A (not) new geometric approach to Ashtekar variables and symmetry reduction

Matteo Bruno

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SAPIENZA
UNIVERSITÀ DI ROMA

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Fiber bundle structure of Ashtekar variables

A Yang-Mills approach

The natural language of Yang-Mills theories is the principal bundle formalism.

Principal G-bundle (P, π, M)

- P a smooth manifold called «fiber bundle»,
- M a smooth manifold called «base manifold»,
- $\pi: P \to M$ a smooth projection,

Such that:

- G has a free and transitive action on $\pi^{-1}(x), x \in M$. Hence, $\pi^{-1}(x) \cong G$,
- $P/G \cong M$.

Fiber bundle structure of Ashtekar variables

Orthonormal frame bundle

$$P^{SO}(M) \downarrow M$$

A principal bundle appears from the tetrad formulation: orthonormal frame bundle

$$P_x^{SO}(M) = \{h: \mathbb{R}^n \to T_x M \mid h \;\; \text{o. p. isometry}\} = \{\text{collection of orthonormal frames in } T_x M\}$$

g metric on $M \iff P^{SO}(M)$

E.g. in n=4 the tetrad e^μ_α is $h(\varepsilon_\alpha) \doteq e_\alpha = e^\mu_\alpha \partial_\mu$ and so $g(e_\alpha, e_\beta) = < \varepsilon_\alpha, \varepsilon_\beta > = \eta_{\alpha\beta}$

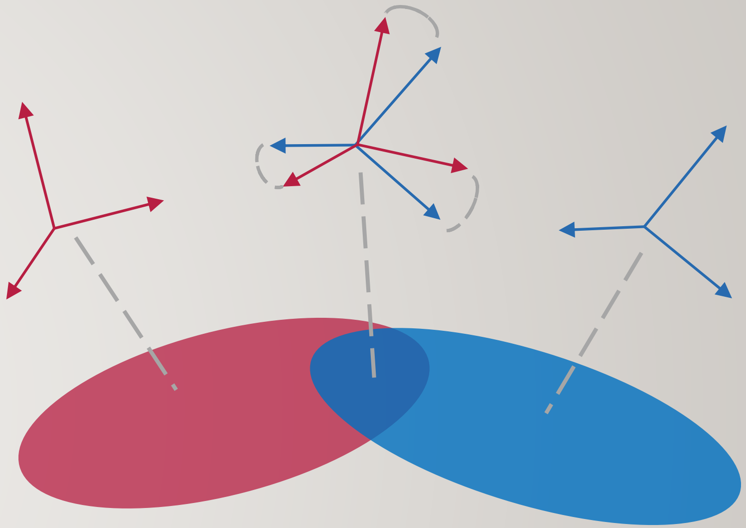
$\{\varepsilon_1, \varepsilon_2, \varepsilon_3, \varepsilon_4\}$ canonical basis of $\mathbb{R}^4$

Fiber bundle structure of Ashtekar variables

The dimension 3 - Dreibein

3-dimensional Riemannian manifold (Σ, q)

$$\begin{array}{c} P^{SO}(\Sigma) \\ \downarrow \\ \Sigma \end{array}$$



$P^{SO}(\Sigma)$ is a principal $SO(3)$-bundle $\Rightarrow$ Gauge group $SO(3)$

A choice of a triad $e_i^a(x)$ is equivalent to the choice of a section $e: \Sigma \to P^{SO}(\Sigma)$, i.e. $\pi(e_x) = x$

$$e_x(\varepsilon_i) = e_i^a(x)\partial_a$$

Fiber bundle structure of Ashtekar variables

The dimension 3 – SU(2) appears

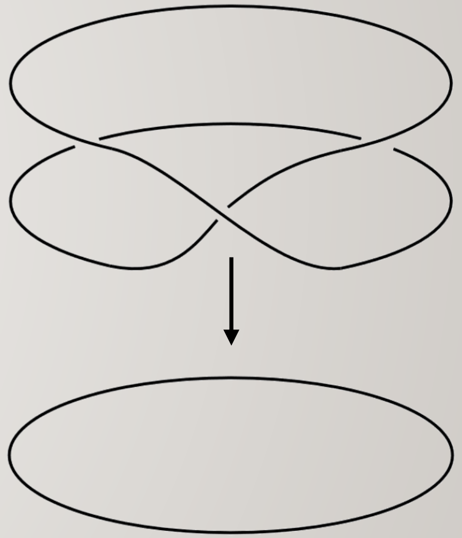
3-dimensional Riemannian manifold (Σ, q)

$$P^{SO}(\Sigma) \\ \downarrow \\ \Sigma$$

Spin Structure

$$P^{Spin}(\Sigma) \\ \downarrow \\ P^{SO}(\Sigma) \\ \downarrow \\ \Sigma$$

- Principal $SU(2)$-bundle $P^{Spin}(\Sigma)$
- Double-covering $\rho: P^{Spin}(\Sigma) \to P^{SO}(\Sigma)$



Lift of the dreibein $\bar{e}$ s.t. $e = \rho \circ \bar{e}$ (It is not unique, but it does not matter)

Fiber bundle structure of Ashtekar variables

The dimension 3 – Connection

3-dimensional spin manifold (Σ, q)

$$P^{Spin}(\Sigma) \\ \downarrow \\ P^{SO}(\Sigma) \\ \downarrow \\ \Sigma$$

A spin connection is represented by a 1-form ω with values in $\mathfrak{su}(2)$ on $P^{Spin}(\Sigma)$

$$\Updownarrow \quad \omega = \rho^* \varpi$$

A metric-compatible connection is represented by a 1-form ϖ with values in $\mathfrak{so}(3)$ on $P^{SO}(\Sigma)$

Ashtekar connection: Local field of a spin connection $A = \bar{e}^* \omega$ is a 1-form with values in $\mathfrak{su}(2)$ on Σ

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The dimension 3 – Electric field

3-dimensional spin manifold (Σ, q)

$$\begin{array}{c} P^{Spin}(\Sigma) \\ \downarrow \\ P^{SO}(\Sigma) \\ \downarrow \\ \Sigma \end{array}$$

Associated vector bundle $ad^* P^{Spin}(\Sigma)$. Fiber is $\mathfrak{su}(2)^*$ equipped with the coadjoint action of $SU(2)$

The **electric field** E can be interpreted as a 2-form on Σ with values in $ad^* P^{Spin}(\Sigma)$. Described locally by:

$$E = \star\, e_a^i dx^a \tau_i$$

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The Equivalence

3-dimensional spin manifold (Σ, q)

$$P^{Spin}(\Sigma) \downarrow P^{SO}(\Sigma) \downarrow \Sigma$$

ADM data:

- Symmetric 2-tensor K
- Two equations:
  1. Codazzi equation in Ricci flat spacetime $\mathcal{D}(q, K) = 0$
  2. Gauss equation in Ricci flat spacetime $\mathcal{H}(q, K) = 0$

Ashtekar connection A Dreibein e	Symmetric 2-tensor K
with equations	*with equations*
$G(A, E) = 0$	
$\mathcal{V}(A, E) = 0$	$\mathcal{D}(q, K) = 0$
$\mathcal{S}(A, E) = 0$	$\mathcal{H}(q, K) = 0$

Necessary and sufficient conditions for K to be the extrinsic curvature of an embedding of Σ in a vacuum Universe

Fiber bundle structure of Ashtekar variables

The Constraints

3-dimensional spin manifold (Σ, q)

$$P^{Spin}(\Sigma) \downarrow P^{SO}(\Sigma) \downarrow \Sigma$$

Reconstruction

$$K = "A - \Gamma"$$

$$\Gamma = e^* \varpi_{LC}$$

Gauss constraint

$$G(A, E) = d_A E$$

Symmetry of K

Diffeomorphism constraint

$$\mathcal{V}(A, E) = d_A^2(\star E)$$

Equiv to supermomentum

Scalar constraint

$$\mathcal{S}(A, E) = [E \wedge \star F] + \cdots$$

Equiv to superhamiltonian

Cosmological sector of Loop Quantum Gravity

A Yang-Mills approach

[Brodbeck '96]
[Bojowald, Kastrup '00]
[MB '24]
[MB '24]

$$P^{Spin}(\Sigma) \downarrow P^{SO}(\Sigma) \downarrow \Sigma$$

Yang-Mills variables

Connection ω is a 1-form on $P^{Spin}(\Sigma)$ with value in the Lie algebra of SU(2)

Dreibein e is a section in $P^{SO}(\Sigma)$

Ashtekar variables

Connection A is the local field $A = e^*\omega$

Electric field E is built from the dreibein $E = \star\, e_a^i dx^a \tau_i$

The problem is to find the cosmological sector of those variables

Cosmological sector of Loop Quantum Gravity

Cosmological hypothesis

Homogeneity: a group S acts transitively and freely on $\Sigma \Rightarrow \Sigma \cong S$

A metric q on S is homogeneous if it is invariant under the action of S, $L_g^* q = q$

$P^{SO}(S)$ is homogeneous if it is S-invariant, i.e. there exists an action ϕ of S via automorphisms s.t.

$$\pi \circ \phi(g) = L_g$$

There exists a unique(!) homogeneous spin structure on S

Homogeneity condition for Ashtekar connection from Wang's theorem $\phi(g)^* \omega = \omega, \ \forall g \in S$

(classified by linear maps $\Lambda: \mathfrak{s} \to \mathfrak{su}(2)$)

The request of homogeneity for ω yields to a homogeneous geometry for Σ

Cosmological sector of Loop Quantum Gravity

Quantum states

Configurational space $\mathcal{A} = \{A \mid A = e^*\omega, \omega \text{ homogeneous}\}$

The set of constraints are the same of LQG

Spin-network states as cylindric functions on $\mathcal{A}$

An important property of curves on S:

every curve can be approximated by piecewise integral curves of invariant vector field

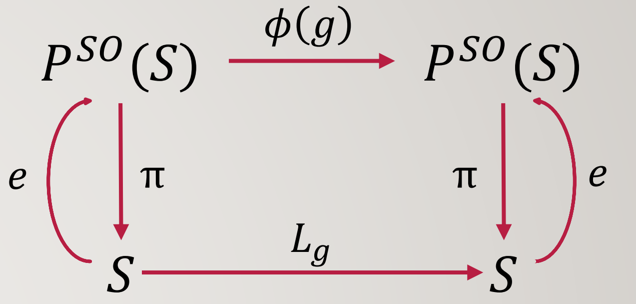
The homogeneous graphs are dense in the set of graphs

Cosmological sector of Loop Quantum Gravity

Invariant spin network

Equation of parallel transport along a homogeneous curve in a homogeneous gauge

$$\dot{u}(t) = \Lambda(v)u(t)$$



$$\phi(g) \circ e = e \circ L_g$$

Holonomy $h_c(A) = u(1)^{-1} = \exp\big(\Lambda(v)\big)$

The holonomies brought by invariant spin-network states are pointwise holonomies

Conclusions

- Ashtekar-Barbero-Immirzi formulation has a rigourous and clear geometric interpretation:

    1. The data are encoded in a spin connection A and a section (gauge) e
    2. The constraints have the same form (but different interpretation) of Yang-Mills' ones

- In this formulation we can find a cosmolological sector using the Wang's theorem

- We are able to perform the loop quantization of that sector and spin networks naturally arise with properties analogous to the usual cosmological states:

    1. the spin networks are homogeneous, namely the curves of the graph are integral curves of invariant vector fields
    2. the invariant states bring pointwise holonomies

Thank you for your attention

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CGM Research Group