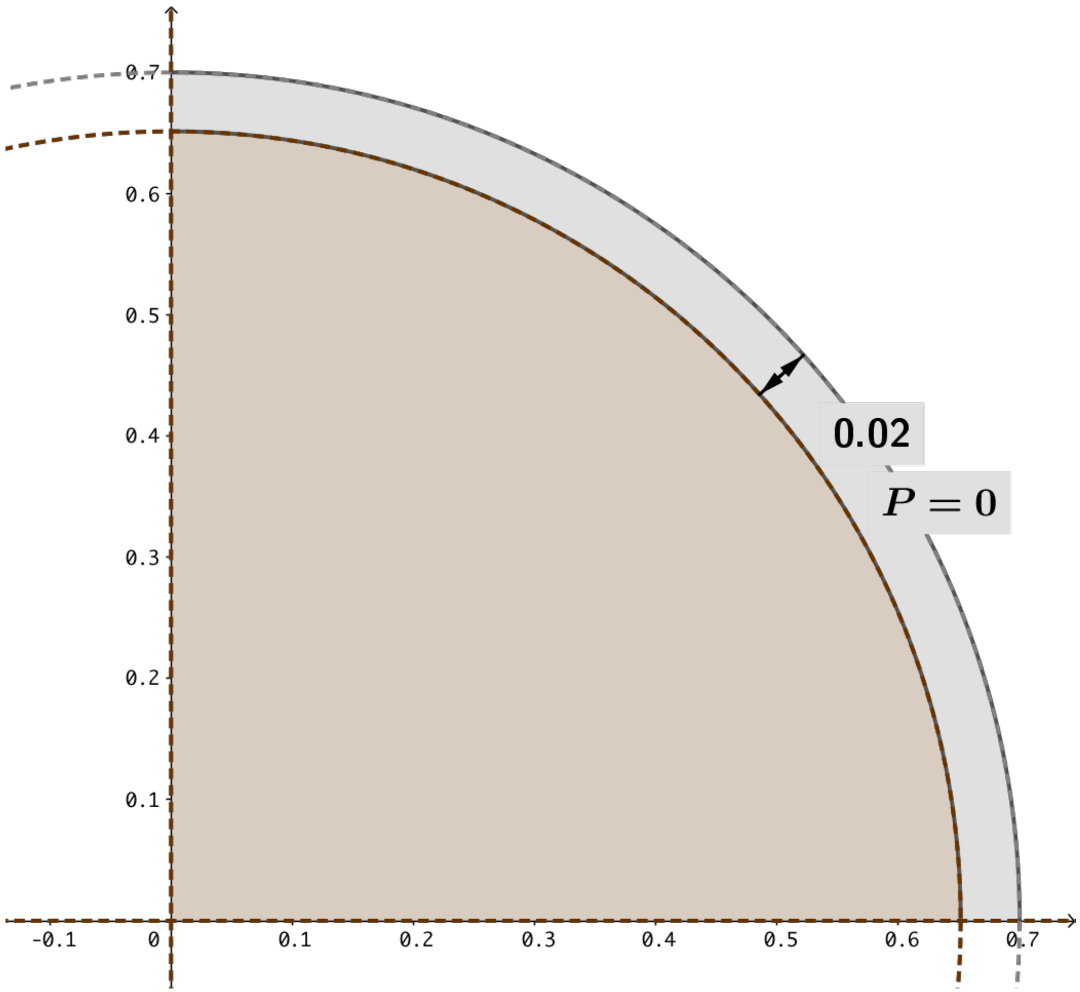
Parameter	Case 1	Case 2
Central rest energy density ρ_c	0.08804	0.14244
Central internal energy e_c	0.05	0.11
Central pressure P_c	0.01	0.02
C_2 constant	0.04179	0.08259
C_4 constant	0.083	0.001

Table: Parameters used in the module idata_TOVstar2 to solve the initial conditions. Case 1 has an initial sound speed of $v_s = 0.76$ and a total radius $R_t = 0.9$, while case 2, $v_s = 0.92$ and $R_t = 1.3$.

Numerical Solutions

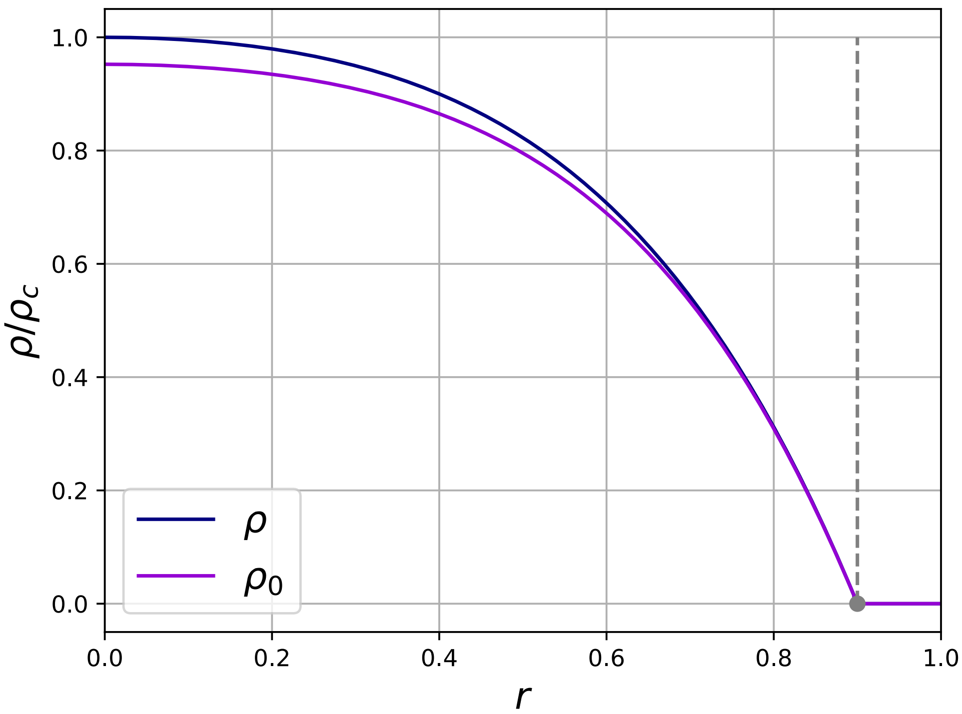
The pressure cancels out before the energy density does. Therefore, there will be a core and a layer of dust.

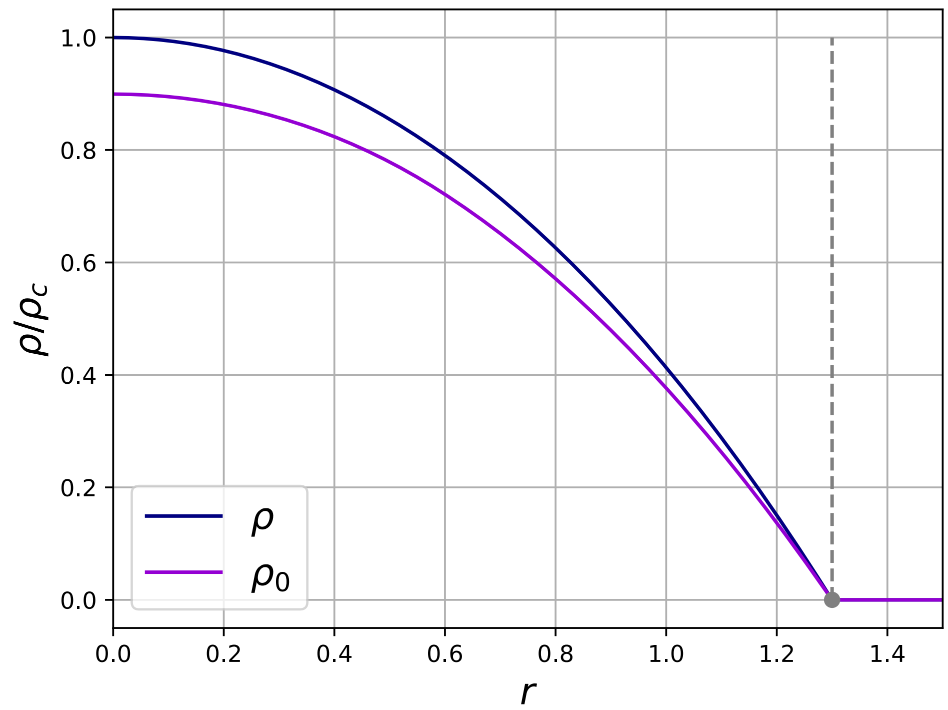
Since there is no force to maintain the dust layer, it must fall in the evolution part.



Numerical Solutions

Energy density profiles,



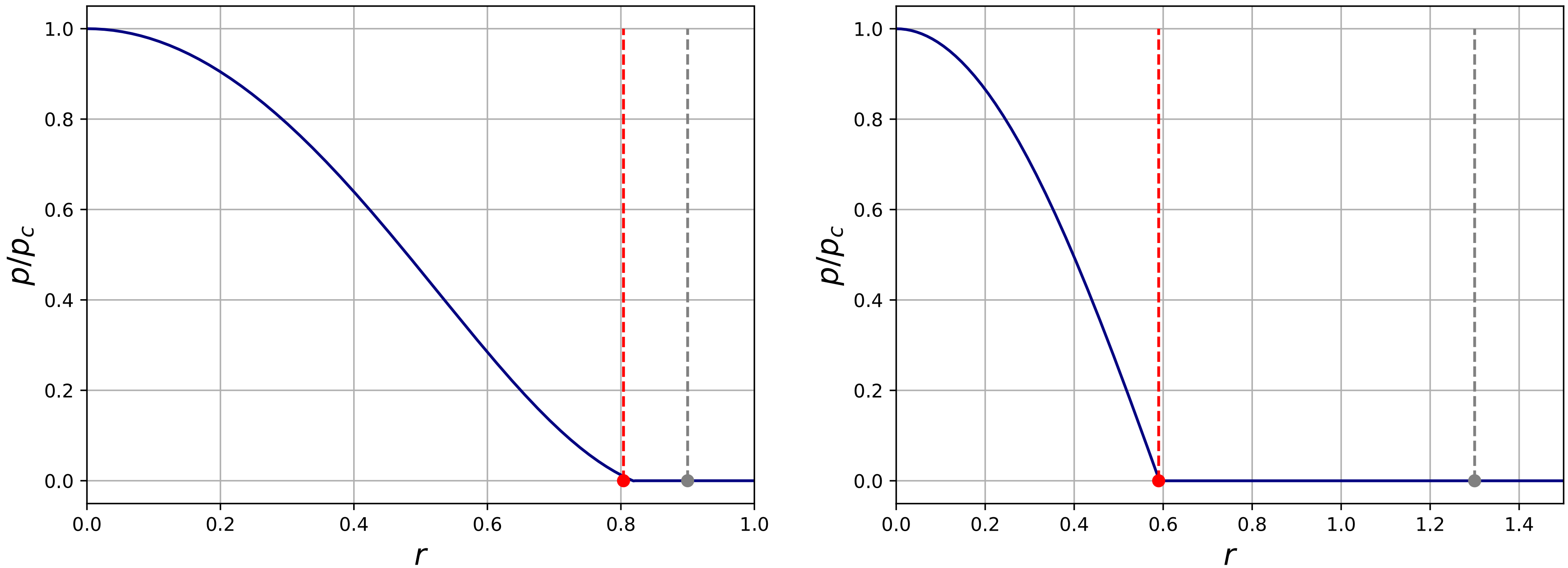


In gray the radius of the object.

Both cases showing decreasing, continuous and finite functions

Numerical Solutions

Pressure profile

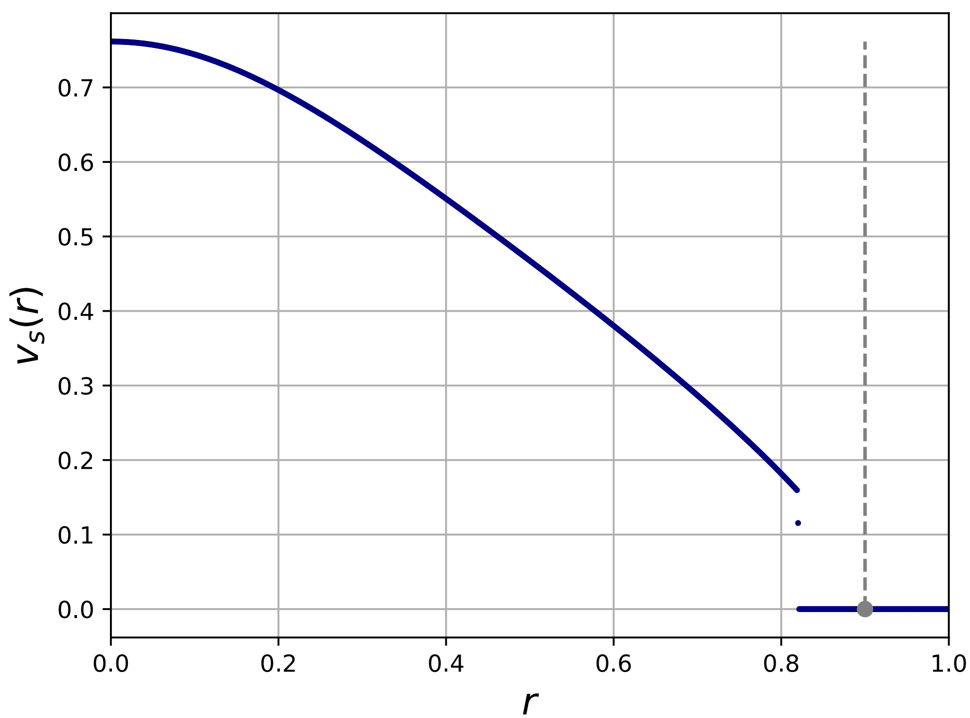


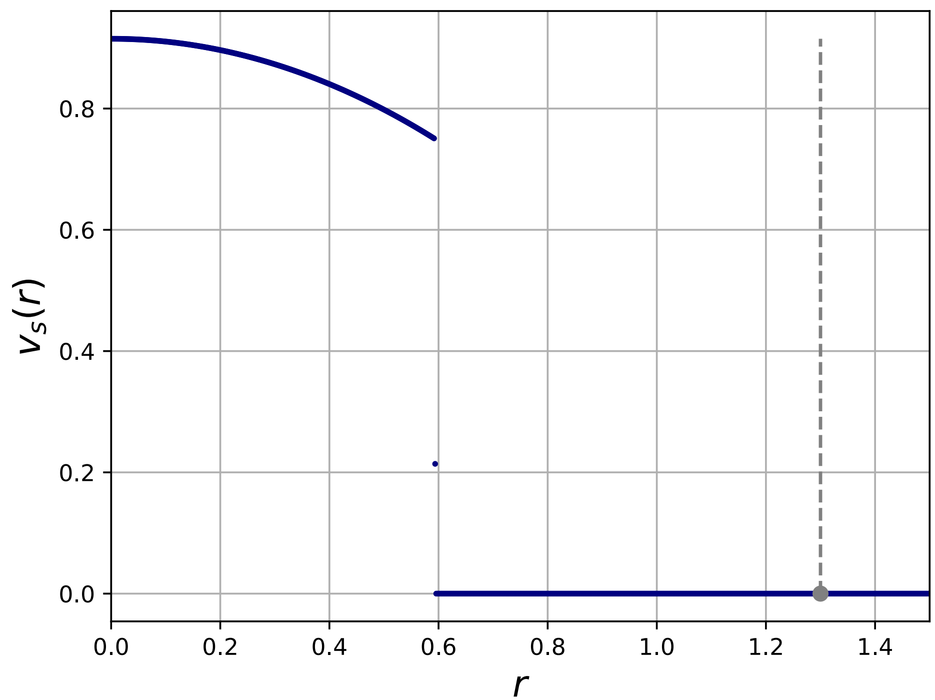
In red the radius of the nucleus.

At this point we see that the pressure cancels out before the density does.

Numerical Solutions

Speed of sound

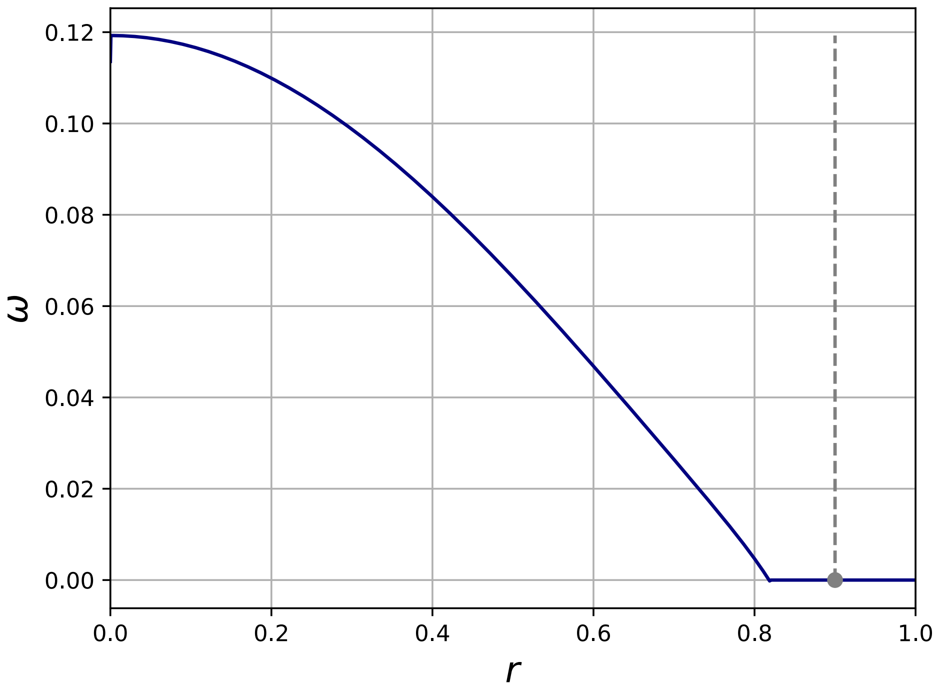


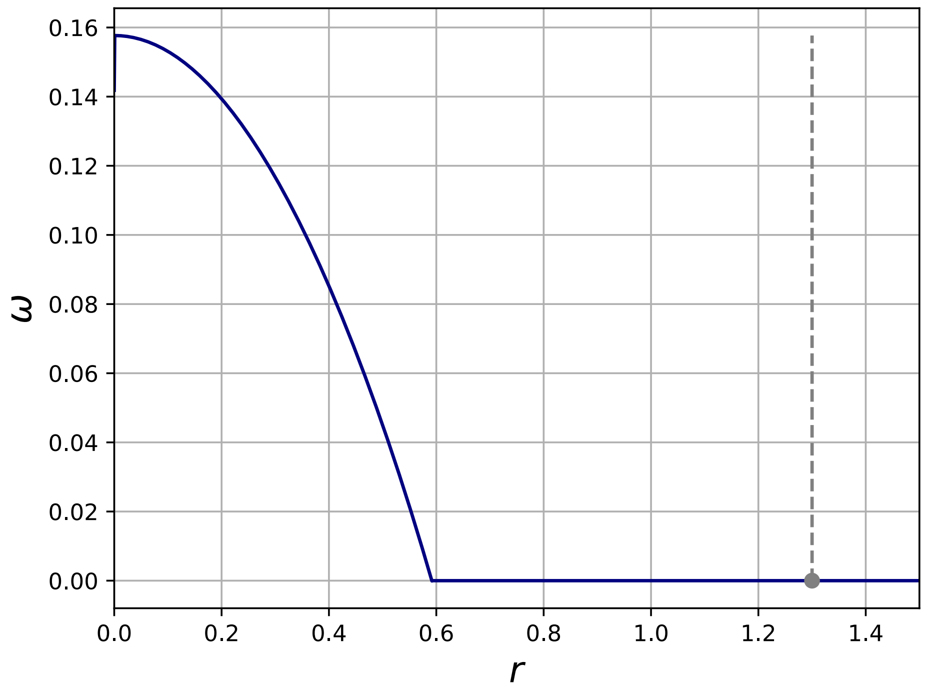


In both cases less than one

Numerical Solutions

The barotropic *function* : $\omega(r) = p(r)/\rho(r)$

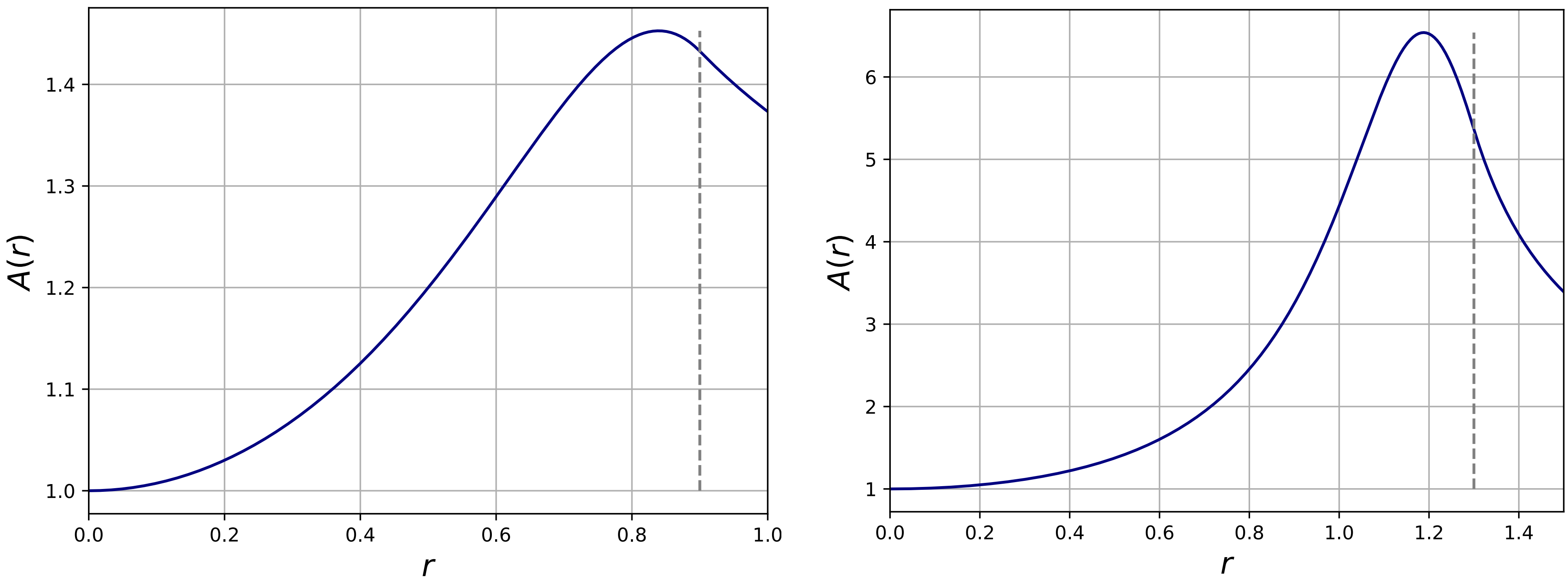




Numerical Solutions

$$A(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} < 9$$

Metric function $A(r)$



In both cases less than 9 as expected.

Neutron Stars

Canonical Neutron Stars

Let's take a canonical neutron star, i.e. a convention for the typical properties of NS.

$$R = 1.2\ km$$

$$\bar{\rho} \approx 10^{17}\ kg/m^3$$

And let's also take the atom's nucleus pressure,

$$p \approx 10^{33}\ N/m^2$$

Canonical Neutron Stars

Let's take a canonical neutron star, i.e. a convention for the typical properties of NS.

$$R = 1.2\ km$$

$$\rho = 2.8 \times 10^{-4}\ km^{-2}$$

And let's also take the atom's nucleus pressure,

$$p = 4.33 \times 10^{-5}\ km^{-2}$$

How many c_2 and c_4 values satisfy the physical conditions?

Canonical Neutron Stars

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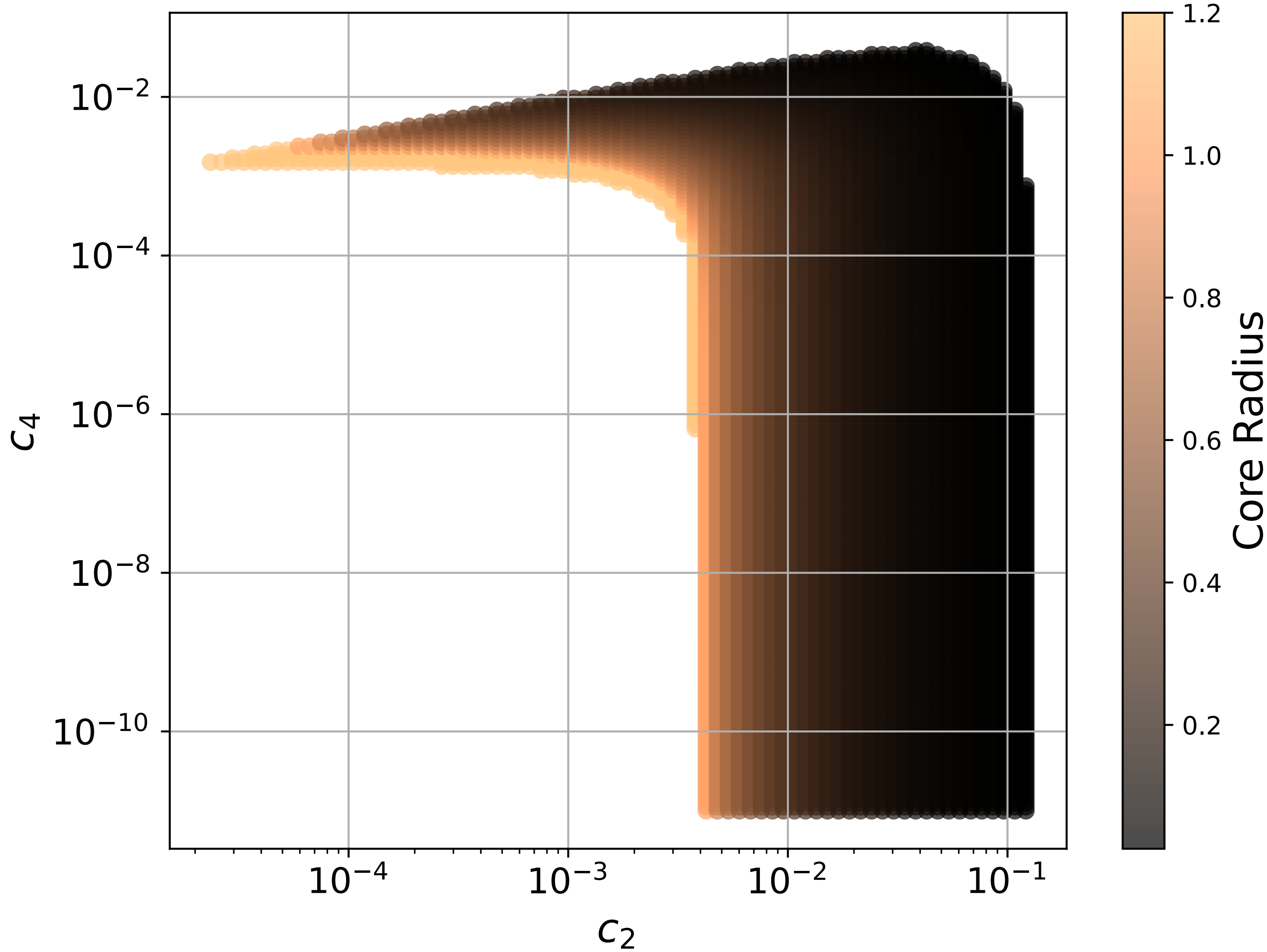

Canonical Neutron Stars

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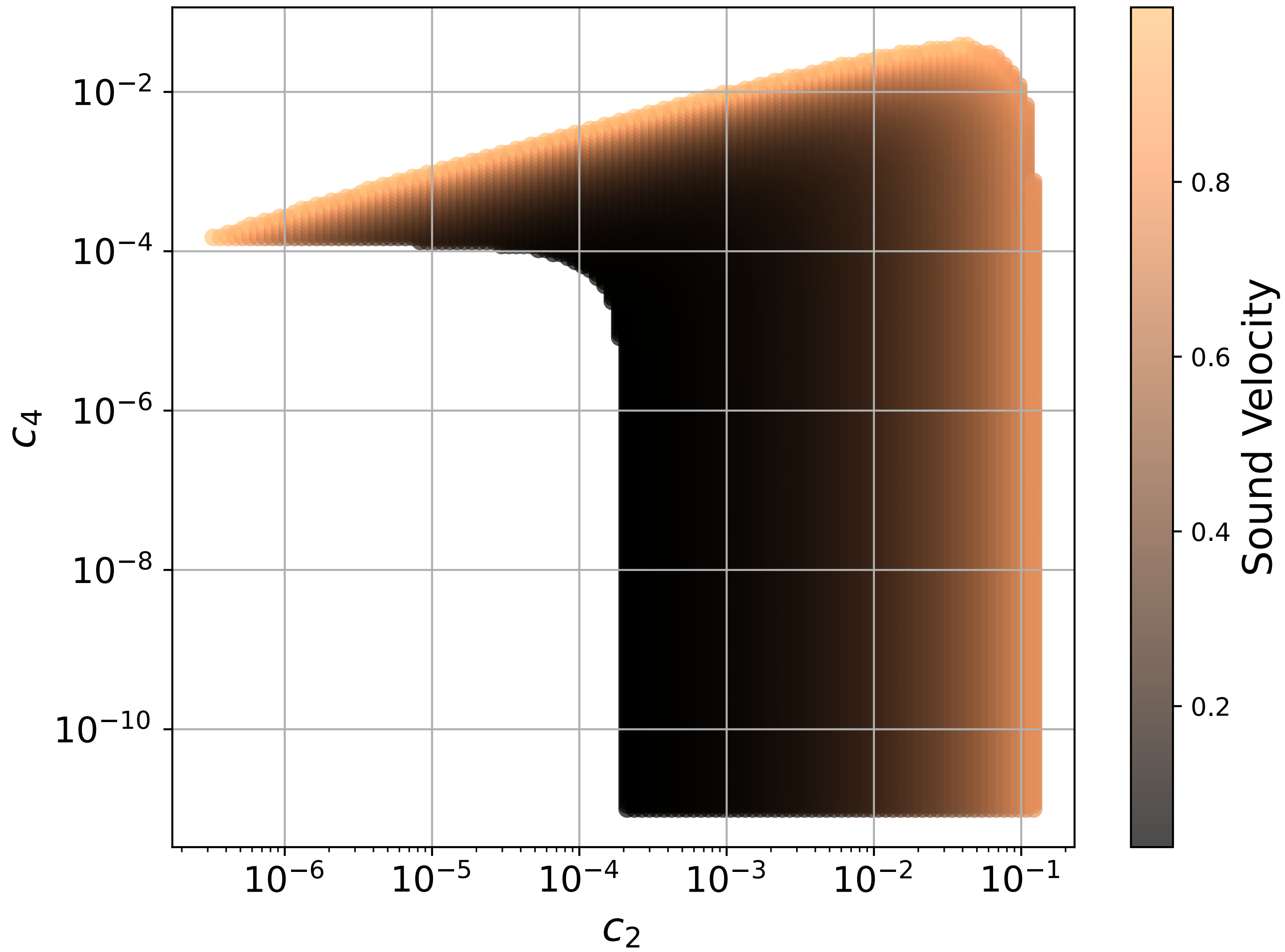


Canonical Neutron Stars

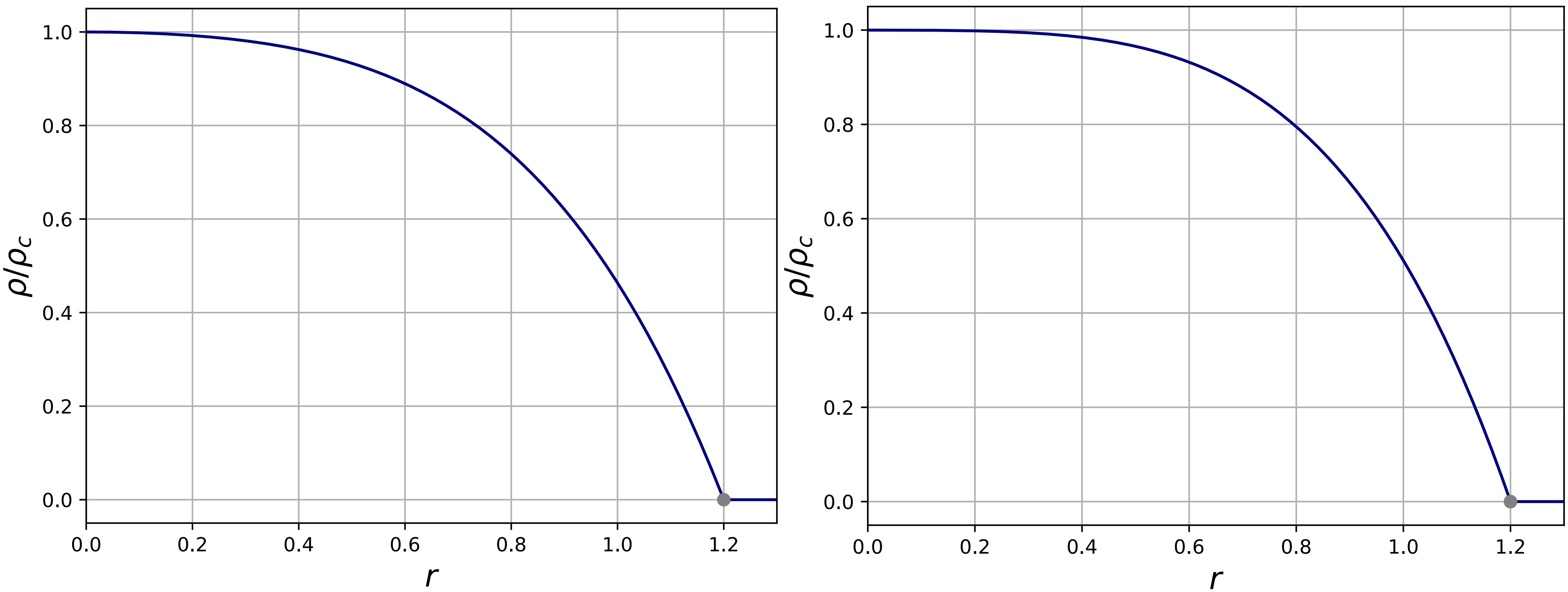
Let's look at two new cases,

Parameter	**Case 1**	**Case 2**
c_2 constant	0.00091	0.00017
c_4 constant	0.00184	0.00379

TABLE II: Parameters used to solve the initial conditions with total radius $R = 1.2$ $[km]$. Case 1 has an initial sound speed of $v_s = 0.76$ and case 2 $v_s = 0.91$. Both cases has a central rest energy density of $\rho_c = 2.87 \times 10^{-4}$ and a central pressure $p_c = 4.33 \times 10^{-5}$

Canonical Neutron Stars

Energy density profiles,

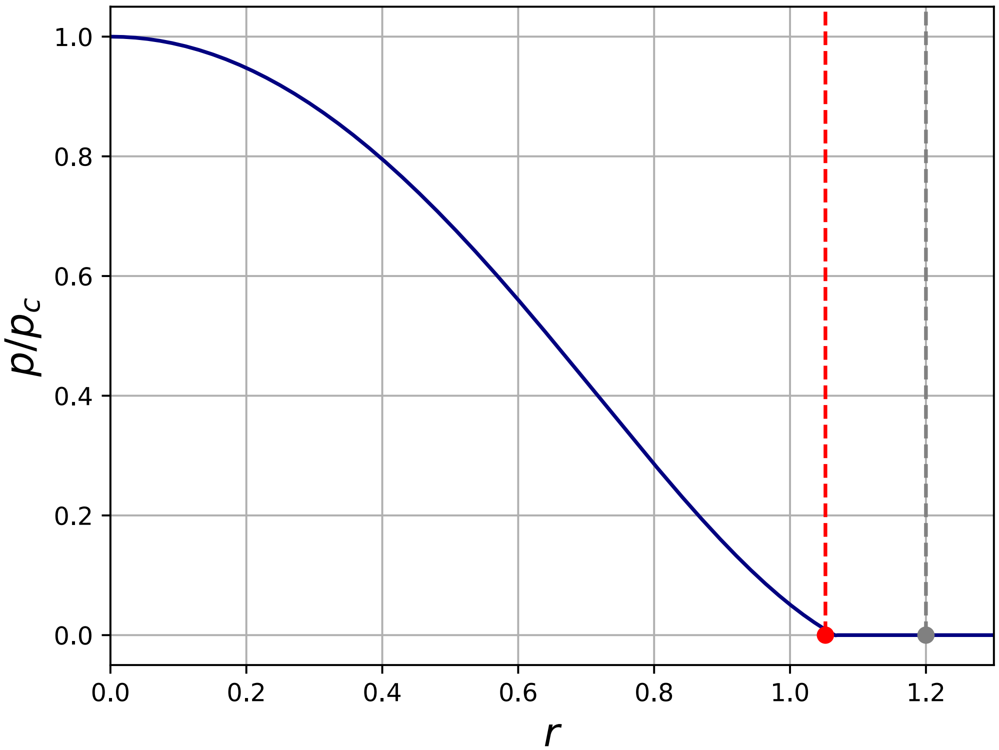


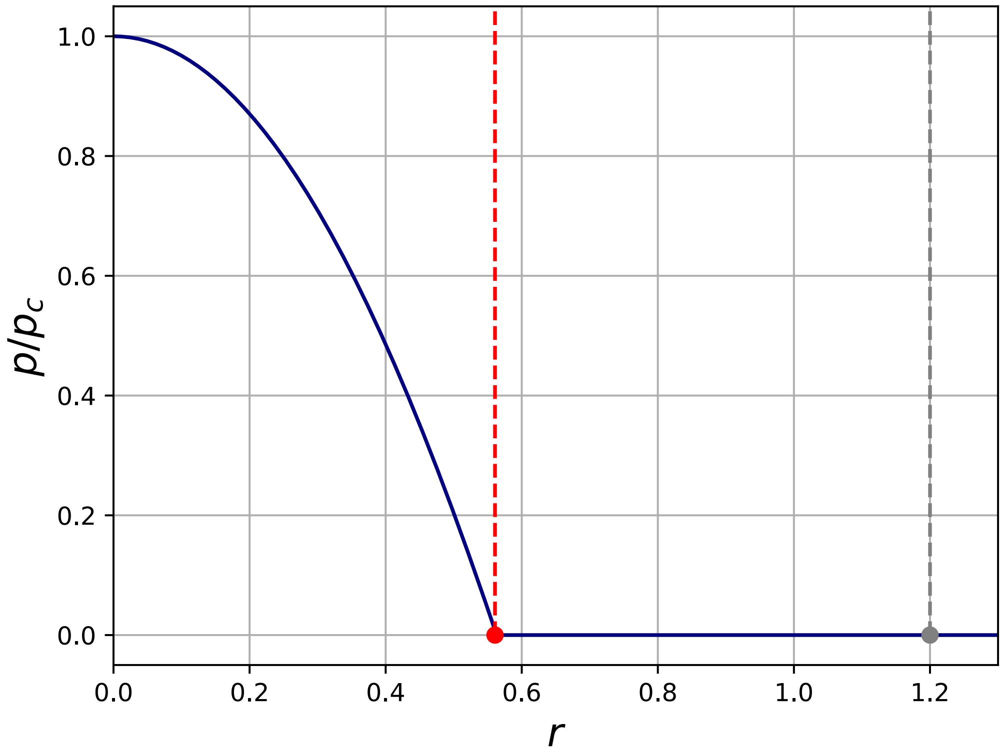
In gray the radius of the object.

Both cases showing decreasing, continuous and finite functions

Canonical Neutron Stars

Pressure profile



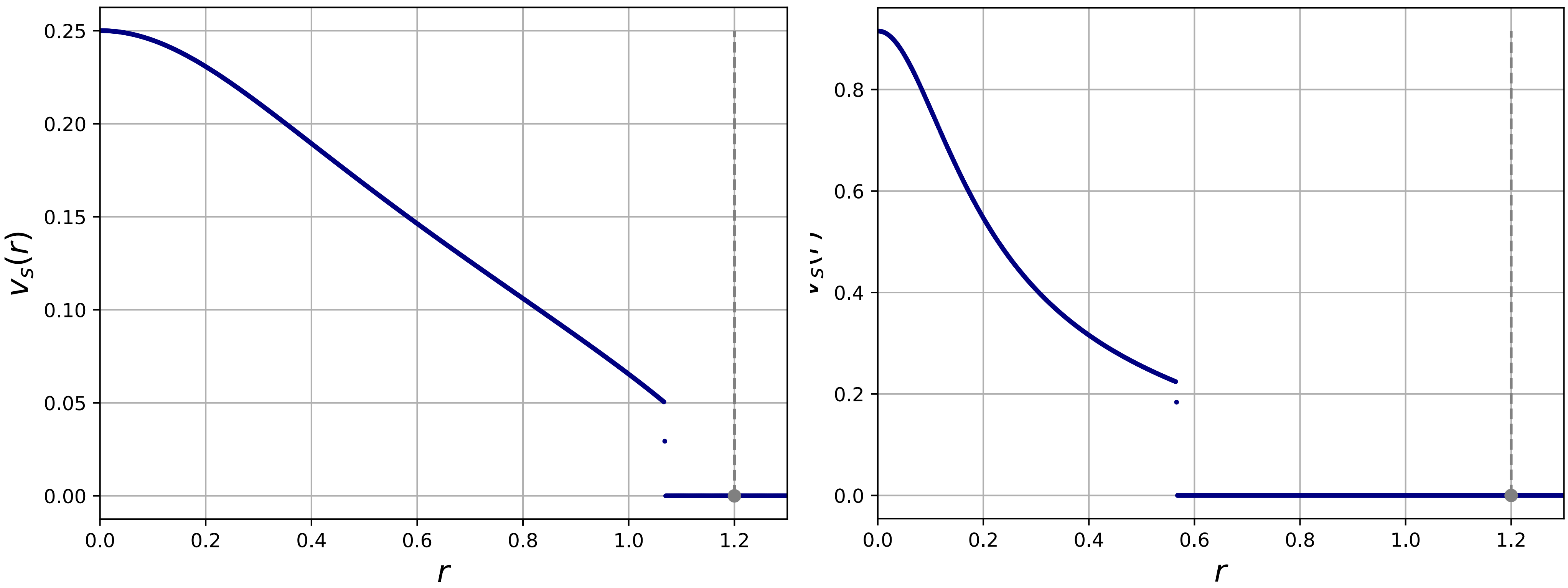


In red the radius of the nucleus.

At this point we see that the pressure cancels out before the density does.

Canonical Neutron Stars

Speed of sound

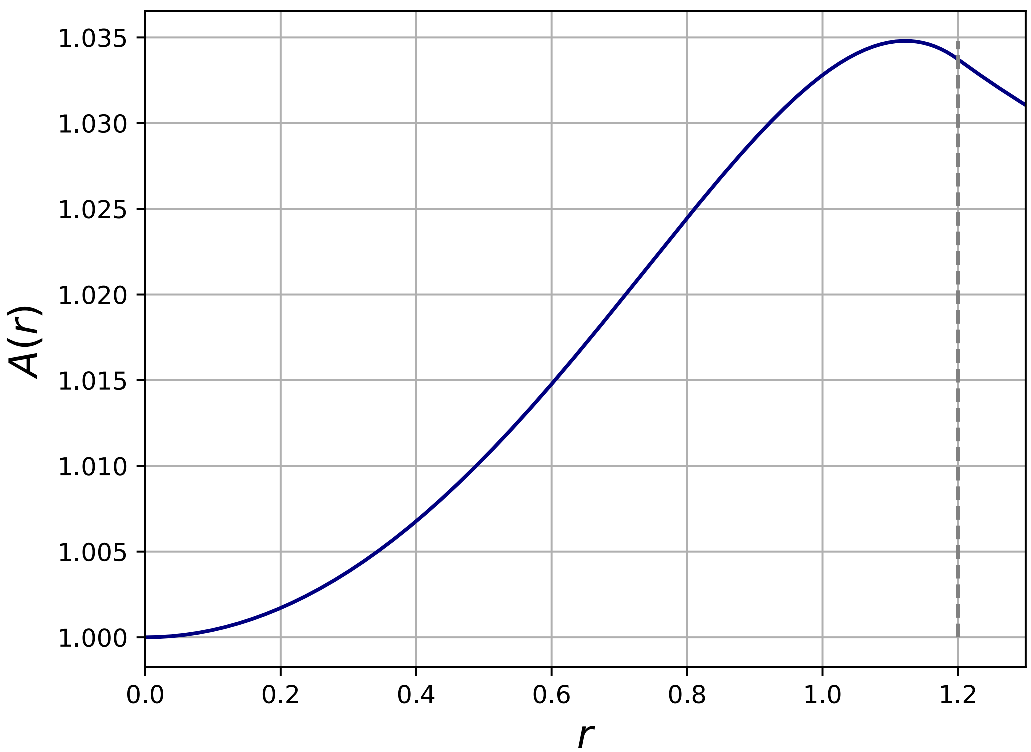


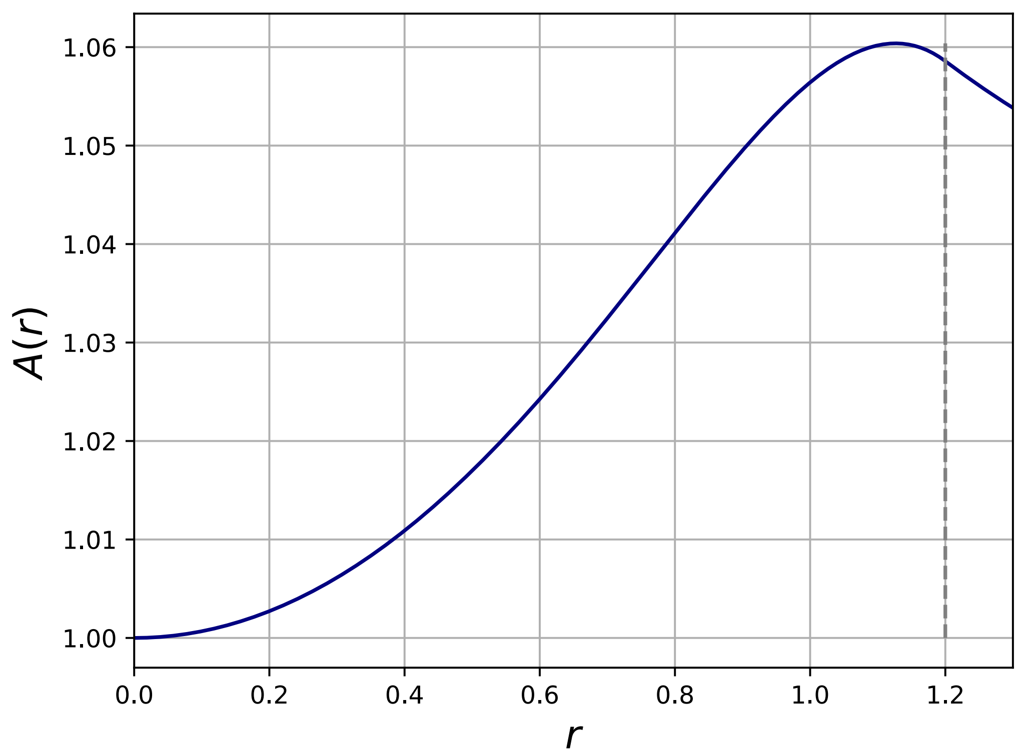
In both cases less than one

Canonical Neutron Stars

Metric function $A(r)$

$$A(r) = \left(1 - \frac{2m(r)}{r}\right)^{-1} < 9$$

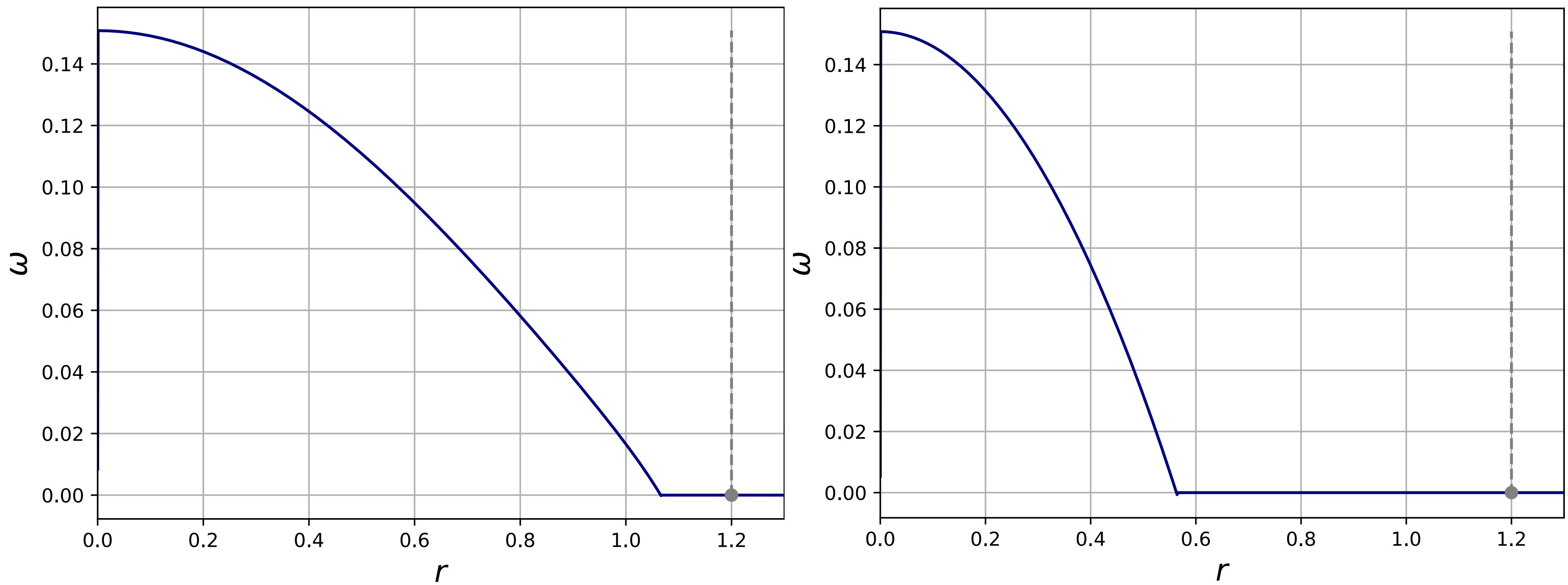




In both cases less than 9 as expected.

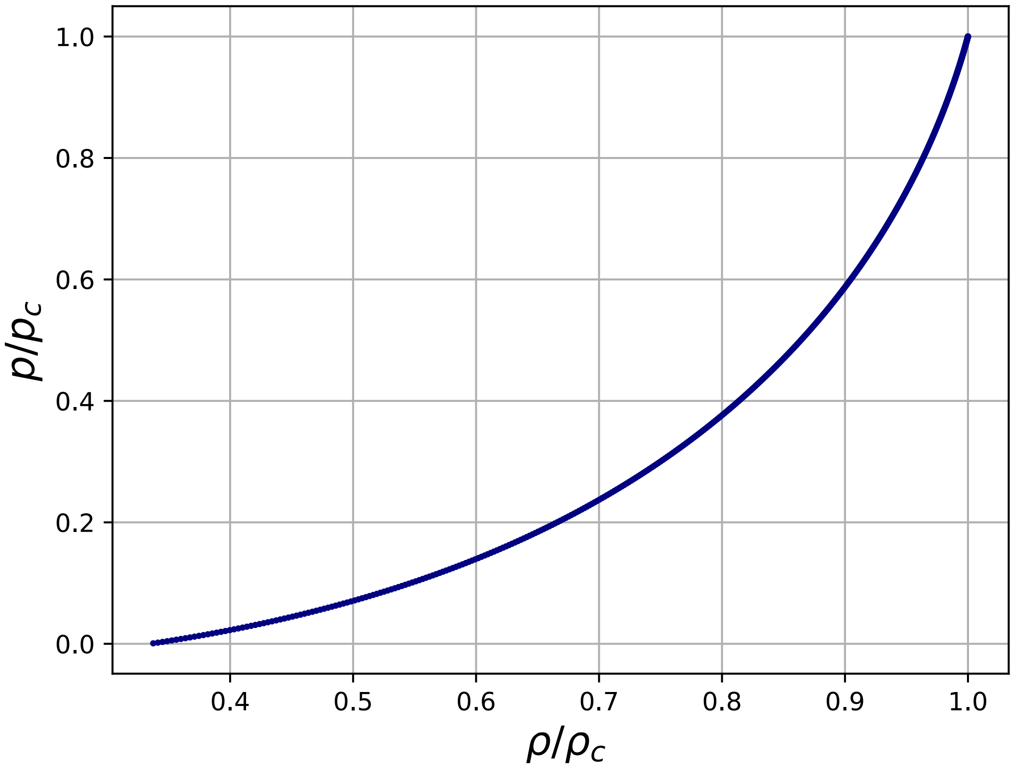
Canonical Neutron Stars

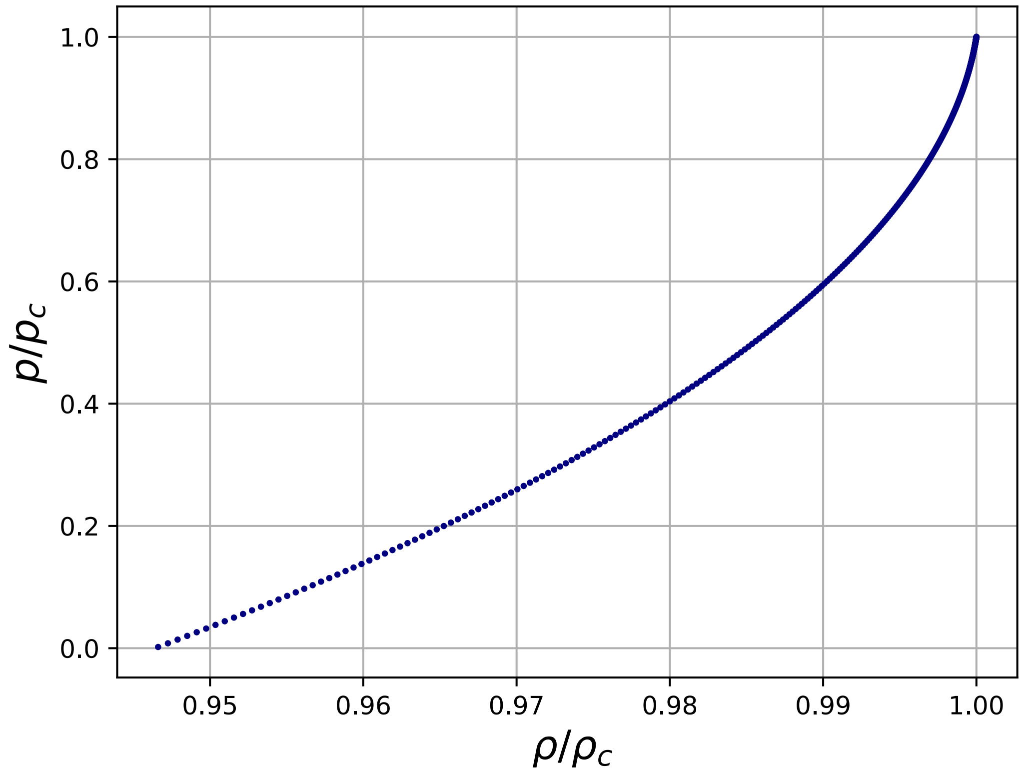
Barotropic *function* $\omega(r) = p(r)/\rho(r)$



Canonical Neutron Stars

EoS of the core $p(\rho)$

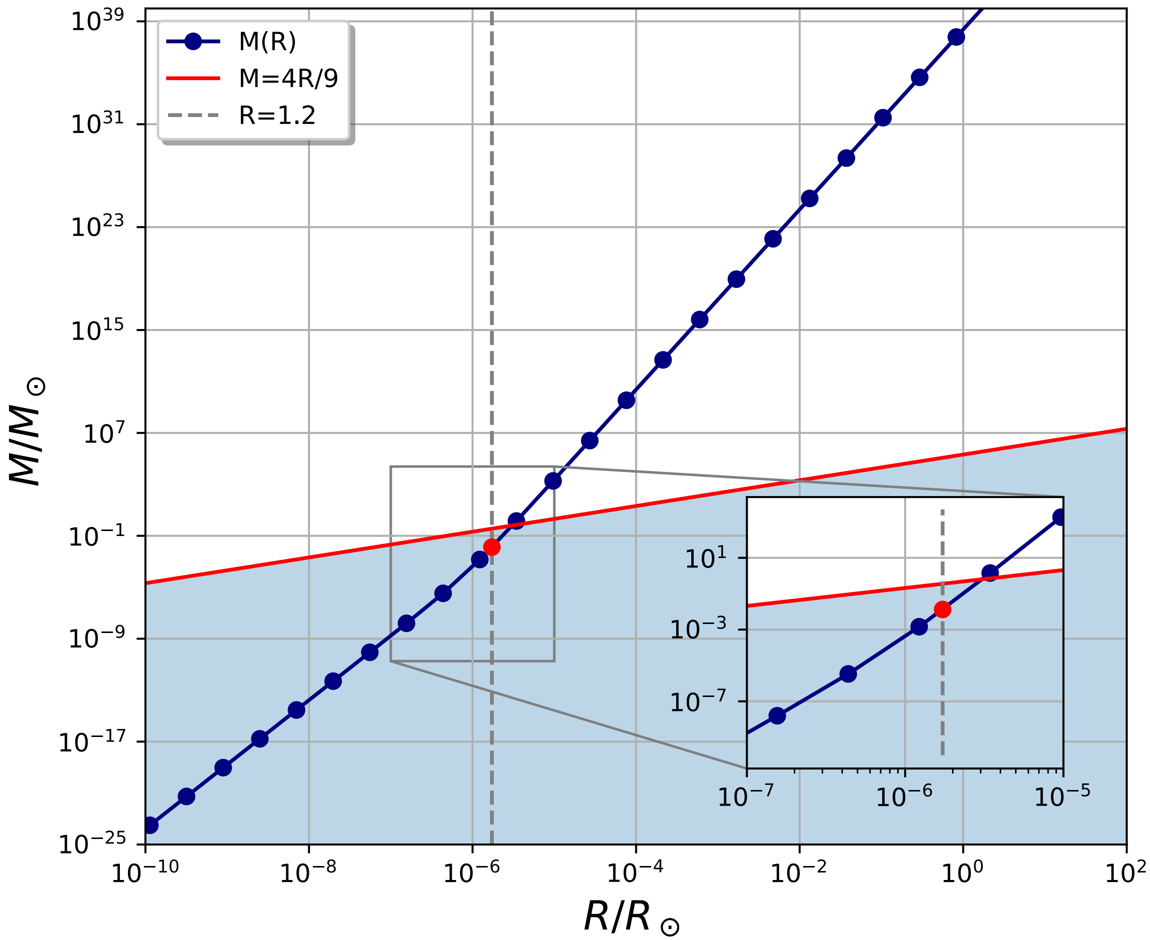




Canonical Neutron Stars

M-R Relation

$$M = 4\pi R^3 \left[\frac{\rho_c}{3} - \frac{c_2}{5}R^2 - \frac{c_4}{7}R^4\right]$$

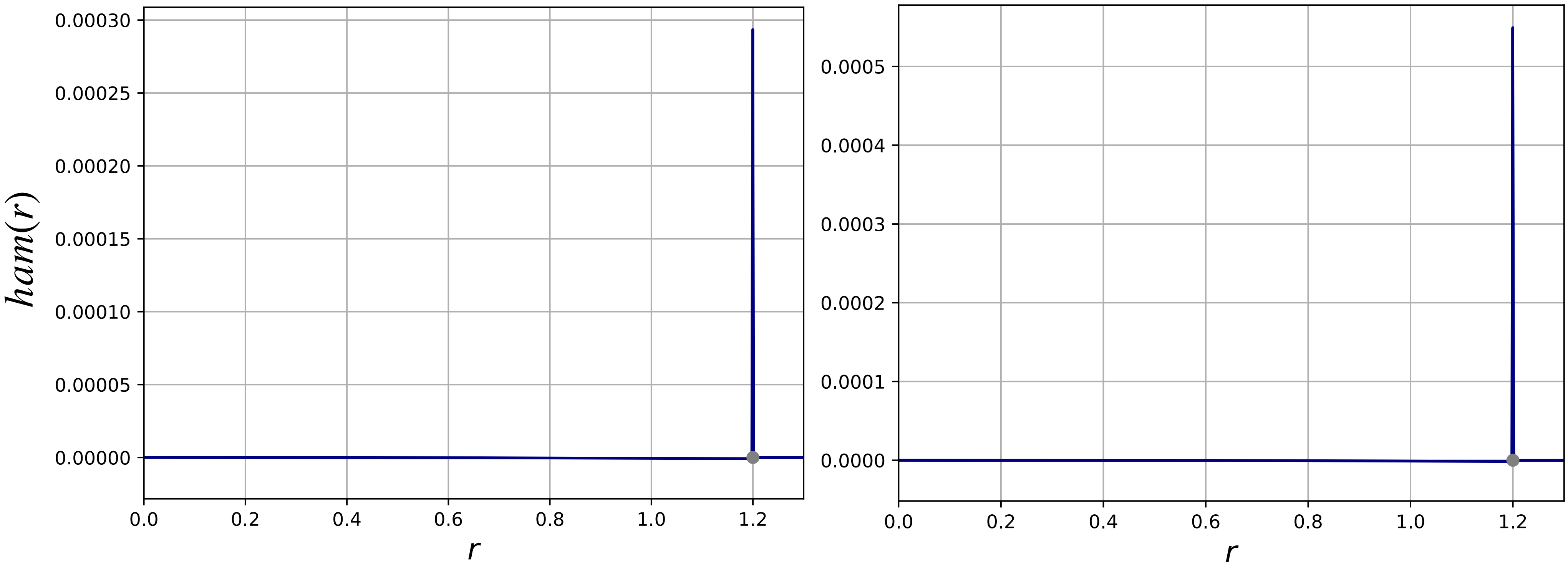


In red the Buchdahl limit.

Canonical Neutron Stars

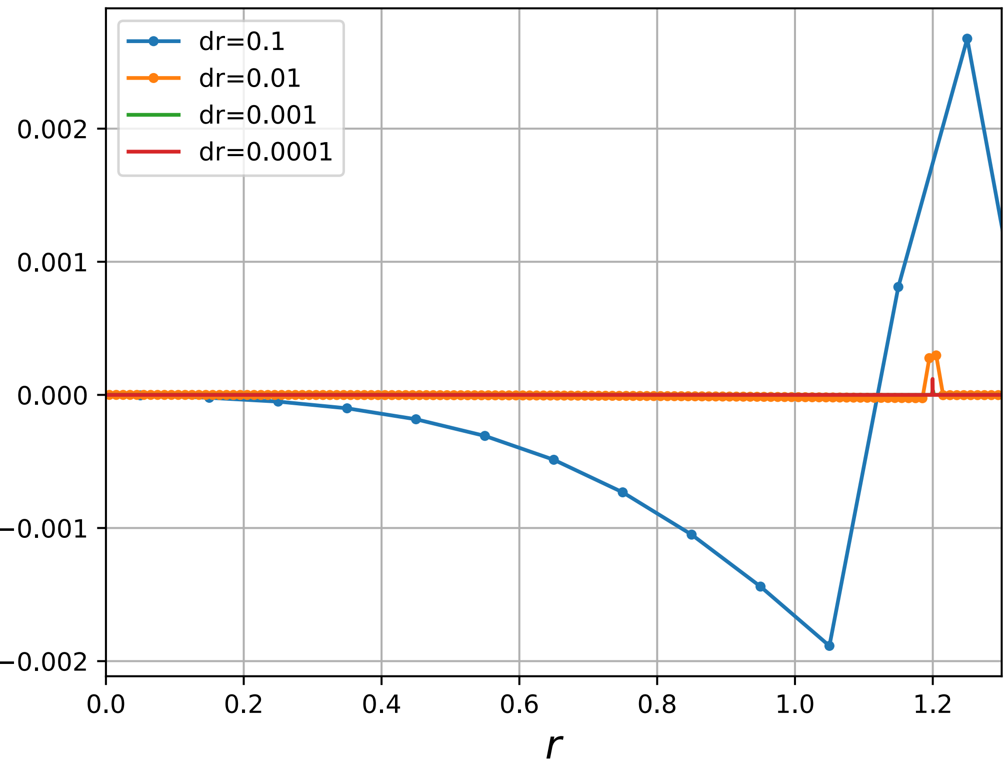
$$\frac{dA(r)}{dr} - A(r)\left[\frac{1-A(r)}{r} + 8\pi r\rho(r)A(r)\right] = 0$$

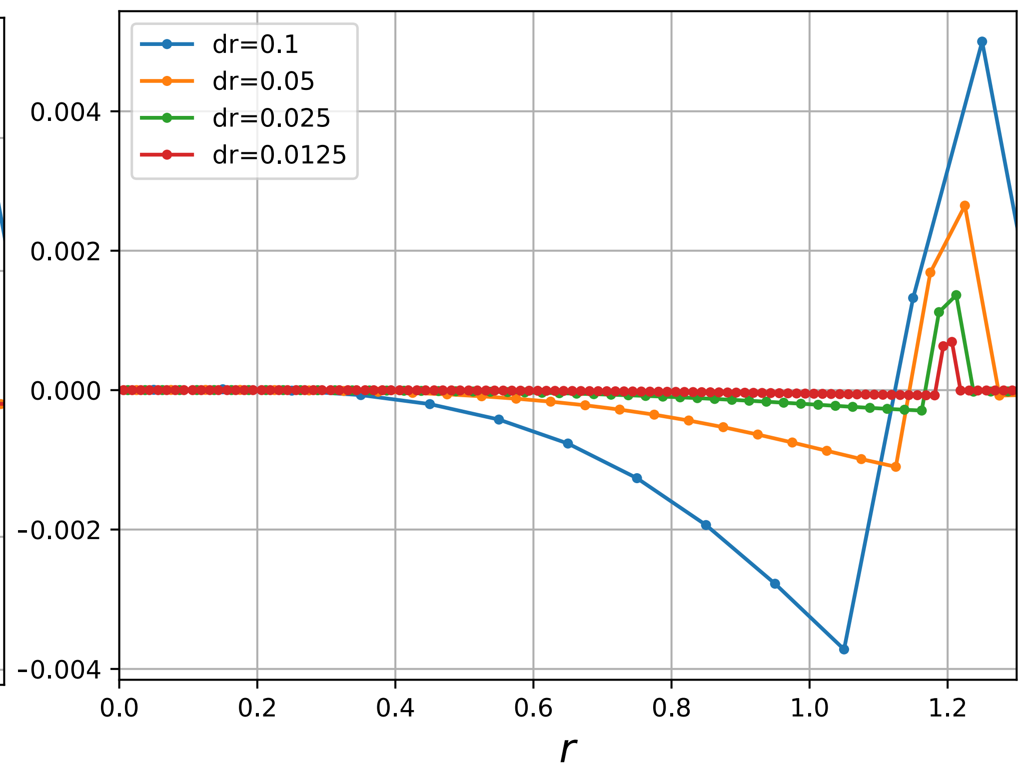
Hamiltonian constriction



Canonical Neutron Stars

Hamiltonian constriction $ham(r)$ for different mesh widths





Evolution

Evolution

For evolution we need an equation of state compatible with the initial conditions;

$$p(r,t) = \omega(r)\rho(r,t)$$

We fixed $\omega(r) = p(r,0)/\rho(r,0)$ at the value in the initial conditions with the pressure and density.

$$p(r,t) = p(\rho(r,t))$$

We could extend the relationship between pressure and density for density values greater than those found in the initial data.

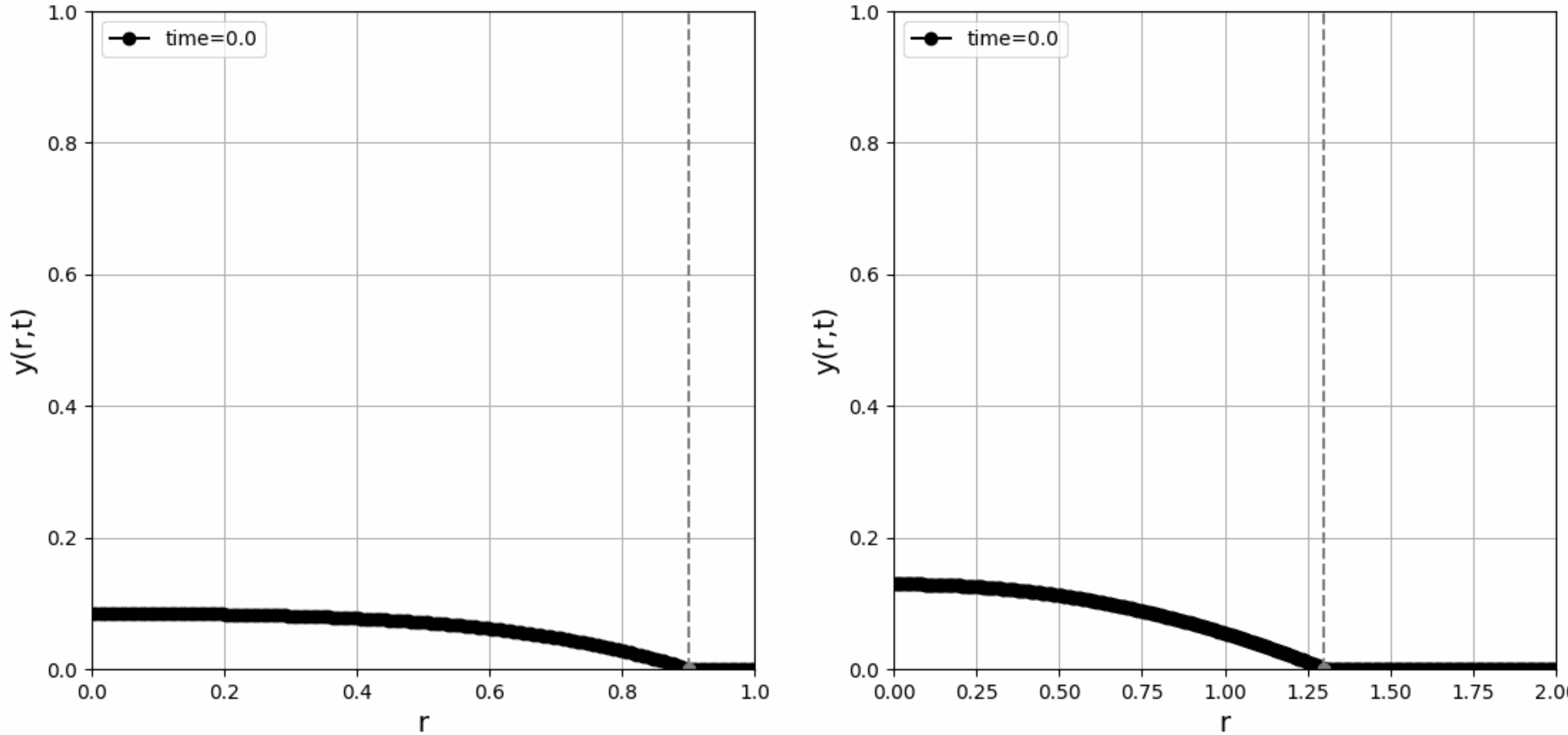
We use the *OllinSphere* code, developed by Dr. Alcubierre and his research team. They use,

$$dl^2 = e^{4\phi(r,t)}\left(A(r,t)dr^2 + r^2 B(r,t)d\Omega^2\right)$$

$$ds^2 = (-\alpha^2 + \beta_r\beta^r)dt^2 + 2\beta_r dtdr + dl^2$$

Evolution (Arbitrary Cases)

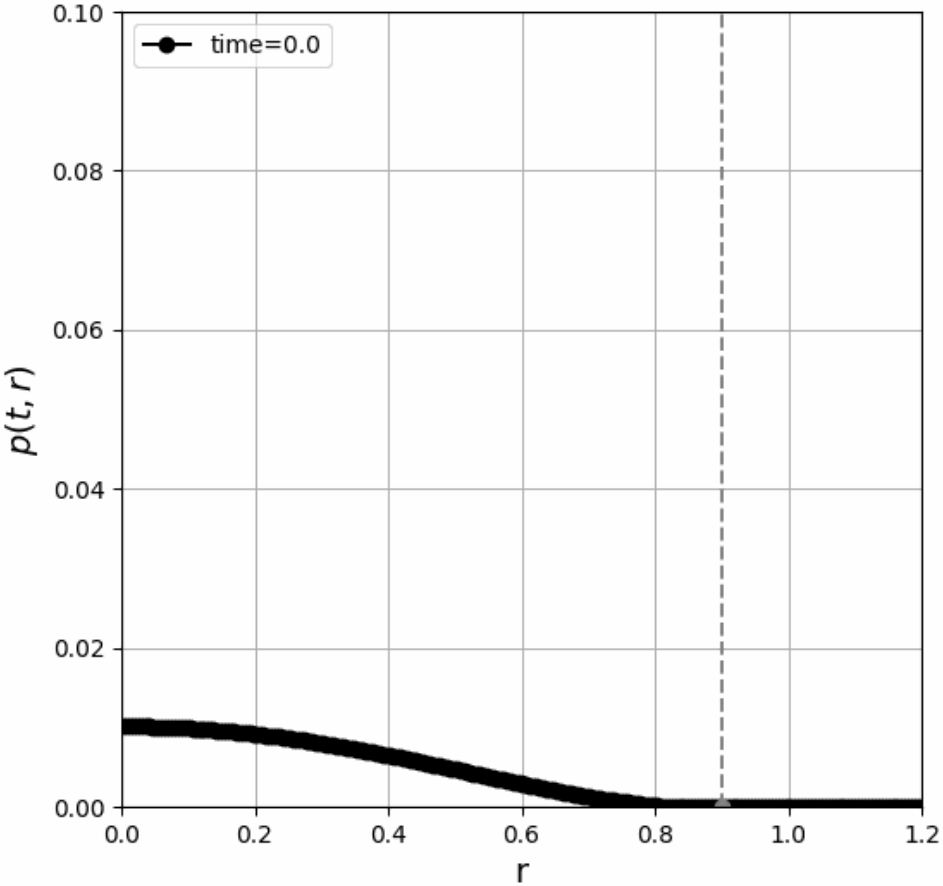
Density Evolution;

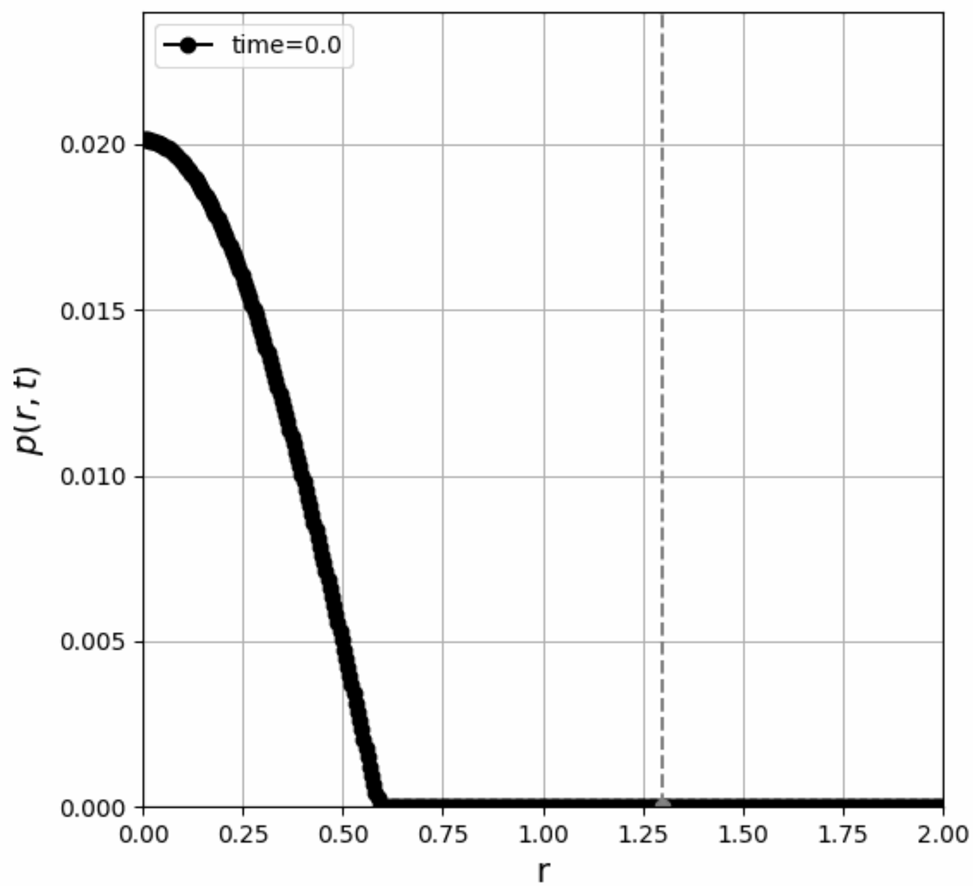


The dust layer falls into the core!

Evolution (Arbitrary Cases)

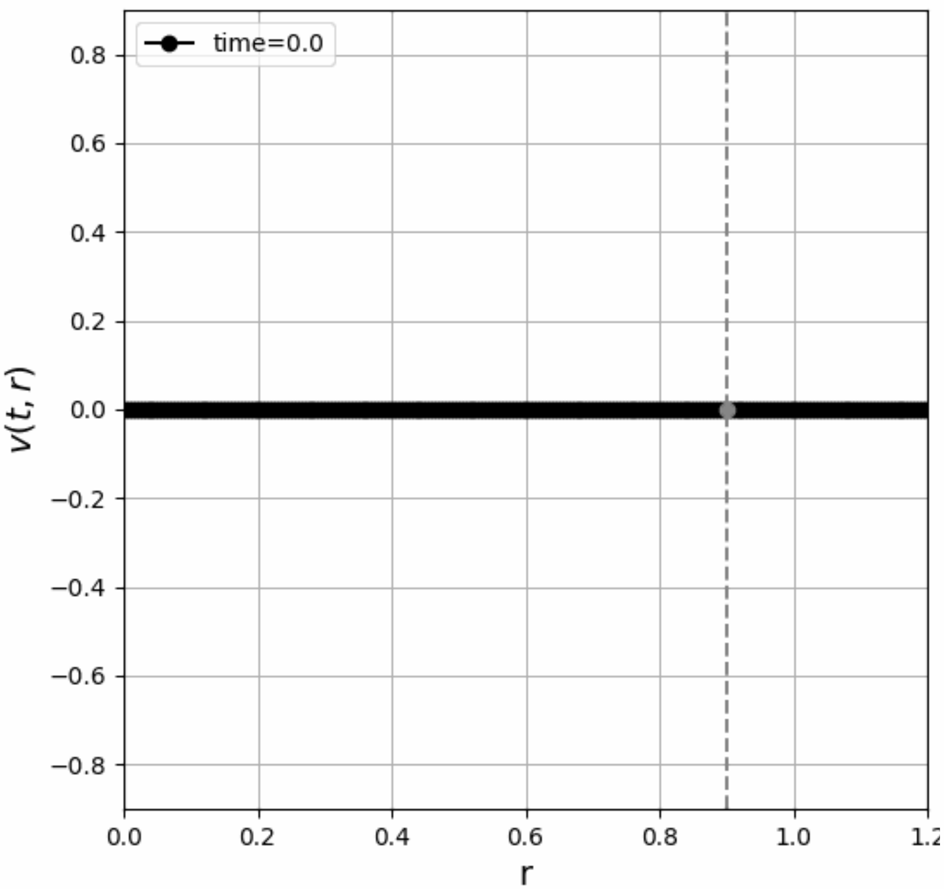
Pressure Evolution;

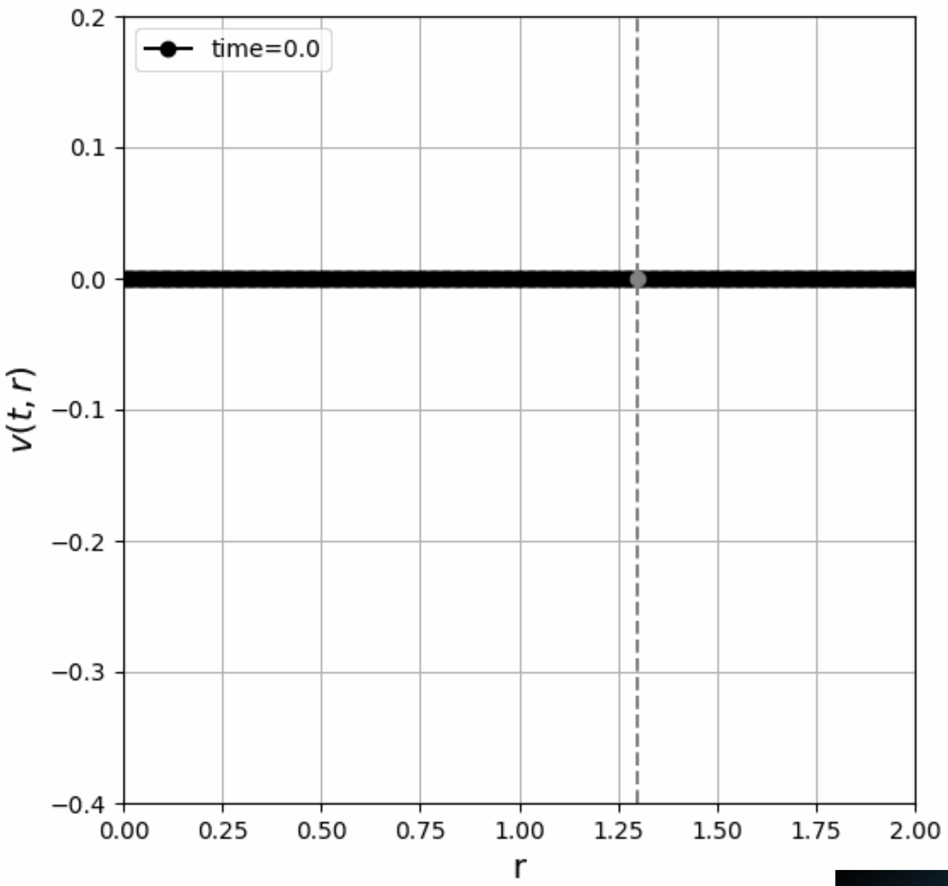




Evolution (Arbitrary Cases)

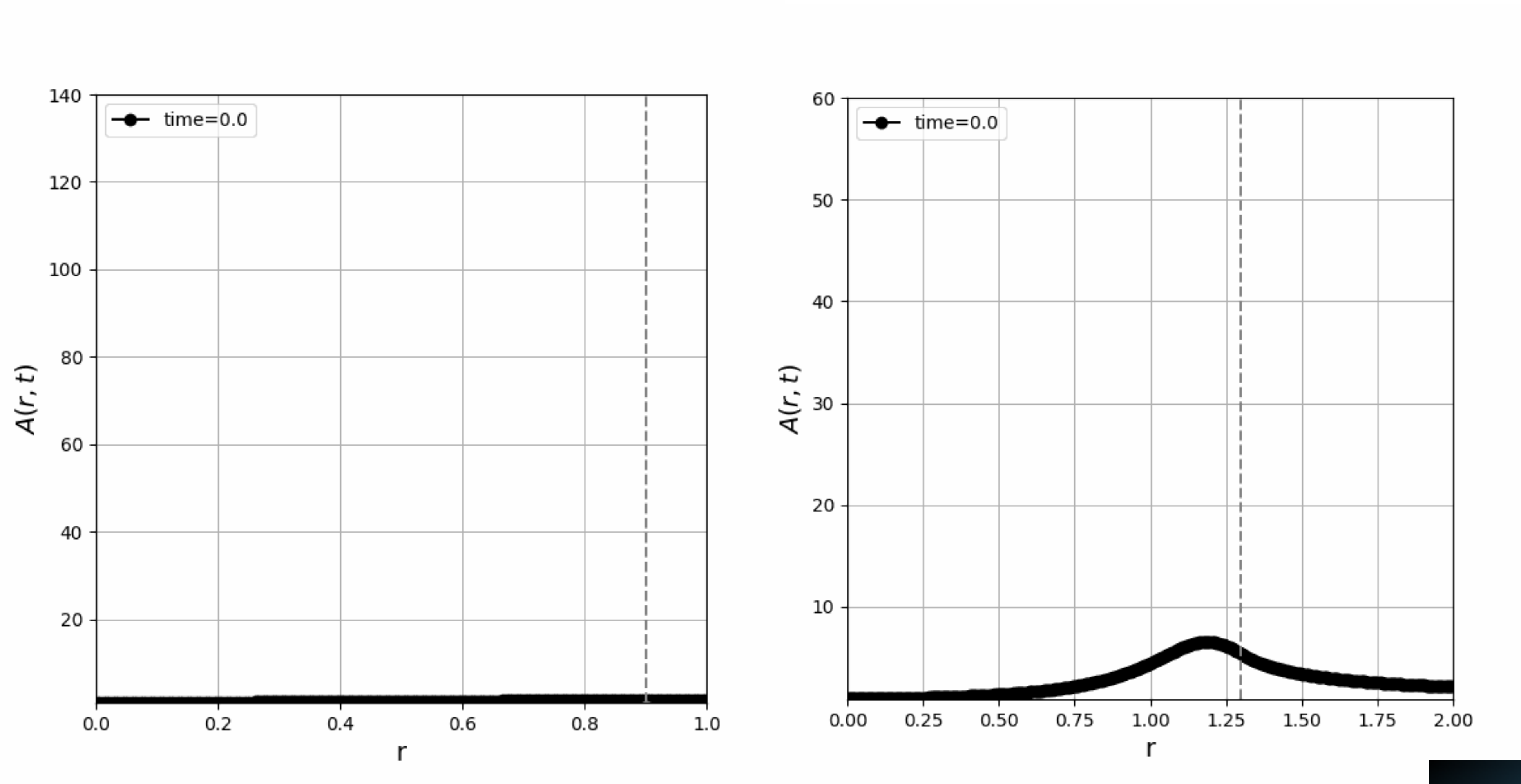
Fluid speed $v(r,t)$;





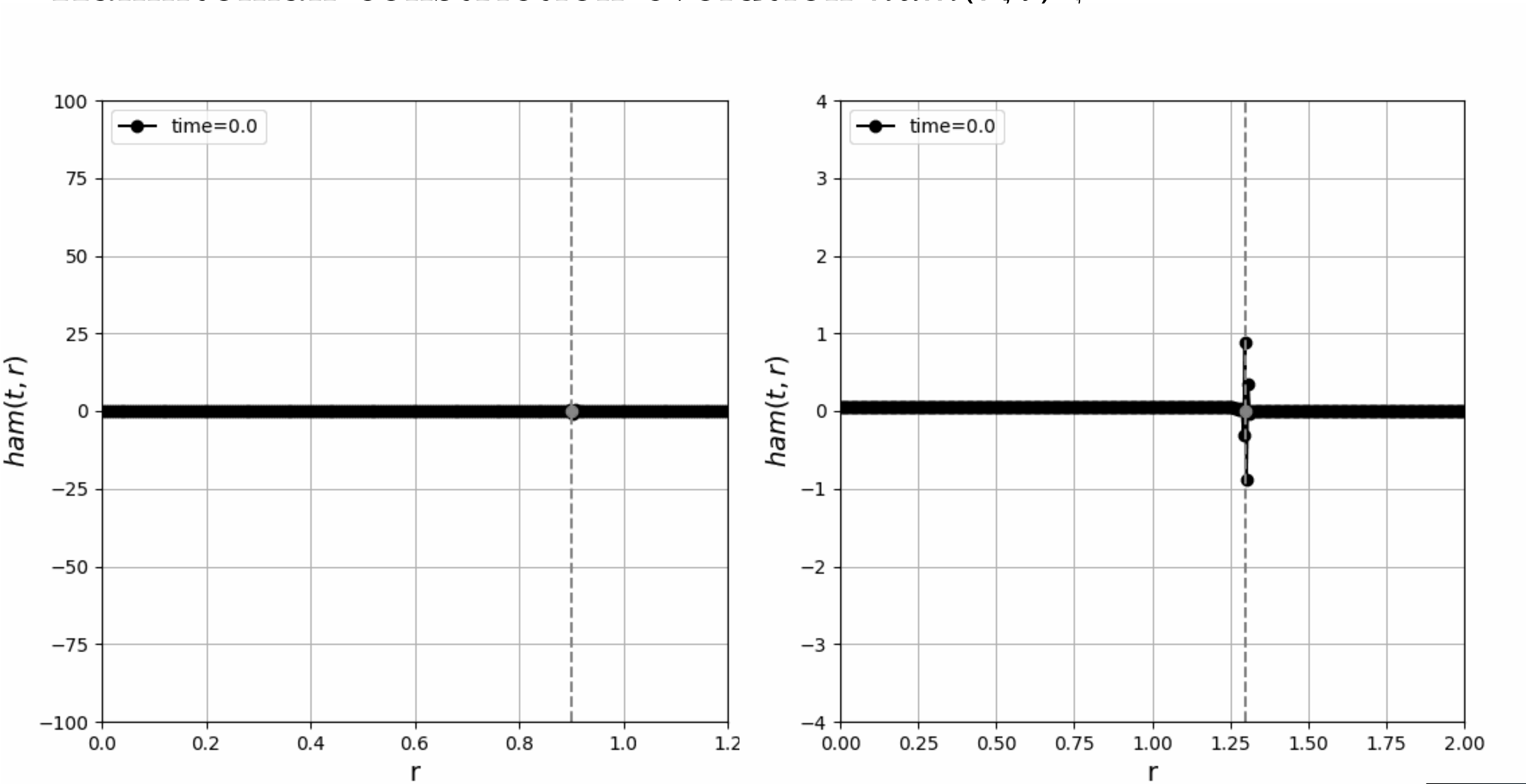
Evolution (Arbitrary Cases)

Metric function Evolution $A(r, t)$;



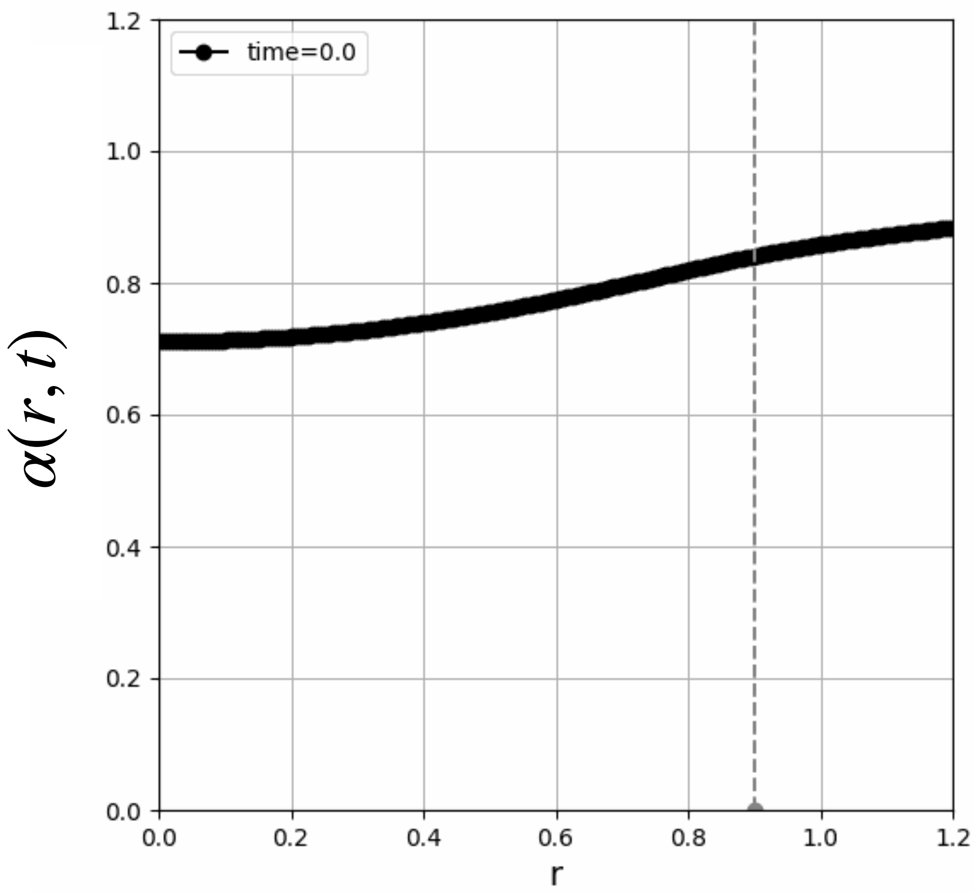
Evolution (Arbitrary Cases)

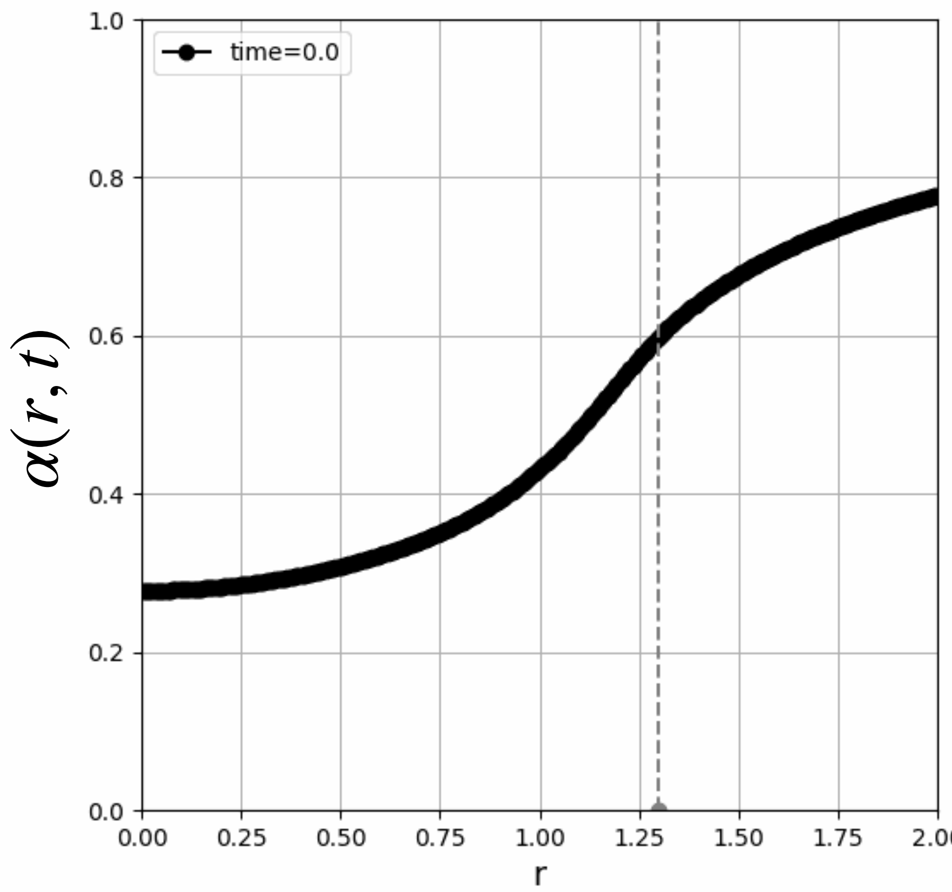
Hamiltonian constriction evolution $ham(r, t)$;



Evolution (Arbitrary Cases)

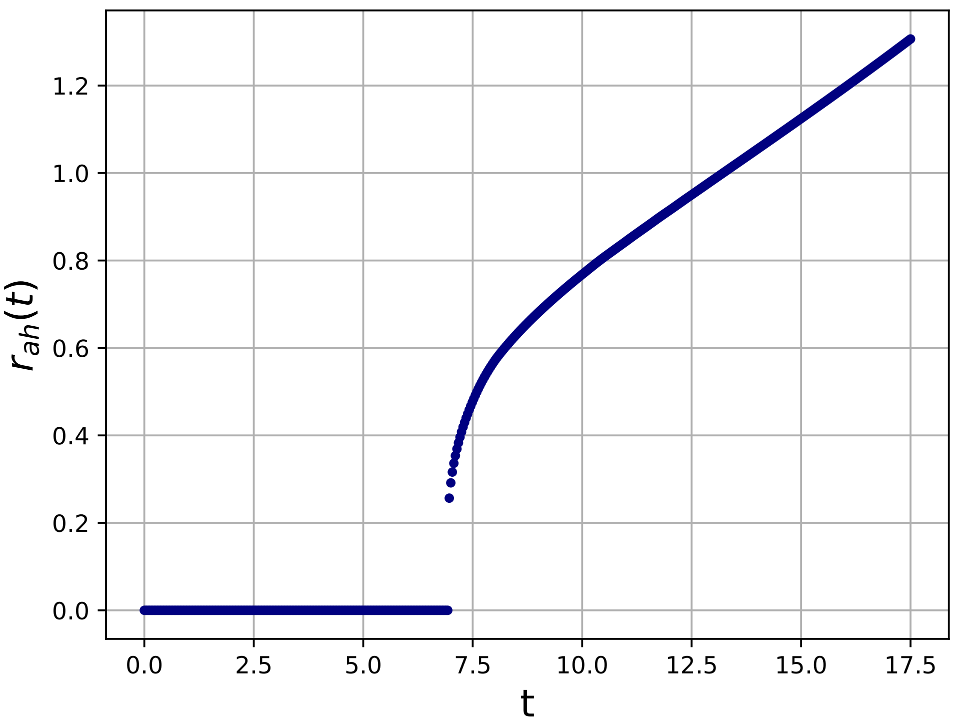
Lapse evolution $\alpha(r,t)$;

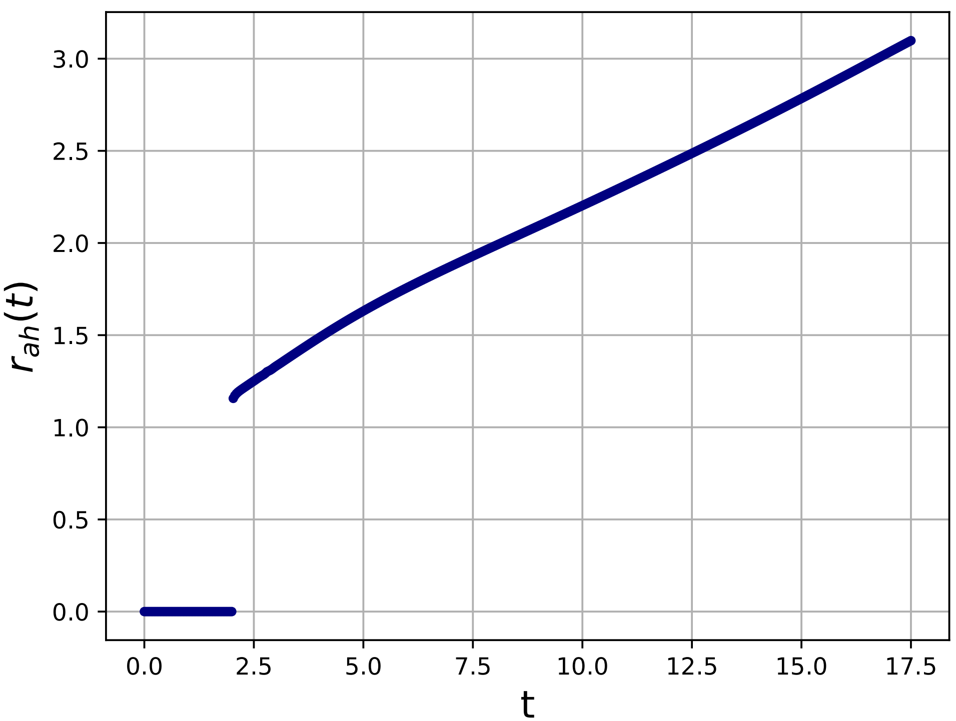




Evolution (Arbitrary Cases)

Radius of the aparent horizon;

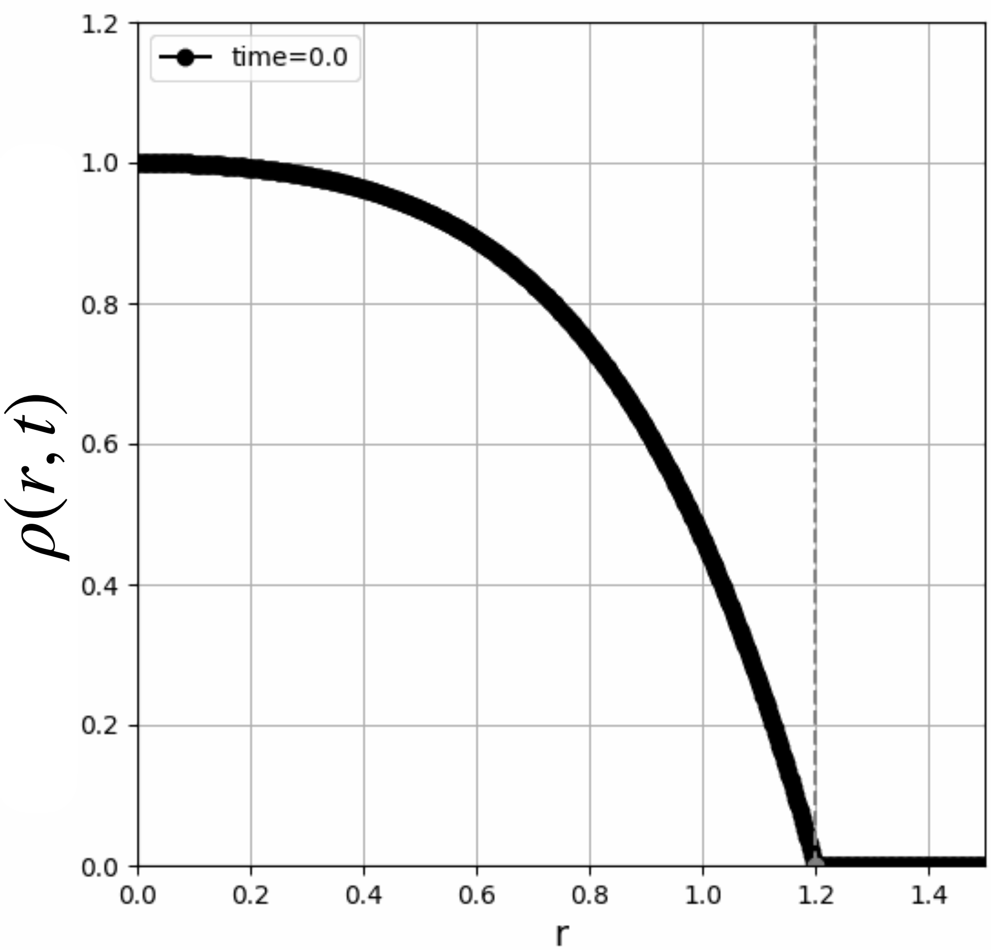


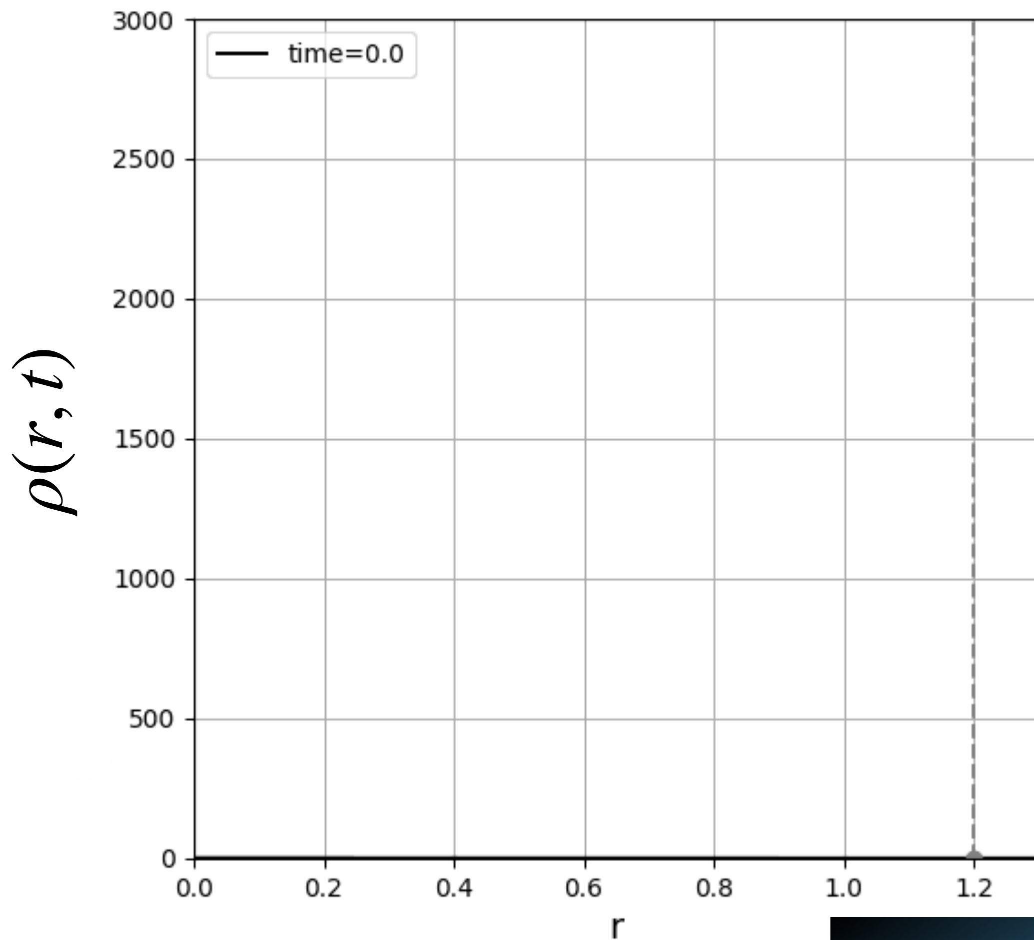


NS Evolution

Evolution (NS Cases)

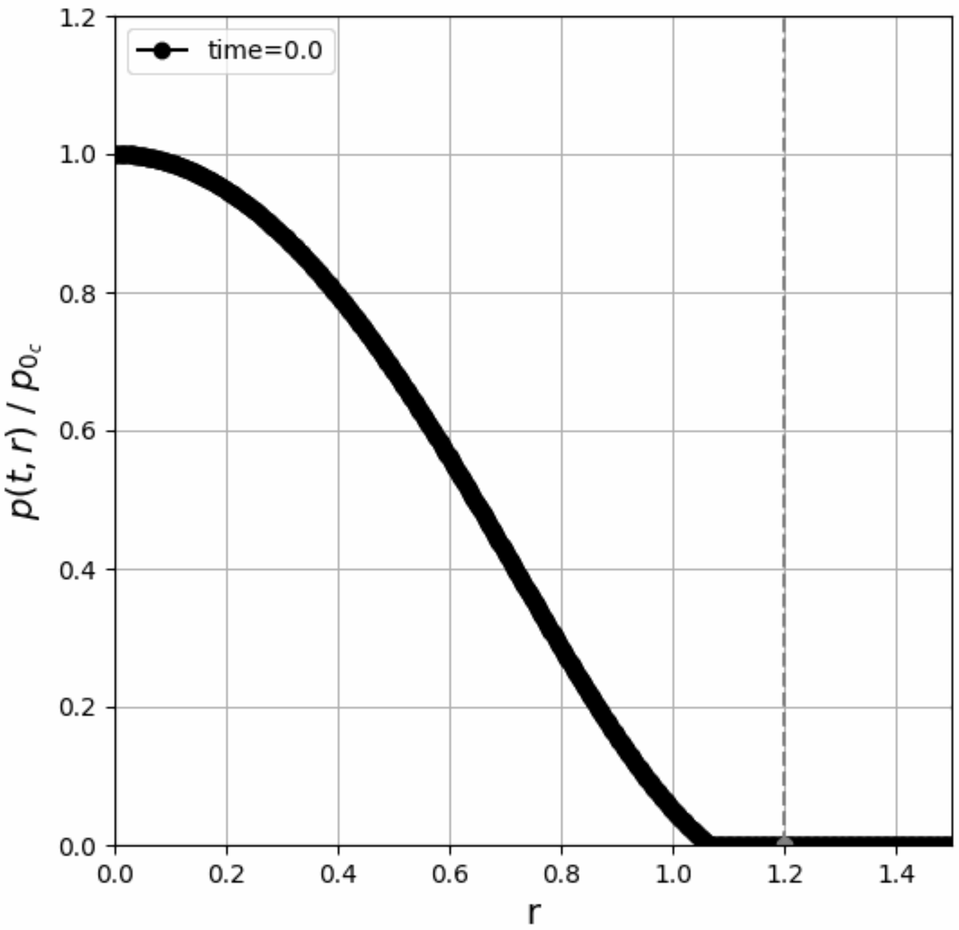
Density Evolution;

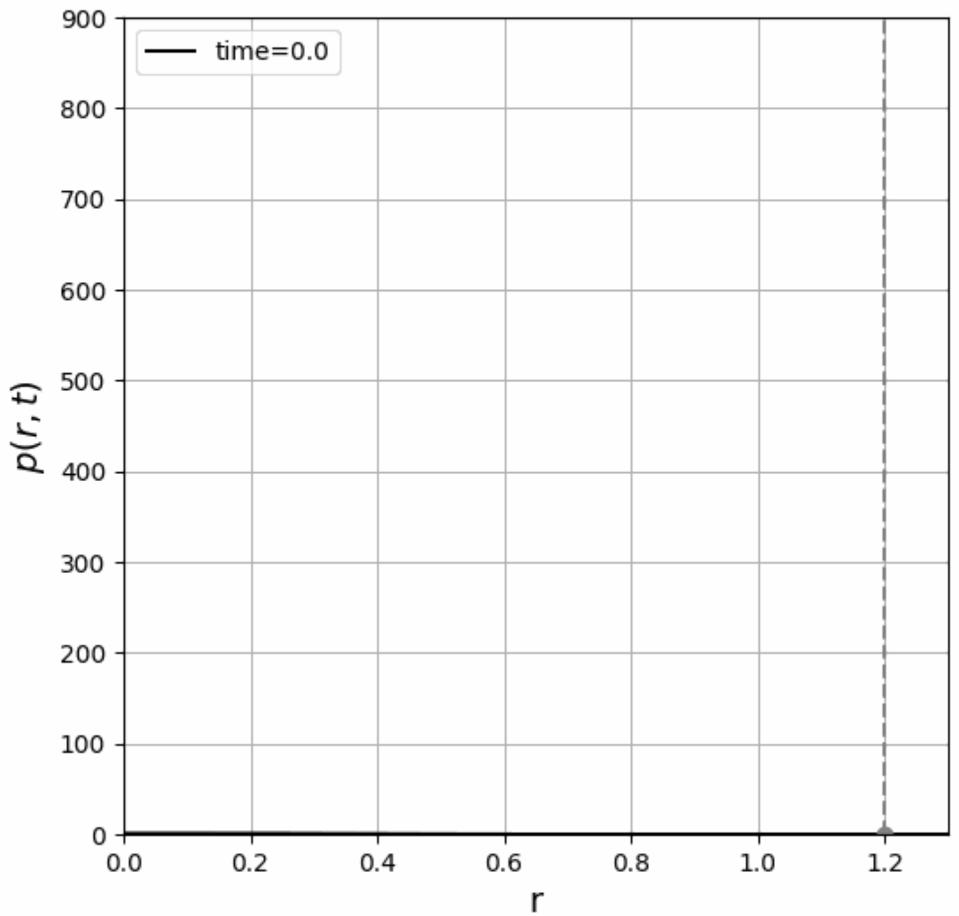




Evolution (NS Cases)

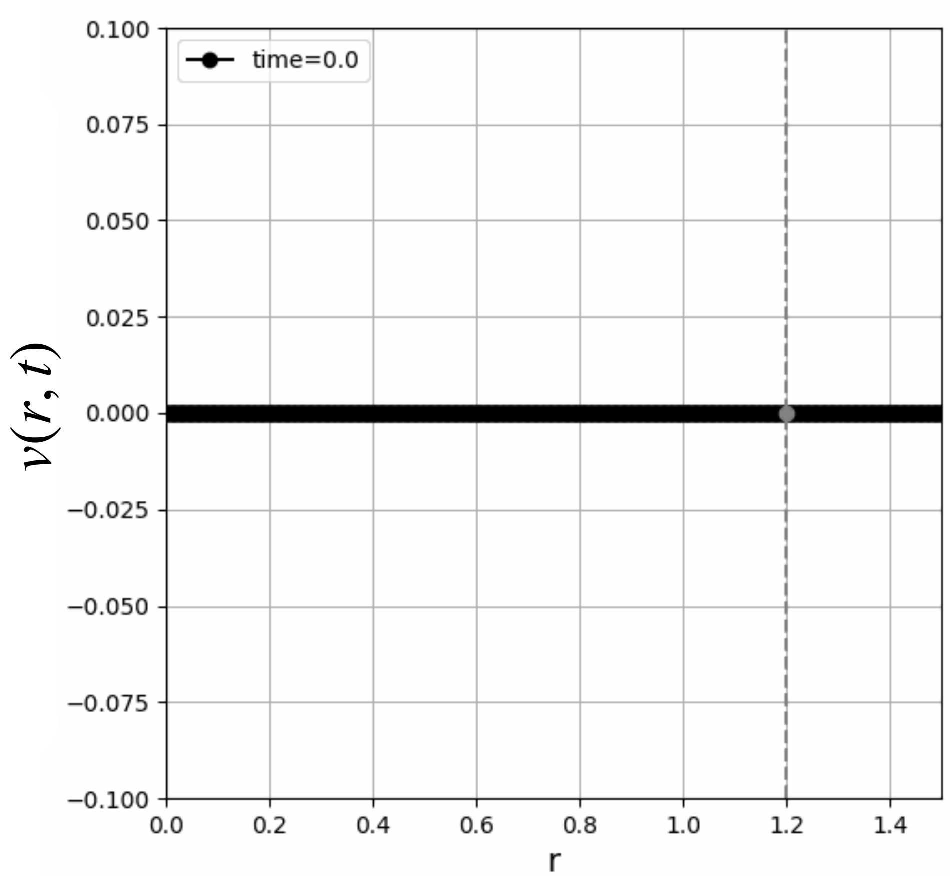
Pressure Evolution;

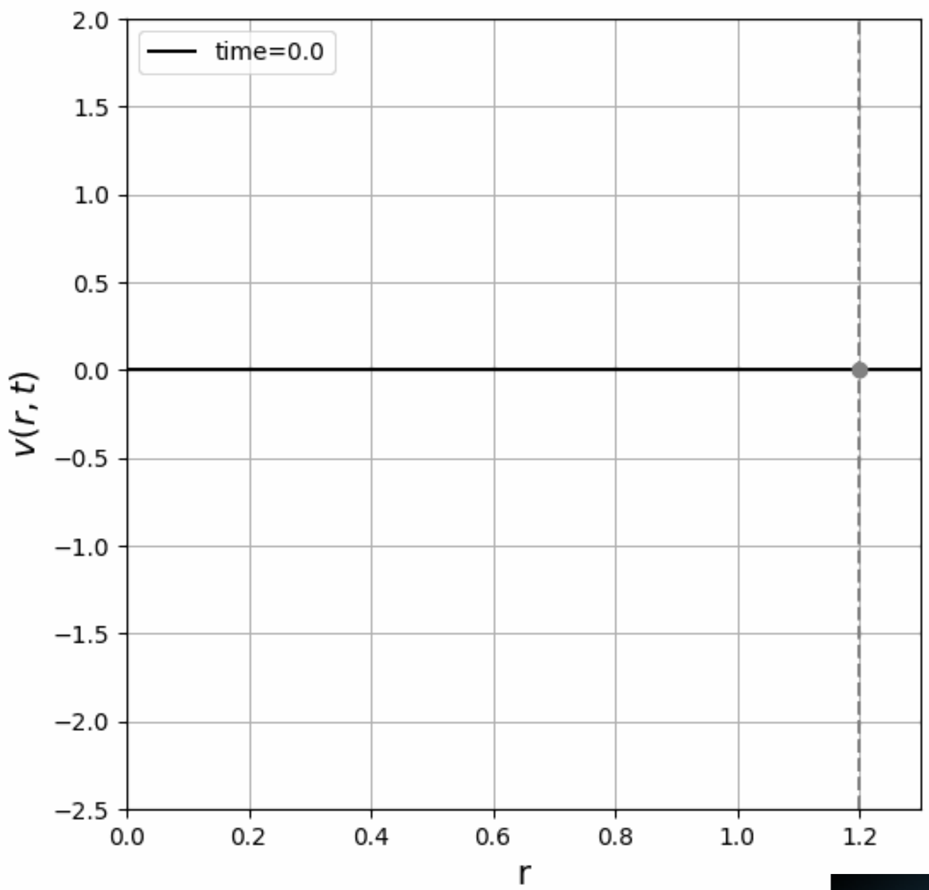




Evolution (NS Cases)

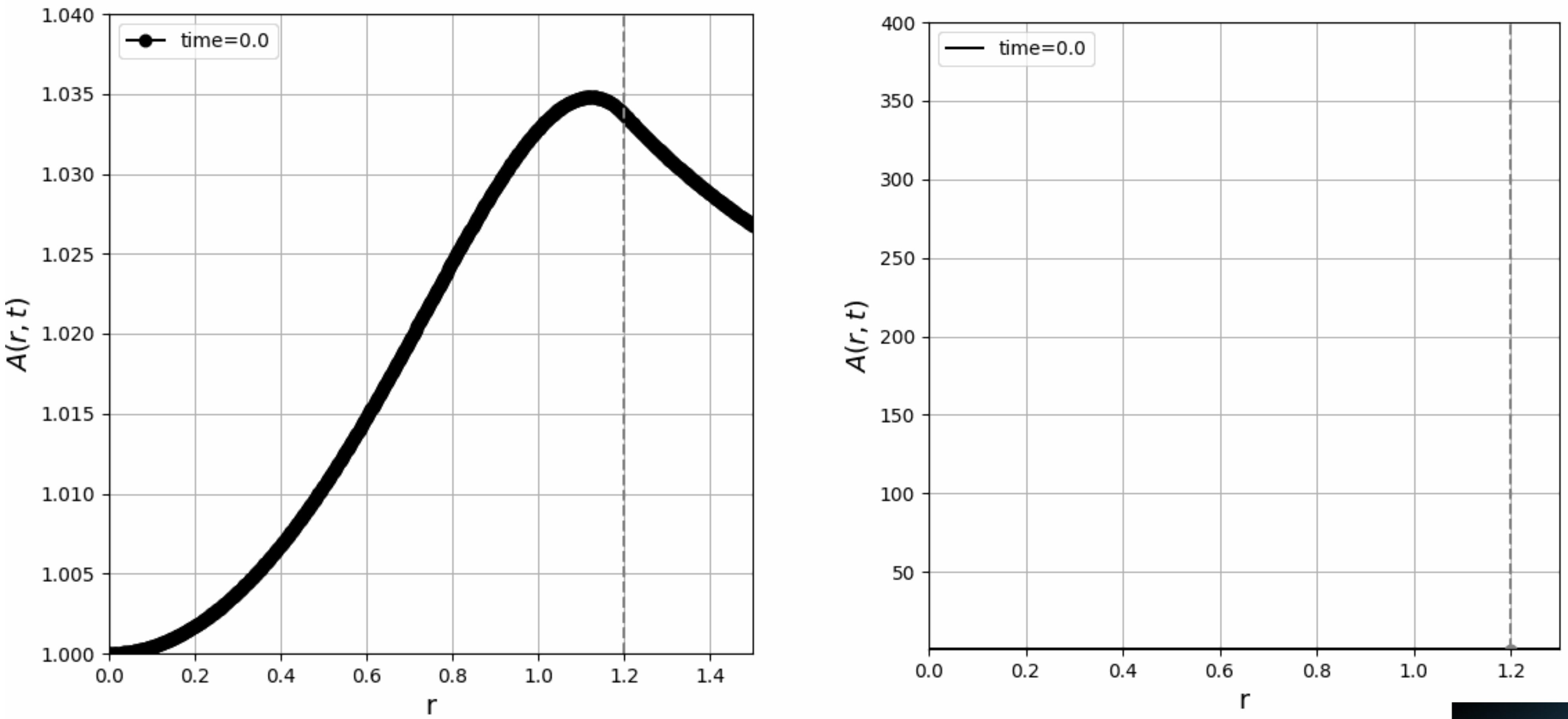
Fluid speed $v(r,t)$;





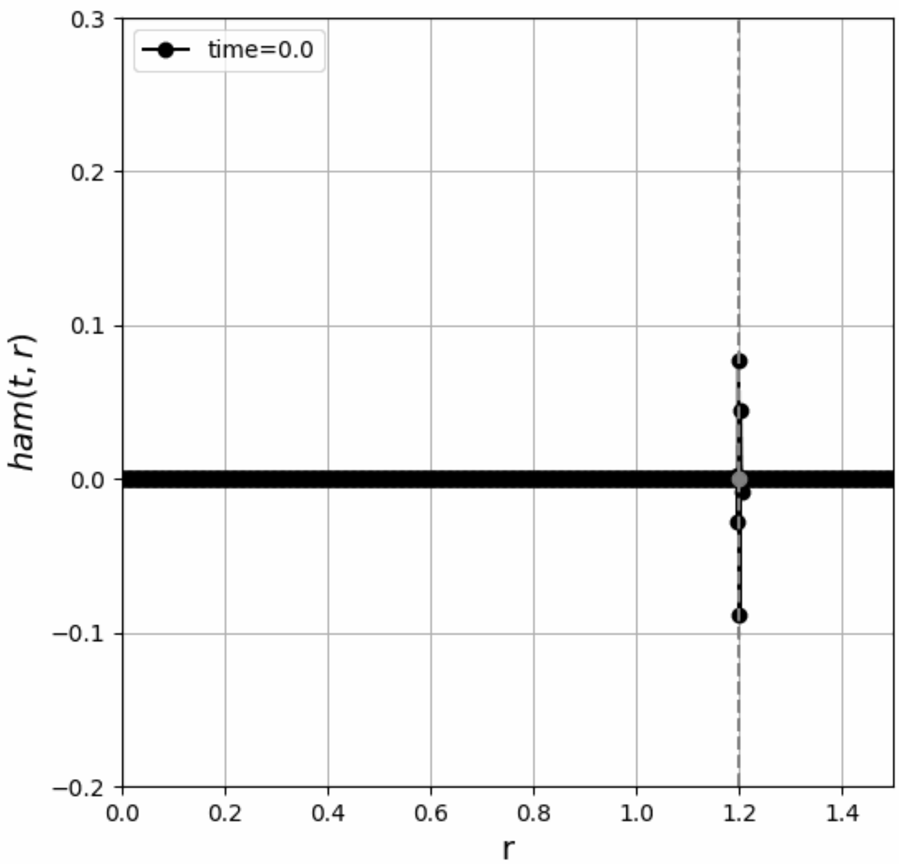
Evolution (NS Cases)

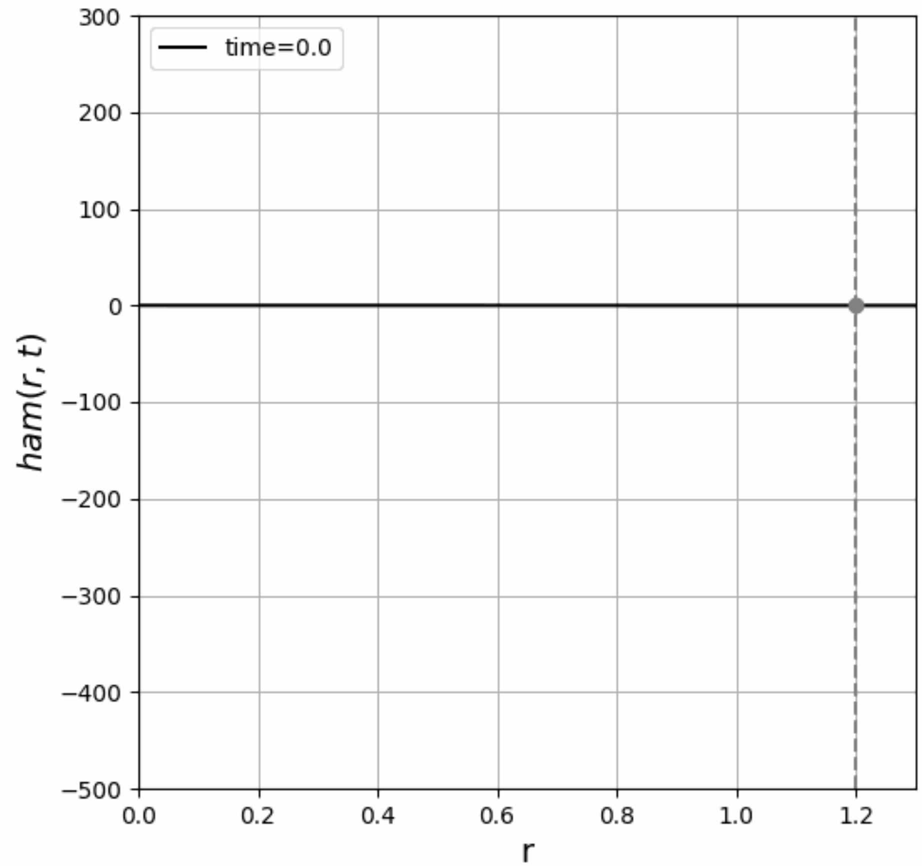
Metric function Evolution $A(r,t)$;



Evolution (NS Cases)

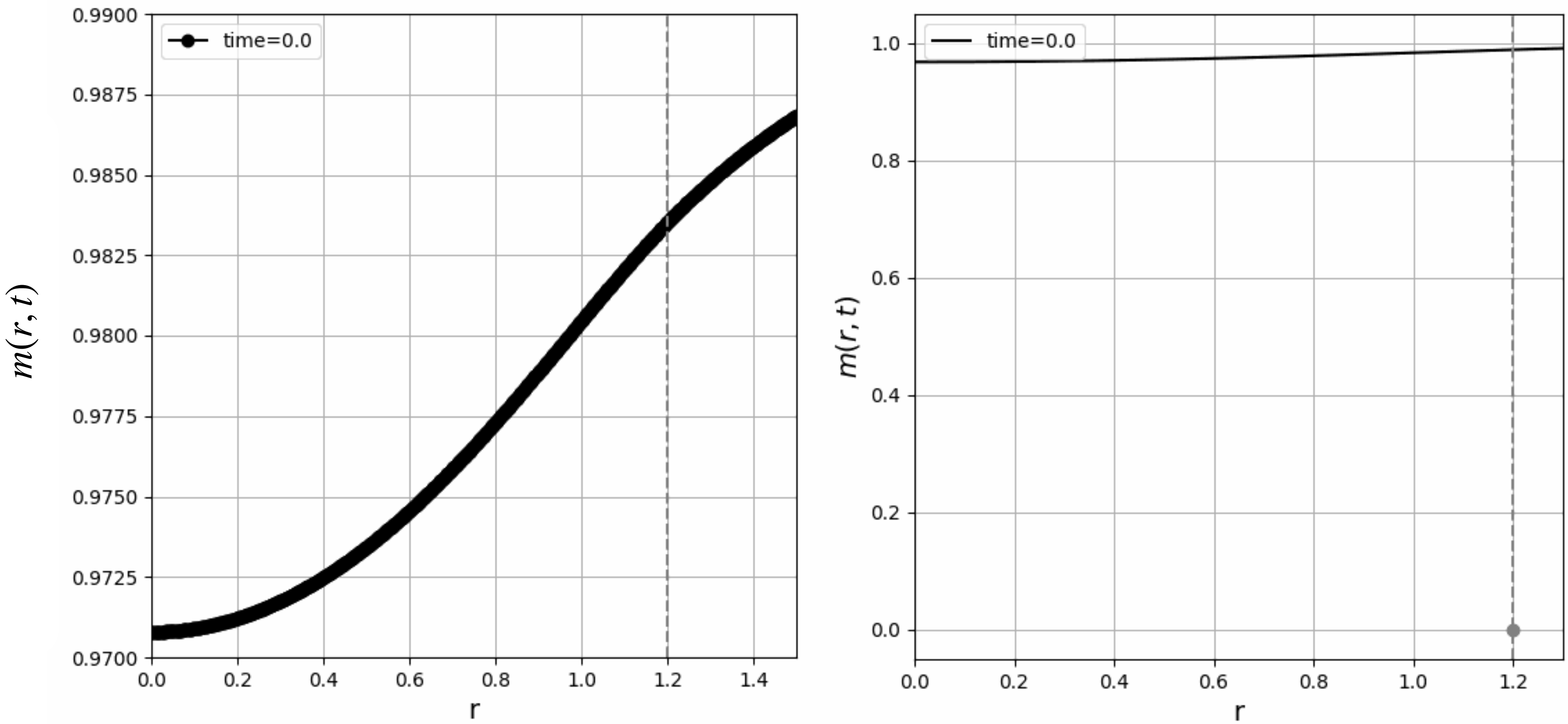
Hamiltonian constriction evolution $ham(r, t)$;





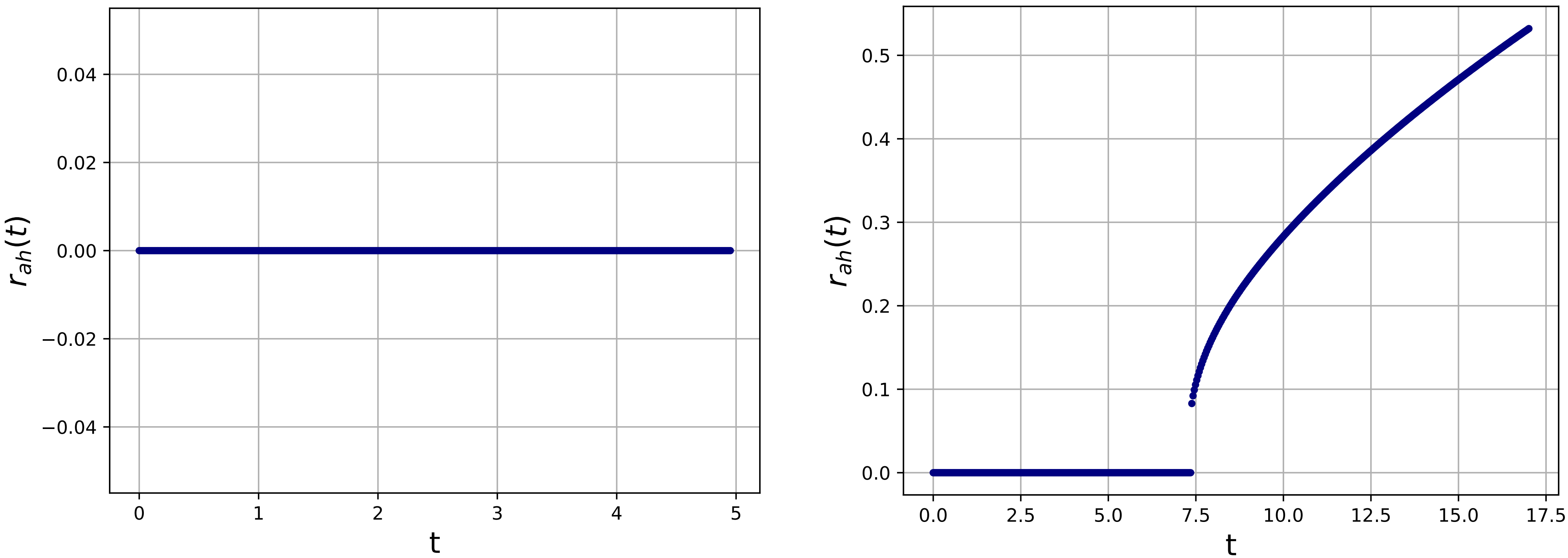
Evolution (NS Cases)

Lapse evolution $\alpha(r,t)$;



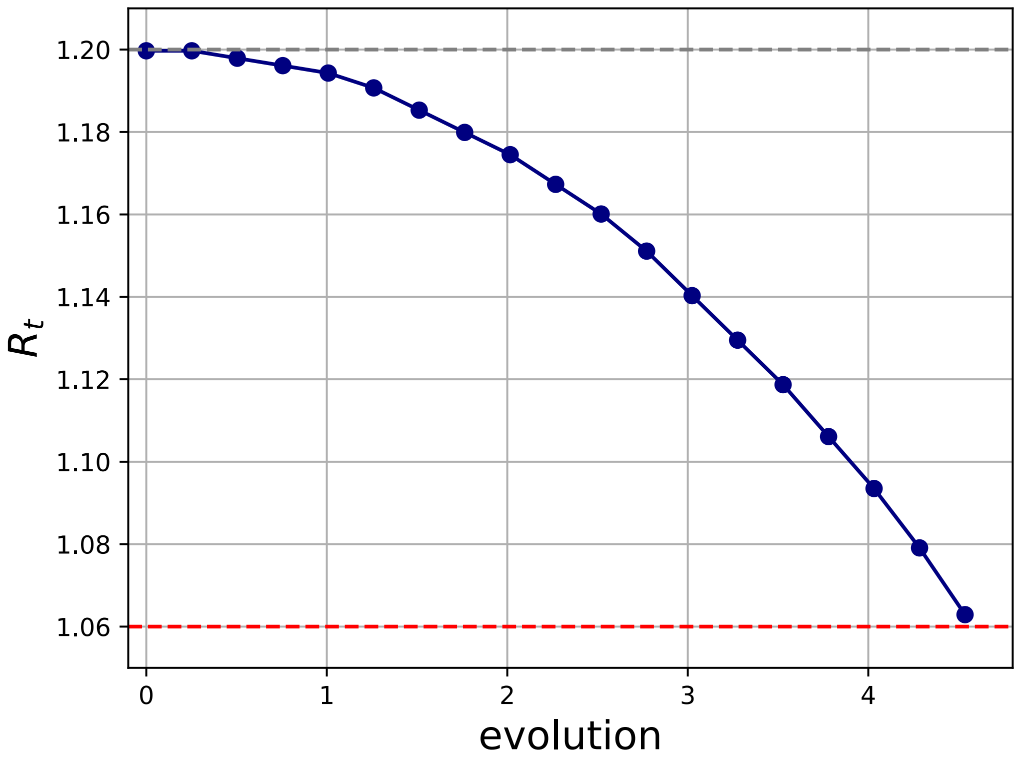
Evolution (NS Cases)

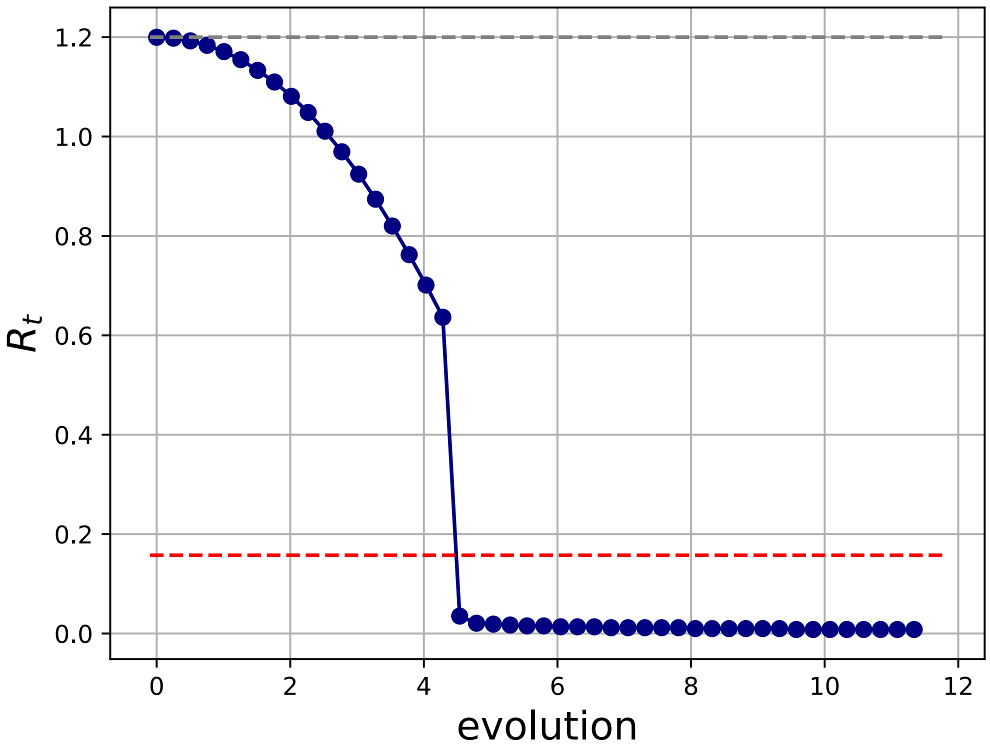
Radius of the aparent horizon;



Evolution (NS Cases)

Total radius





Conclusions

Conclusions

- We solve the TOV equations with an initial density of the perfect fluid given by a polynomial function of the radial coordinate that is regular everywhere within the fluid.
- The model is consistent with the Buchdahl limit and the speed of sound, even when using realistic values of astrophysical objects such as neutron stars.
- The model is also consistent with the C^3 matching conditions, where both pressure and energy density must be canceled at the matching hypersurface.
- During the evolution, the dust layer falls as expected, generating a case of a completely collapsed object in some cases.

Thanks!

Gracias!

Grazie!

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Appendix A. The Elementary test.

- Isotropy of the pressure. $G_r^r = G_\theta^\theta$
- Regularity at the origin.
- Positive definiteness of the energy density and pressure at the origin.
- Vanishing of the pressure at some finite radius.
- Monotonic decrease of the energy density and pressure with increasing radius.
- Subluminal sound speed.
- The sound speed monotonically decreases with radius.

Appendix B. The Buchdalh limit.

The Buchdahl limit tells us that the compactness ratio must be less than 8/9 within a physically reasonable equation of state, This applies not only to the star's surface, but also to its interior [Chávez and Sarbach 2021], that is,

$$\frac{2m(r)}{r} < \frac{8}{9} \qquad \& \qquad M < \frac{4}{9}R$$