Asymptotically Schwarzschild Solutions in f(R) Extension of General Relativity[1]

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[1] F. Scali and O. F. Piattella arXiv:2406.04417.

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Two scales two problems

- $f(R)$ challenges:

(i) provide a viable cosmological model[3] (radiation - matter - late times inflation).

(ii) Compatibility with solar system experiments and observations (red-shift, precession of Mercury etc.).

(!) $f(R) \to$ GR at solar system scales (not mandatory)

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- Idea:
  a) constrain $f(R)$ from solar system physics;
  b) then apply to Cosmology.

- Question: how can we build a $f(R)$-class compatible with the local tests, with no preliminary assumptions on f?

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Physical setting

- Spherically symmetric, static source

$$ds^2 = -e^{\nu(r)}dt^2 + e^{\mu(r)}dr^2 + r^2 d\Omega^2. \quad (1)$$

- Modified Einstein equations (EE) in vacuum ($\phi \equiv \frac{\partial f(R)}{\partial R}$):

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- Modified Einstein equations (EE) in vacuum ($\phi \equiv \frac{\partial f(R)}{\partial R}$):

$$\begin{aligned}
&\frac{2\phi}{r}\nu_r + \frac{2\phi}{r}\mu_r + \mu_r\phi_r + \nu_r\phi_r - 2\phi_{rr} = 0, \\
&\frac{2\phi}{r^2}e^{\mu} + \phi\nu_{rr} - \frac{\phi}{2}\nu_r\mu_r + \frac{\phi}{2}\nu_r^2 + \frac{\phi}{r}\nu_r + \frac{\phi}{r}\mu_r + \phi_r\nu_r - \frac{2\phi_r}{r} - \frac{2\phi}{r^2} = 0, \\
&e^{\mu}f(R) + \phi\nu_{rr} - \frac{\phi}{2}\nu_r\mu_r + \frac{\phi}{2}\nu_r^2 + \frac{2\phi}{r}\nu_r + \mu_r\phi_r - \frac{4\phi_r}{r} - 2\phi_{rr} = 0.
\end{aligned} \qquad (2)$$

Asymptotically Schwarzschild solutions

- Full compatibility with GR in the weak field limit ($r >> R_s$)

$$\begin{aligned} \phi(r) &= 1 + \sigma(r), & \lim_{r/R_s \to \infty} \sigma(r) &= 0 \\ \nu(r) &= \ln\left(1 - \frac{2MG}{r}\right) + g(r), & \lim_{r/R_s \to \infty} \frac{r}{R_s} g(r) &= 0, \\ \mu(r) &= -\ln\left(1 - \frac{2MG}{r}\right) + m(r), & \lim_{r/R_s \to \infty} \frac{r}{R_s} m(r) &= 0. \end{aligned} \tag{3}$$

- σ, g, m analytic + polar singularity in $r = 0$:

$$\begin{aligned} \sigma(r) &= \sum_{n=1}^{+\infty} \frac{\alpha_n^\sigma}{r^n} + \sum_{m=0}^{+\infty} \beta_m^\sigma r^m, \\ g(r), m(r) &= \sum_{i=2}^{+\infty} \frac{\alpha_i^{g,m}}{r^i} + \sum_{j=0}^{+\infty} \beta_j^{g,m} r^j, \end{aligned} \tag{4}$$

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Leading correction to Schwarzschild

- Question:

$$\phi = 1 + \frac{c_1}{r^n} + \mathcal{O}\left(\frac{1}{r^{n+1}}\right), \;\; n \geq 1, \;\; c_1 << R_s \times R_*^{n-1} \qquad (5)$$

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- Answer: Insert into the modified EE + Laurent expansion:

$$\begin{aligned} \nu(r) &= \ln\left(1 - \frac{2MG}{r}\right) - \frac{c_1}{r^n} + \mathcal{O}\left(\frac{1}{r^{n+1}}\right), \\ \mu(r) &= -\ln\left(1 - \frac{2MG}{r}\right) - n\frac{c_1}{r^n} + \mathcal{O}\left(\frac{1}{r^{n+1}}\right), \\ n &\geq 2. \end{aligned} \tag{6}$$

$$\Longrightarrow \boxed{ds^2 = ds^2_{\text{Schwarzschild}} - \frac{c_1}{r^n}(-dt^2 + ndr^2).} \tag{7}$$

Recovering $f(R)$

- Correction to the scalar curvature:

$$R = 3n(n-1)\frac{c_1}{r^{n+2}} + \mathcal{O}\left(\frac{1}{r^{n+3}}\right). \tag{8}$$

- Using $\phi(R(r)) = \frac{\partial f(R)}{\partial R} \implies$

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(i) $c_1 = [c_1] \times l^n$.

(ii) Not analytical in $R = 0$[4].

(iii) Physical domain: spherical symmetry, low curvature, some regularity of the metric at $r \to \infty$.

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PPN parameters

- Post Newtonian approximation[5] (isotropic coordinates):

$$ds^2 = -\left(1 - \frac{2MG}{\rho} + 2\beta\frac{(MG)^2}{\rho^2}\right)dt^2 + \left(1 + 2\gamma\frac{MG}{\rho}\right)(d\rho^2 + \rho^2 d\Omega^2),$$

- Schwarzschild: $\gamma = \beta = 1$.
- $f(R)$: $\begin{cases} n > 2 \implies \gamma = \beta = 1, \\ n = 2 \implies \gamma = 1,\ \beta = 1 - \frac{c_1^{(2)}}{2(MG)^2} \end{cases}$

[5] C. Misner, K. Thorne, J. Wheeler, and D. Kaiser, Gravitation (Princeton University Press, 2017).

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(i) Cassini bound[6]: $\gamma - 1 = (2.1 \pm 2.3) \times 10^{-5}$.

(ii) Precession of Mercury[7]: $\beta - 1 = (-4.1 \pm 7.8) \times 10^{-5}$.

$$\implies |c_1|^{(2)} \lesssim 1.2 \times 10^{-6}\ \text{mm}^2.$$

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Gravitational red-shift

$$z \equiv \frac{\nu_1 - \nu_2}{\nu_1} = 1 - \sqrt{\frac{1 - \frac{2MG}{r_1}}{1 - \frac{2MG}{r_2}} + \frac{1}{2}\frac{c_1}{r_1^n}\left(1 - \frac{r_1^n}{r_2^n}\right)}. \qquad (10)$$

- Sunlight red-shift[8] $z = \frac{1}{c}(638 \pm 6) ms^{-1}$, GR contribution $\sim \frac{1}{c} 633 ms^{-1}$

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$$\Longrightarrow \frac{1}{2}\frac{|c_1|}{r_1^n}\left(1 - \frac{r_1^n}{r_2^n}\right) \lesssim 1\frac{ms^{-1}}{c}.$$

n	$\sqrt[n]{\frac{\|c_1\|}{(2MG)^n}}$
2	19.14
3	441.24
4	2118.45

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Cosmology

$$f(R) = R + \frac{1}{2}|c_1|^{\frac{2}{n+2}} \frac{n+2}{(n+1)(3n^2-3n)^{\frac{n}{n+2}}} |R|^{2\frac{n+1}{n+2}}.$$

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• Add Cosmological Constant (embed in Schwarzschild-de Sitter background)

$$\implies f(R) = R - 2\Lambda + \frac{1}{2}|c_1|^{\frac{2}{n+2}} \frac{n+2}{(n+1)(3n^2-3n)^{\frac{n}{n+2}}} |R-4\Lambda|^{2\frac{n+1}{n+2}}.$$

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(!) Able to produce late times inflation (but not very useful).

• Correct use: constrain exact $f(R)$'s working in Cosmology (local limit).

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Future research

- Change symmetries (same $f(R)$ structure?).
- Stability analysis[10].
- Use the modified Newtonian potential $\rightarrow$ application to galaxies.
- Apply same bottom-up approach in Cosmology, check consistency with local $f(R)$.

[10] M. D. Seifert, Physical Review D 76, (2007).

Backup 1: Brans-Dicke

- $S = \frac{1}{2k} \int_{\mathcal{M}} d^4x \sqrt{-g}(\phi R - V(\phi))$

$$\implies \nabla^2\phi - \frac{dV_{eff}(\phi)}{d\phi} = 0, \quad \frac{dV_{eff}(\phi)}{d\phi} \equiv \frac{1}{3}\left(\phi \frac{dV(\phi)}{d\phi} - V(\phi)\right).$$

- $f(R) = R + \alpha |R|^k, \;\; \alpha > 0, \;\; k > 1,$

$$\implies V_{eff}(\phi) = \frac{\alpha}{3}\frac{k-1}{2k-1}\left(\frac{|\phi - 1|}{\alpha k}\right)^{\frac{k}{k-1}} [(2-k)\phi + 3(k-1)]. \quad (11)$$

(i) $k = 2$, Starobinsky, global minimum at $\phi = 1$.

(ii) $1 < k < 2$, local minimum at $\phi = 1$, unbounded for $\phi \to -\infty$

(iii) $k > 2$, local minimum at $\phi = 1$, unbounded for $\phi \to +\infty$

(!) Our case $\frac{3}{2} < k < 2$.

Backup 2: gravitational lensing

- $f(R)$ contribution to gravitational lensing

$$\delta\alpha|_{f(R)} = \frac{1}{2}\frac{c_1}{r_0^n}\frac{\sin(\alpha)}{\cos(\alpha)}(1-\sin^n(\alpha)).$$

$\delta\alpha$ deflection angle at Earth position.

α angle between apparent direction of the star and the Sun.

r_0 distance of closest approach.

(!) Zero for $\alpha = 0, \pi, \frac{\pi}{2}$.

Backup 3: Shapiro delay

- $f(R)$ contribution to the Shapiro delay

$$2\Delta\tau(r_E, r_2)|_{f(R)} = \frac{c_1}{r_0^{n-1}}\frac{r_0}{r_E}\left[\frac{1-2\left(\frac{r_0}{r_E}\right)^{n-2}+\left(\frac{r_0}{r_E}\right)^{n}}{\sqrt{1-\left(\frac{r_0}{r_E}\right)}} \pm (r_e \to r_2)\right].$$

$\Delta\tau$ contribution to proper time elapsed on Earth.

r_0 distance of closest approach.

r_E Earth radial position.

Backup 4: precession of Mercury

- $f(R)$ contribution to the Precession of Mercury ($\varphi(a) = 0$)

$$\varphi(b)|_{f(R)} = \frac{\pi}{4M_{\odot}G}\frac{c_1}{a^{n-1}}(1+k)\sum_{j=0}^{n-2}\binom{n}{j+2}\frac{(2j+1)!}{(j!)^2}\left(\frac{k-1}{4}\right)^j .$$

$\varphi(b)$ contribution to angle swept at the aphelion.

a perihelion.

k $\equiv \frac{a}{b}$.