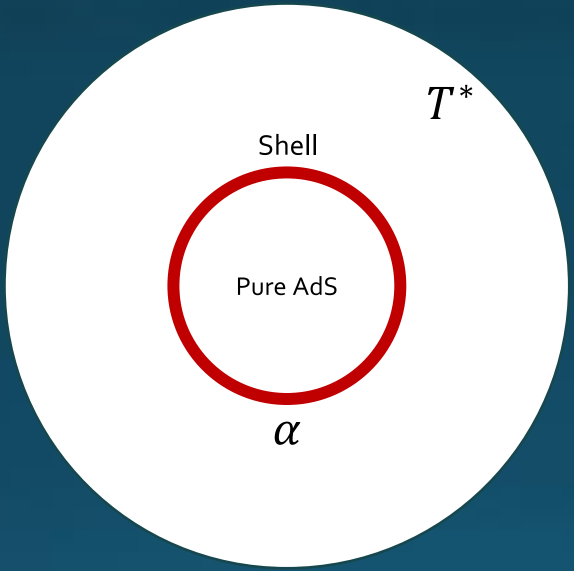
The canonical ensemble of a self-gravitating matter thin shell in asymptotically AdS



Tiago Vasques Fernandes

with: Francisco Gandum & José P. S. Lemos

Instituto Superior Técnico, CENTRA, GRIT

UIDB/00099/2020
RD0970

XVII Marcel Grossmann Meeting, Pescara 2024

Motivation

- Statistical ensembles through Euclidean path integral approach;
  - [Gibbons & Hawking, 1977];
- Hawking and Page phase transition;
  - [Hawking & Page, 1983];
- Connection between matter thermodynamics and gravity;
  - [Martinez & York, 1989], [Martinez, 1996], [Lemos & Zaslavskii, 2023];

Euclidean Path Integral Approach

The amplitude of the quantum states at different times is given by

$$< \boldsymbol{g}(0), \phi(0) | \boldsymbol{g}(t_1), \phi(t_1) > = < \boldsymbol{g}(0), \phi(0) | e^{-i \int_0^{t_1} \mathcal{H} dt} | \boldsymbol{g}(0), \phi(0) > = \int_{\boldsymbol{g}(0), \phi(0)} D\boldsymbol{g}\, D\phi\, e^{i\, I_L[\boldsymbol{g}, \phi]}\, .$$

We consider the thermal partition function $Z[\beta] = Tr\left(e^{-\,\beta\, H}\right)$.

We assume the quantum states are stationary.

We perform the complex analytic extension of metrics with identification $t \to -i\,\tau$, with $\tau \in [0,2\pi]$.

Then, $Z[\beta] = \int_{\boldsymbol{g}(0)=\boldsymbol{g}(2\pi), \phi(0)=\phi(2\pi)} D\boldsymbol{g}\, D\phi\, e^{-\, I_E[\boldsymbol{g}, \phi]}$ if ϕ is bosonic,

where $\int_0^{2\pi} (g^{\tau\tau})^{-\frac{1}{2}}\, d\tau = T^{-1}$ and $T[\partial M]^{-1} = \beta$

Partition Function of Matter and Gravity

We split the path integral as

$$Z[\beta_*] = \int_{\boldsymbol{g}(0)=\boldsymbol{g}(2\pi)} D\boldsymbol{g}\; e^{-I_g[\boldsymbol{g}]} \int_{\phi(0)=\phi(2\pi)} D\phi\; e^{-\,I_m[\boldsymbol{g},\phi]}\,,$$

In principle, one cannot perform the path integral over matter in a closed form.

Yet, we consider spherically symmetric metrics and matter is described by **spherically symmetric shell**. Then

$$e^{-\int_M \mathcal{F}[\boldsymbol{g}]\sqrt{g}\,d^4x} = \int_{\phi(0)=\phi(2\pi)} D\phi\; e^{-\,I_m[\boldsymbol{g},\phi]}\,,$$

where $\mathcal{F}[\boldsymbol{g}]$ is the free energy density of the shell, with $d\mathcal{F} = s_m dT_m + \frac{\mathcal{F}+p_m}{n}dn$. T_m is local temperature, s_m is entropy density, p_m is pressure and $n = \frac{\mathcal{N}}{\sqrt{\mathrm{h}\,\mathrm{h}^{\tau\tau}}}$, with h the induced metric.

The partition function is $Z[\beta_*] = \int_{g(0)=g(2\pi)} Dg\; e^{-I_{eff}[g]}$. With $I_{eff}[g] = I_g[g] + \int_M \mathcal{F}[\boldsymbol{g}]\sqrt{g}\,d^4x$

We must give an equation of state.

Action and metric

Action:

$$I_{eff} = -\int_{M\setminus\{A\}} \left(\frac{R+\frac{6}{l^2}}{16\,\pi}\right)\sqrt{g}\, d^4x - \frac{1}{8\pi}\int_{\partial M}(K)\sqrt{\gamma}d^3x - \int_A \left(-\frac{[K]}{8\pi} - \mathcal{F}[\boldsymbol{h}]\right)\sqrt{h}\, d^3x - I_0$$

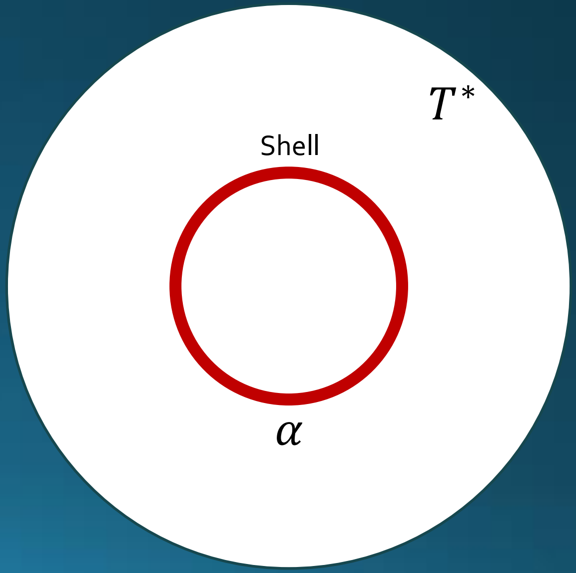
with A being the shell, l the AdS length, K the extrinsic curvature, $[K] = K_2 - K_1$.

Metric:

$$ds_1^2 = \frac{b_1^2(y)b_2^2(y_m)}{b_1^2(y_m)}d\tau^2 + \ v_1(y)^2dy^2 + r(y)^2d\Omega^2 \qquad , \tau \in [0,2\pi[\ \ , y \in]0, y_m] ,$$

$$ds_2^2 = \ b_2(y)^2d\tau^2 + \ v_2(y)^2dy^2 + r(y)^2d\Omega^2 \qquad , \tau \in [0,2\pi[\ \ , y \in]y_m, +\infty[.$$

Shell at $r(y_m) = \alpha$.



Boundary Conditions:

$y = 0$: $b(0)$ is finite, $\frac{b_1'}{v_1}(0) = 0, r(0) = 0$. “Flat” conditions.

$y = +\infty$: Asymptotic AdS conditions $\frac{b_2(y)}{r(y)} \to \frac{\beta_*}{2\pi l}$.

Zero Loop approximation

Minimize the action in variations of $b \longrightarrow$ Hamiltonian constraints which splits into $G^\tau_\tau = 0$ for $M \backslash \{A\}$ and junction conditions at A

$$G^\tau_\tau = 0 \quad \to \quad \frac{r'}{v_1} = f_1(r) = 1 + \frac{r^2}{l^2} \;, \text{ “Pure AdS”.}$$

$$\frac{r'}{v_2} = f_2(r) = 1 + \frac{r^2}{l^2} - \frac{\tilde{r}_+}{r}\left(1 + \frac{\tilde{r}_+^2}{l^2}\right), \text{ with } r' = \frac{dr}{dy} \;, \text{ “Schw-AdS”.}$$

Junction Cond. $\to \quad m(T_m, \alpha) = m(\tilde{r}_+, \alpha) := \alpha\left(\sqrt{f_1(\alpha)} - \sqrt{f_2(\alpha)}\right)$

Now pick an equation of state $T_m(m, \alpha)$, the temperature of shell is then $T_m(m(\tilde{r}_+, \alpha), \alpha)$.

Zero Loop approximation

Hamiltonian constraints – no dependence on $b_{1,2}(y)$!! (T_m has been determined)
One can choose $y(r)$ coordinate change. It only changes $\alpha = r(y_m)$.

The function v_1 is fully determined, while v_2 is a functional of $\tilde{r}_+$.

Therefore $DrDv_1Dv_2 = D\tilde{r}_+D\alpha$.

Partition function $Z[\beta_*] = \int D\tilde{r}_+ D\alpha \; e^{-I^*[\beta_*;\tilde{r}_+,\alpha]}$, with

$I^*[\beta_*;\tilde{r}_+,\alpha] = \frac{\beta_*}{2}(\tilde{r}_+ + \tilde{r}_+^3/l^2) \; - S_m(m(\tilde{r}_+,\alpha),\alpha)$,

with an equation of state

$dS_m = T_m^{-1}(m(\tilde{r}_+,\alpha),\alpha)dm(\tilde{r}_+,\alpha) + T_m^{-1}(m(\tilde{r}_+,\alpha),\alpha)\, p_m(m(\tilde{r}_+,\alpha),\alpha)d\alpha$.

Zero Loop approximation - Equation of state

We choose an equation of state

$$S_m = c_0 \left(l^{5/4} l_c^{1/4}\right) \left(\frac{m}{l}\right)^{\frac{3}{4}} \left(\frac{4\pi\alpha^2}{l^2}\right)^{\frac{1}{4}},$$

l_c is the "Compton wavelength"

From the differential, one gets

$$T_m^{-1} = c_0 \left(l^{1/4} l_c^{1/4}\right) \left(\frac{m}{l}\right)^{-\frac{1}{4}} \left(\frac{4\pi\alpha^2}{l^2}\right)^{\frac{1}{4}},$$

$$p_m = \frac{1}{3l} \left(\frac{m}{l}\right) (4\pi\alpha^2)^{-1} \; .$$

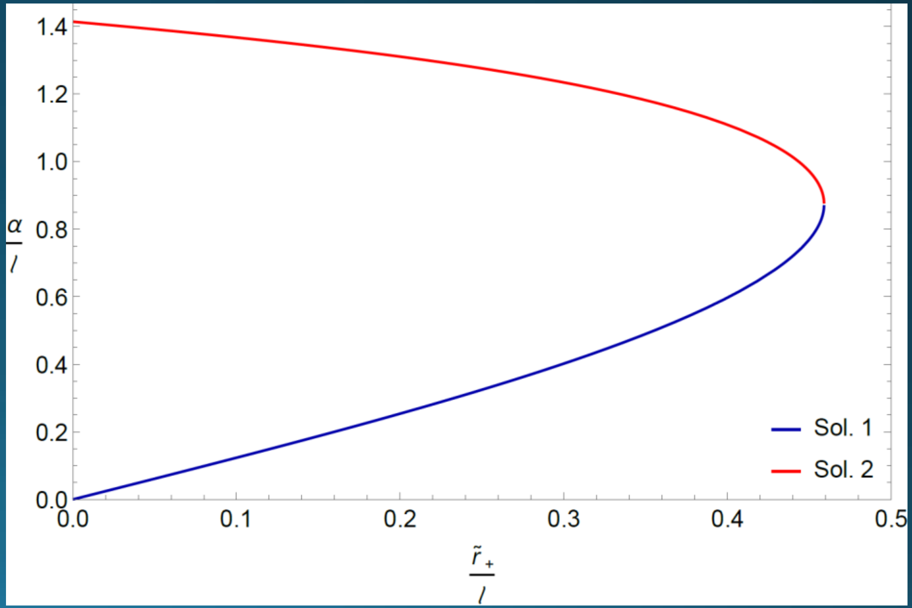
Zero Loop approx. – Stationary points p.1

Find the stationary points of the reduced action I^*

$$\frac{\partial I^*}{\partial \alpha} = 0 \quad \rightarrow \quad p_m(m(\tilde{r}_+, \alpha), \alpha) = p_{GR}(\tilde{r}_+, \alpha) \text{ with}$$

$$p_{GR}(\tilde{r}_+, \alpha) := \frac{1}{8\pi\alpha}\left(\frac{-2+3f_1(\alpha)+f_2(\alpha)}{2\sqrt{f_2(\alpha)}} - \frac{-1+2f_1(\alpha)}{\sqrt{f_1(\alpha)}}\right).$$

Two solutions: critical value $\frac{\alpha_c}{l} \approx 0.87.$



Stability $\partial_\alpha(p_{GR} - p_m) \geq 0$ happens when $\partial_{\tilde{r}_+}\alpha \leq 0$ in this specific case.

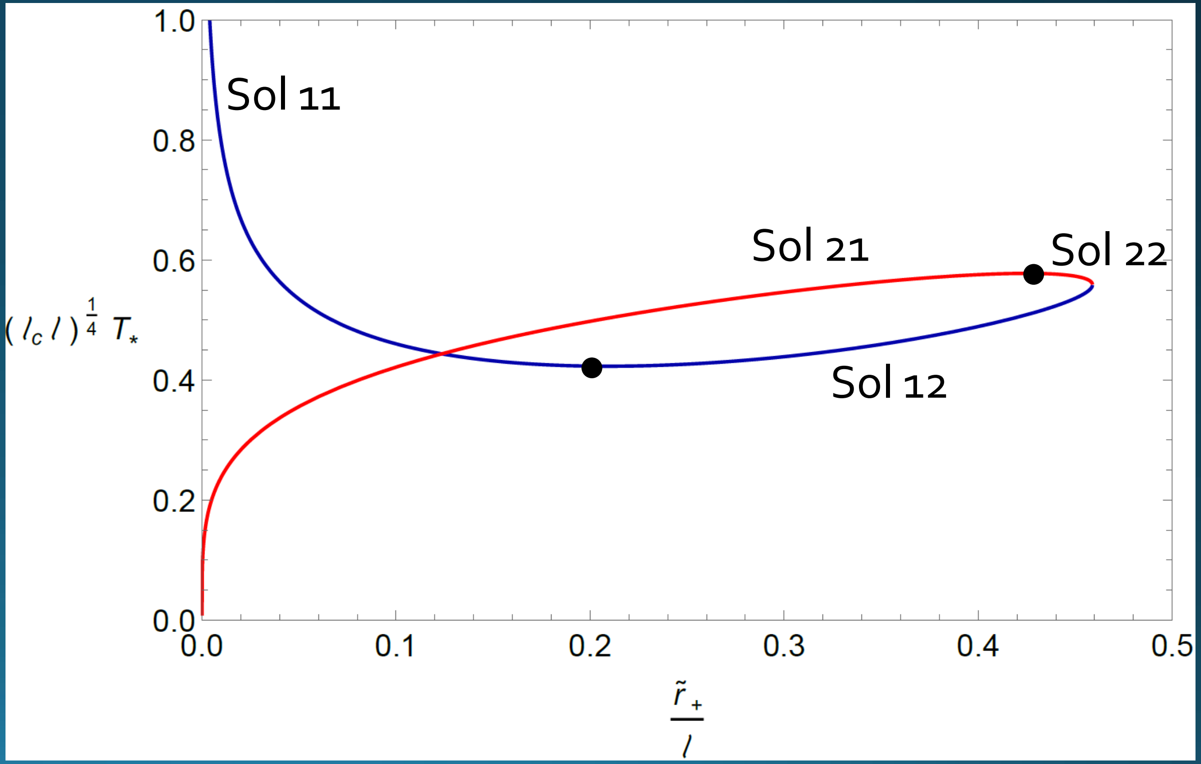
This is precisely mechanical stability!!

Zero Loop approx. – Stationary points p.2

Find the stationary points of the reduced action I^*

$$\frac{\partial I^*}{\partial \tilde{r}_+} = 0 \quad \rightarrow \quad \beta_* = \frac{1}{T_m(m(\tilde{r}_+,\alpha),\alpha)\sqrt{f_2(\tilde{r}_+,\alpha)}} \; ,$$

There are four solutions, two for each $\alpha(\tilde{r}_+)$ solution.



Stability $\partial_{T_*}\tilde{r}_+ \geq 0$.

Sol11 and Sol12 are black hole like, but mechanically unstable.

Sol21 is fully stable!

Thermodynamics

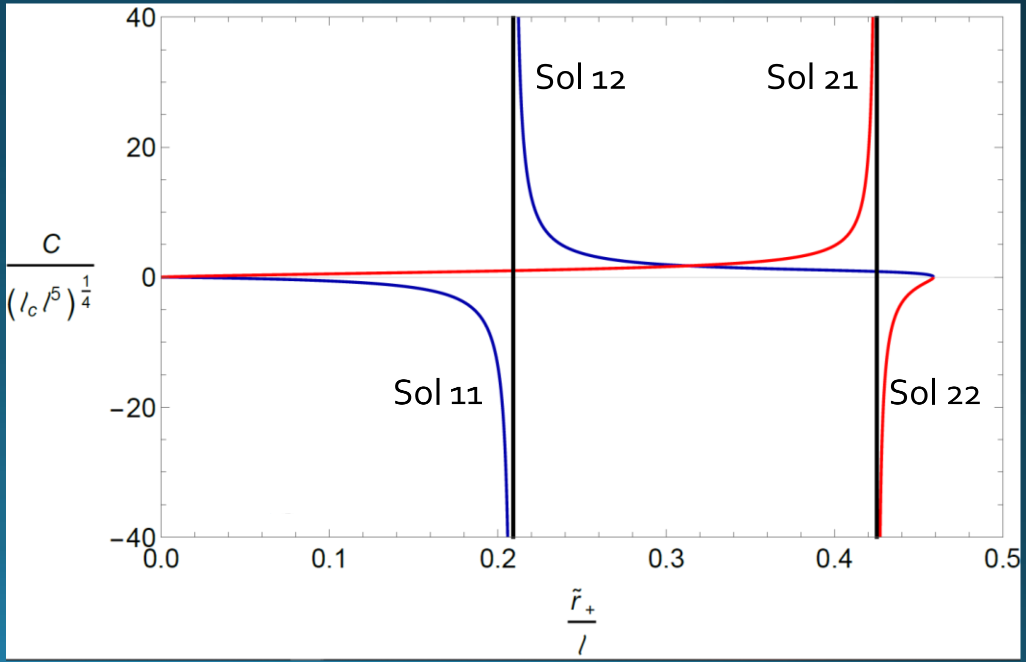
The zero loop approximation relates free energy to the reduced action

$$\beta_* F = I^* = \frac{\beta_*}{2}(\tilde{r}_+ + \tilde{r}_+^3/l^2) - S_m[m(\tilde{r}_+,\alpha),\alpha]\text{ , with } \tilde{r}_+[T_*] \text{ and } \alpha[\tilde{r}_+[T_*]]\text{ .}$$

The energy is given by $E = \frac{1}{2}(\tilde{r}_+ + \tilde{r}_+^3/l^2)$
and entropy $S = E - \beta_* F = S_m[T_*]$.

The thermodynamic stability is given by the heat capacity $C \geq 0$.

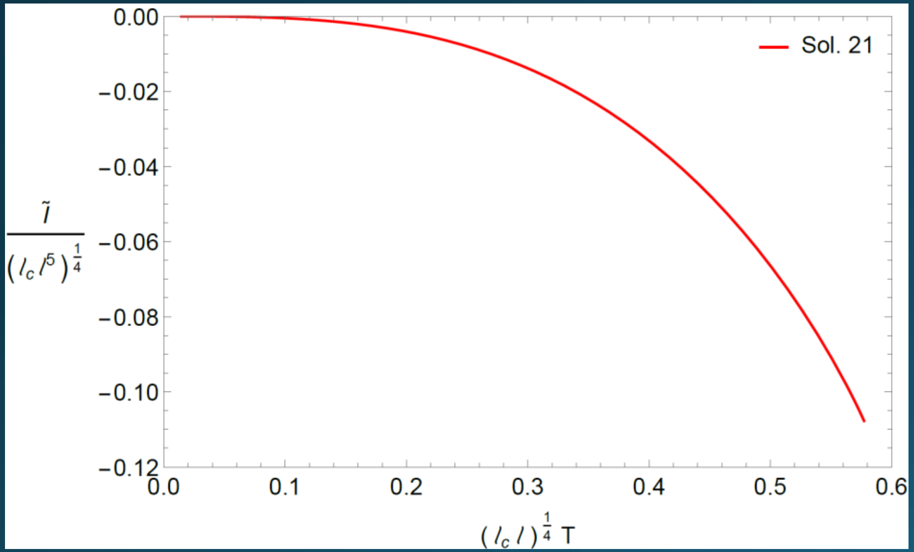
There is no information about internal mechanical stability.
But if one introduces a chemical potential,
this information may be recovered.



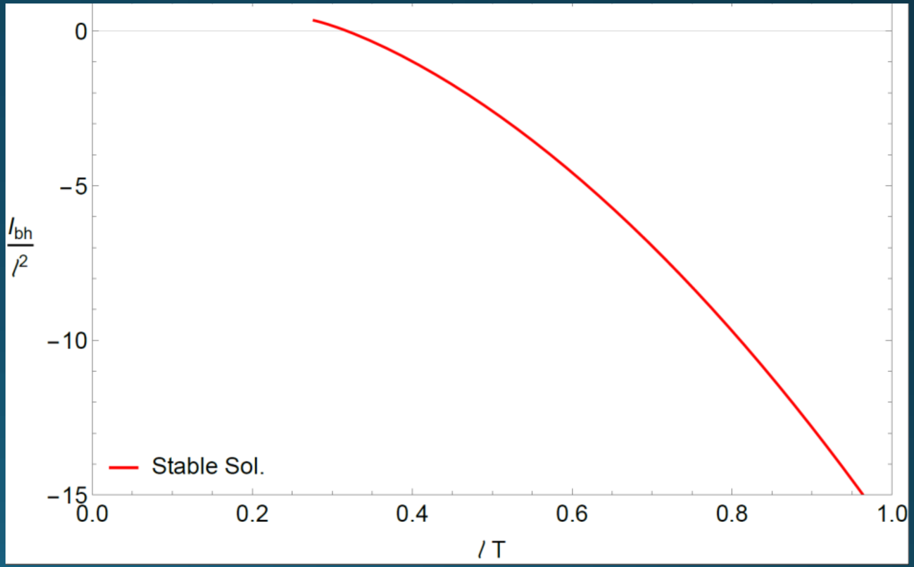
Phase transitions (Preliminary work)

Consider two states, matter shell AdS and stable black hole AdS, which one has lower free energy?

Action of the stable shell



Action of the stable black hole



If $2.10\left(\frac{l^3}{l_c}\right)^{\frac{1}{4}} > 1$, then there is a region of temperatures that both solutions exist. There should be a temperature T_{int} at which the free energy of both is the same.
From $0 < T_* < T_{int}$ shell is favorable, from $T_{int} < T_*$ black hole is favorable.

Summary and Conclusions

Construction of the canonical ensemble of a matter shell in AdS was done and it depends on the equation of state.

For a specific equation of state, there are four solutions to the shell, only one is stable.

The canonical ensemble requires mechanical stability of the shell, yet this does not seem acessible from the full ensemble. One maybe has to introduce a chemical potential.

The entropy of the ensemble is the entropy of the shell.

The shell solution acts as version of hot AdS.