Does no-hair theorem fail in the modified theories of gravity?

Semin Xavier
Department of physics, IIT Bombay

collaboration with
Alan Sunny, S. Shankaranarayanan
Class.Quant.Grav. 41 (2024) 13, 135002(arXiv:2303.04684)

Seventeenth Marcel Grossmann Meeting
(Talk in the parallel session)

12 July 2024

Modified theories of gravity vs GR

- Vacuum GR

$$\mathcal{L}_{EH} = R$$

- Spherically symmetry
  - Birkhoff's theorem;
    e.g. Schwarzschild black hole
- Axial symmetry;
  - No-hair theorem;
    e.g. Kerr black hole

Schwarzschild black hole

$$R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}\Big|_{r_H=2GM} = \frac{12}{r_H^4}$$

- Modified action

$$\mathcal{L}_{MG} = R + \alpha R^2 + \beta R_{\mu\nu}R^{\mu\nu} + \gamma R_{\mu\nu\sigma\lambda}R^{\mu\nu\sigma\lambda}$$

- E.O.M are complex
- ~~Birkhoff's theorem~~
  - Particular $f(R)$ [SX et al.,CQG-225006]
- No-hair theorem....?

$f(R)$ model and solution

Modified Einstein-Hilbert action:

$$S = \frac{1}{16\Pi G} \int d^4x \sqrt{-g} f(R) \quad f(R) = (\alpha_0 + \alpha_1 R)^p \quad \alpha_0, \alpha_1 > 0 \quad p > 1. \tag{1}$$

- Case $p = 2$
- Line-element up to $\mathcal{O}(\chi^2)$ ($\chi = a/m$, spin parameter)

$$\begin{aligned} ds^2 = & -e^{\mu(\rho)} \left[U(\rho) + \chi^2 V(\rho) \cos^2\theta\right] d\tau^2 + \frac{1}{U(\rho)} \left[1 - \chi^2 \left(\frac{V(\rho)}{U(\rho)} + \frac{\sin^2\theta}{\rho^2}\right)\right] d\rho^2 \\ & - 2\chi V(\rho) \rho^2 \sin^2\theta \, d\tau \, d\phi + \left[\rho^2 + \chi^2 \cos^2\theta\right] d\theta^2 + \rho^2 \sin^2\theta \left[1 + \frac{\chi^2}{\rho^2} + \chi^2 V(\rho) \sin^2\theta\right] d\phi^2 \end{aligned} \tag{2}$$

$g_{\phi\phi} > 0 \implies V(\rho) > 0$. $\tau = t/M, \rho = r/M$.

[Hartle & Thorne(1968)]

Slowly rotating Kerr metric

The slowly rotating approximation upto quadratic order of χ;

$$
\begin{aligned}
ds^2 \approx & -\left(1-\frac{2}{\rho}+\frac{2\chi^2\cos^2\theta}{\rho^3}\right)d\tau^2-\frac{4\chi\sin^2\theta}{\rho}d\tau d\phi \\
& +\left[\frac{1}{1-\frac{2}{\rho}}-\frac{\chi^2}{\rho^2\left(1-\frac{2}{\rho}\right)^2}\left(1-\left[1-\frac{2}{\rho}\right]\cos^2\theta\right)\right]d\rho^2+\left(\rho^2+\chi^2\cos^2\theta\right)d\theta^2 \\
& +\left[\rho^2+\chi^2\left(1+\frac{2}{\rho}\sin^2\theta\right)\right]\sin^2\theta d\phi^2 \qquad (3)
\end{aligned}
$$

Slowly-rotating black hole solution (SRBH) in $f(R)$

Vacuum $f(R)$ field equations

$$f^{'}(R)R_{\mu\nu} - \frac{1}{2}f(R)g_{\mu\nu} - \nabla_\mu\nabla_\nu f^{'}(R) + g_{\mu\nu}\Box f^{'}(R) = 0 \tag{4}$$

Simplifying field equations;

$$\frac{W_0(\rho)}{4\,\alpha_1^2\,\rho^5\,U(\rho)\,[\Phi(\rho)-2]^2\,[\Phi(\rho)+4]^2}\,\chi^2 + \frac{W_1(\rho)\,W_2(\rho)}{2\,\alpha_1^2\,\rho^3\,U(\rho)\,[\Phi(\rho)-2]\,[\Phi(\rho)+4]} = 0 \tag{5}$$

where, $\Phi(\rho) = \rho[\mu'(\rho) + [\ln\, U(\rho)]']$.

- $W_1(\rho) = \mathcal{O}[U(\rho), \mu(\rho)]$ and $W_2(\rho) = \mathcal{O}[U(\rho), \mu(\rho)]$
- $W_0(\rho) = \mathcal{O}[U(\rho), \mu(\rho), V(\rho)]$
- Evaluate $U(\rho)$ and $\mu(\rho)$ by solving $W_1(\rho) = 0$ or $W_2(\rho) = 0$ *without the knowledge of* $V(\rho)$

Details...

Slowly-rotating black hole solutions (SRBH) in $f(R)$

- The solutions are ($W_2(\rho) = 0$);

$$U(\rho) = 1 - \frac{C_1}{\rho} + \frac{C_0}{\rho^2} + \frac{\rho^2}{12\kappa^2}, \quad C_0 = 0, \quad \kappa = \alpha_0/\alpha_1 \tag{6}$$

$$\frac{\mu(\rho)}{2} = \ln\left[1 + N_1\left(\int \rho^{-1/2}\left(\rho^3 + 12\,\rho\,\kappa^2 - 12\,C_1\,\kappa^2\right)^{-3/2} d\rho\right)\right], \quad N_1 < 0 \tag{7}$$

$$V(\rho) = \sum_{j=0}^{\infty} \frac{C_{j+2}}{\rho^{j+2}} \tag{8}$$

where C_1, C_0, N_1 and C_{j+2} are integration constants.

- Multiple slowly-rotating black hole solutions (SRBH) for vacuum

Comparing asymptotically non-flat SRBH in GR

Asymptotically non-flat in GR

- $U(\rho) = 1 - \frac{C_1}{\rho} + \frac{\rho^2}{12\kappa^2}$

  $\mu(\rho) = 0$

  $V(\rho) = \frac{\tilde{C}}{\rho^3}$

- Unique

Infinite non-flat SRBH in modified gravity

- $W_2(\rho) = 0$;
  - $U(\rho) = 1 - \frac{C_1}{\rho} + \frac{\rho^2}{12\kappa^2}$

    $\mu(\rho) \neq 0$

    $V(\rho) = \frac{C_{j+2}}{\rho^{j+2}} \quad j > 1$
  - Infinite choice for $U(\rho)$ and $\mu(\rho)$
- $W_1(\rho) = 0...?$
- Arbitrary $V(\rho)$

Properties of the SRBH solution, $C_3 \neq 0$ (Event horizon)

- Event horizon ($g^{\rho\rho} = 0$);

$$\rho_H = \frac{H^{2/3}\,\left(2\,\kappa^2\right)^{1/3} - 2^{5/3}\,\kappa^{4/3}}{H^{1/3}}\,; H = 2\sqrt{9 + 4\,\kappa^2} + 6 \tag{9}$$

- The ρ_H depends on κ^2 as compared to the SR Kerr. $\kappa^2 = 0.33$, matches to GR.*

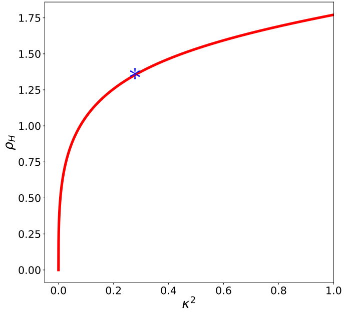


Figure 1: Plot of ρ_H as a function of κ^2 for $\chi = 0.1$ and $C_3 = 1$.

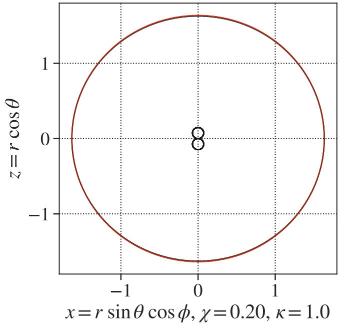
Properties of the SRBH solution, $C_3 \neq 0$ (Ergosphere)

- Ergosphere ($g_{\tau\,\tau} = 0$):

$$\frac{1}{12\,\kappa^2}\rho^5 + \rho^3 - 2\rho^2 + C_3\,\chi^2\,\cos^2(\theta) = 0 \tag{10}$$

The event horizon is very close to the ergosphere for SRBHs

- Red-curve $\rightarrow$ event-horizon, two black curves $\rightarrow$ ergosphere. Variation due to $\chi^2\,\cos\theta$ is visible in the inner part of the ergoregion



Properties of the SRBH solution, $C_3 \neq 0$ (Effective potential)

- The effective potential ($V_{\text{eff}} = 0 = \partial_r V_{\text{eff}}$) $\implies$ circular orbits in the equatorial plane. The circular orbits in the SRBH in $f(R)$ are smaller than Kerr.

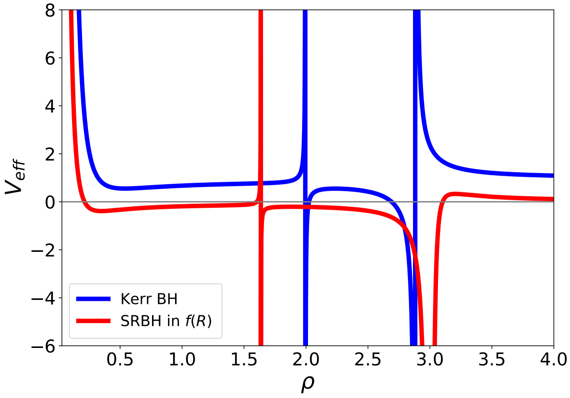


Figure 2: Plot of V_{eff} of a test particle in the equatorial plane for $\mu(\rho) = 0$.

Discussion and Conclusions

- For a class of $f(R)$ models, we obtained asymptotically non-flat, infinite number of SRBH solutions.
- Two branches ($W_1(\rho) = 0$ or $W_2(\rho) = 0$) of SRBH solutions and a collapse of a star might lead to a BH in either one of these branches.
- In GR, no-hair theorems concern mostly black hole solutions with flat asymptotics. Situation can be different in the case of modified theories of gravity.
- Analyzed the kinematical properties of the SRBHs.
- Interested to know whether one can obtain non-singular BH solutions from the other branch $W_1(\rho) = 0$.

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Thank You

$f(R)$ model and solution EOM by eliminating terms greater than χ^2:

$$\mathcal{G}^{\tau}_{\tau} \equiv T_1[U(\rho), V(\rho), \mu(\rho)] \simeq T_1^{(0)}(\rho) + T_1^{(II)}(\rho,\theta)\chi^2 = 0 \tag{11a}$$

$$\mathcal{G}^{\rho}_{\rho} \equiv T_2[U(\rho), V(\rho), \mu(\rho)] \simeq T_2^{(0)}(\rho) + T_2^{(II)}(\rho,\theta)\chi^2 = 0 \tag{11b}$$

$$\mathcal{G}^{\theta}_{\theta} \equiv T_3[U(\rho), V(\rho), \mu(\rho)] \simeq T_3^{(0)}(\rho) + T_3^{(II)}(\rho,\theta)\chi^2 = 0 \tag{11c}$$

$$\mathcal{G}^{\rho}_{\theta} \equiv T_4[U(\rho), V(\rho), \mu(\rho)] \simeq T_4^{(II)}(\rho,\theta)\,\chi^2 = 0 \tag{11d}$$

$$\mathcal{G}^{\tau}_{\phi} \equiv T_5[U(\rho), V(\rho), \mu(\rho)] \simeq T_5^{(I)}(\rho,\theta)\,\chi = 0\,, \tag{11e}$$

Features of field equations

1. T_1 and T_3 contain fourth-order derivatives of $U(\rho), \mu(\rho)$ and $V(\rho)$, while T_2 and T_4 contain third-order derivatives of $U(\rho), \mu(\rho)$ and $V(\rho)$. T_5 contains third-order derivatives of $U(\rho)$ and $\mu(\rho)$, and second-order derivative of $V(\rho)$.
2. T_1, T_2 and T_3 do not contain terms in first-order in χ, i.e. $T_i^{(0)}(\rho) + T_i^{(II)}(\rho)\chi^2 + \mathcal{O}(\chi^3)$, where $i = 1, 2$ and 3. Additionally, these three components' χ independent terms depend only on $U(\rho)$ and $\mu(\rho)$.
3. T_4 only contains second-order in χ, i. e. $T_4^{(II)}(\rho)\,\chi^2 + \mathcal{O}(\chi^3)$, while T_5 only contains first-order in χ, i. e, $T_5^{(II)}(\rho)\,\chi + \mathcal{O}(\chi^3)$.
4. It is possible to express $\mathcal{G}^{\phi}_{\tau}$ and $\mathcal{G}^{\theta}_{\rho}$ in terms of T_1, T_2, T_3, T_4 and T_5.
5. Since the equations of motion contain fourth-order derivatives of $U(\rho)$, $V(\rho)$, and $\mu(\rho)$, an exact solution will have four independent constants.

Slowly-rotating black hole space-time(SRBH)

back