4D-TExS: A new 4D lattice-QCD equation of state with extended density coverage

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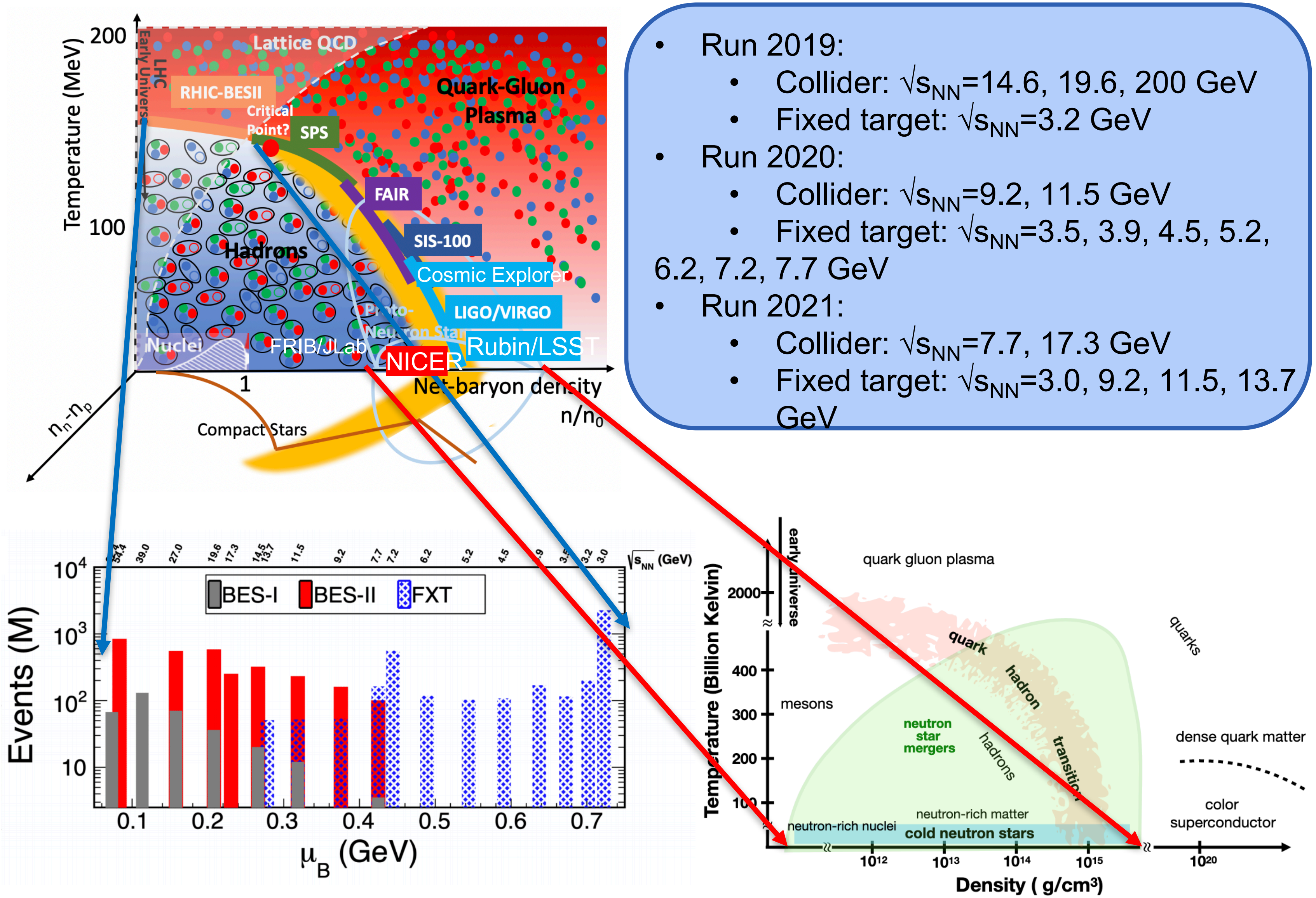
Motivation

- Is there a critical point on the QCD phase diagram?
- What are the degrees of freedom in the vicinity of the phase transition?
- Where is the transition line at high density?
- What are the phases of QCD at high density?
- What is the nature of matter in the core of neutron stars?



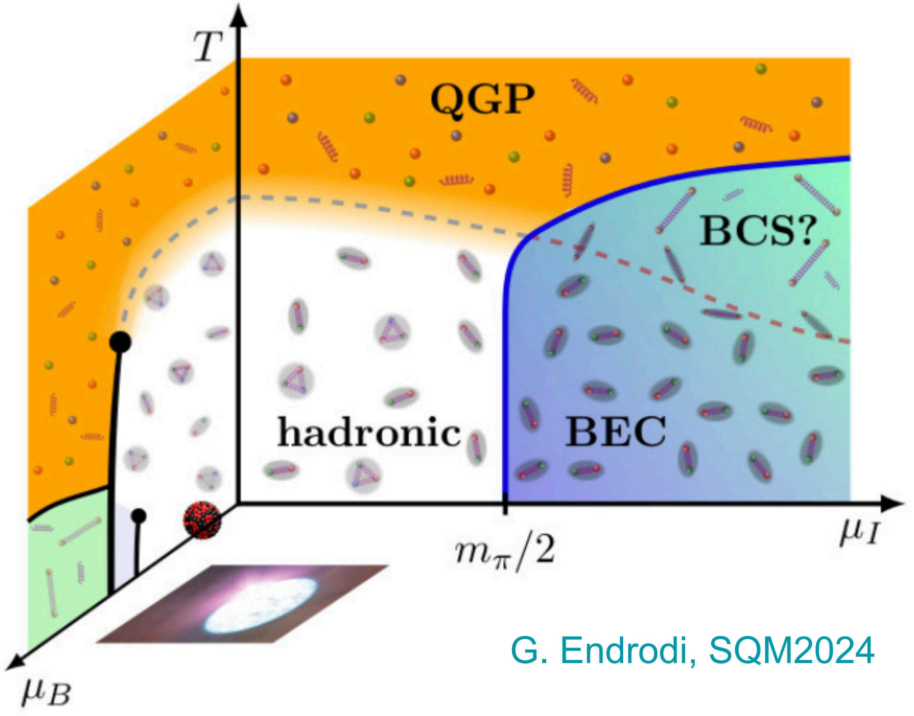
- Run 2019:
  - Collider: $\sqrt{s_{NN}}$=14.6, 19.6, 200 GeV
  - Fixed target: $\sqrt{s_{NN}}$=3.2 GeV
- Run 2020:
  - Collider: $\sqrt{s_{NN}}$=9.2, 11.5 GeV
  - Fixed target: $\sqrt{s_{NN}}$=3.5, 3.9, 4.5, 5.2, 6.2, 7.2, 7.7 GeV
- Run 2021:
  - Collider: $\sqrt{s_{NN}}$=7.7, 17.3 GeV
  - Fixed target: $\sqrt{s_{NN}}$=3.0, 9.2, 11.5, 13.7 GeV

Equation of state in 4D

- QCD has three conserved charges: Baryon Number B, Strangeness S and Electric Charge Q (or Isospin I)

- Heavy-ion collisions
  - Global S=0
  - Global Q=0.4B
  - Local fluctuations in the fluid cells with finite S and Q≠0.4B possible

- Neutron Stars
  - Global Q=0 for stability
  - Strangeness is most likely not in equilibrium
  - Finite isospin density



G. Endrodi, SQM2024

- Cosmological trajectories
  - Large lepton flavor asymmetries possible → large asymmetries between quark flavors (lead to finite B, Q, S values)
  - How would the critical point move in the 4D phase diagram?
  - First-order cosmological phase transition could lead to stable strange quark matter droplets + gravitational waves similar to those observed recently by NANOGrav

Gao & Oldengott, PRL (2022)

A. R. Bodmer, PRD (1971); E. Witten, PRD (1984); F. Di Clemente, C. R. et al, 2404.12094

See talk by Francesco Di Clemente today at 5 p.m.

Lattice QCD EoS in 4D from Taylor expansion

- Among the different ways to calculate the EoS of nuclear matter, lattice QCD is the most accurate way to get thermodynamics directly from first principles

- To reach finite density, one can expand using Taylor series to circumvent the fermion sign problem

J. Noronha-Hostler, C. R. et al., PRC (2019); A. Monnai et al., PRC (2019)

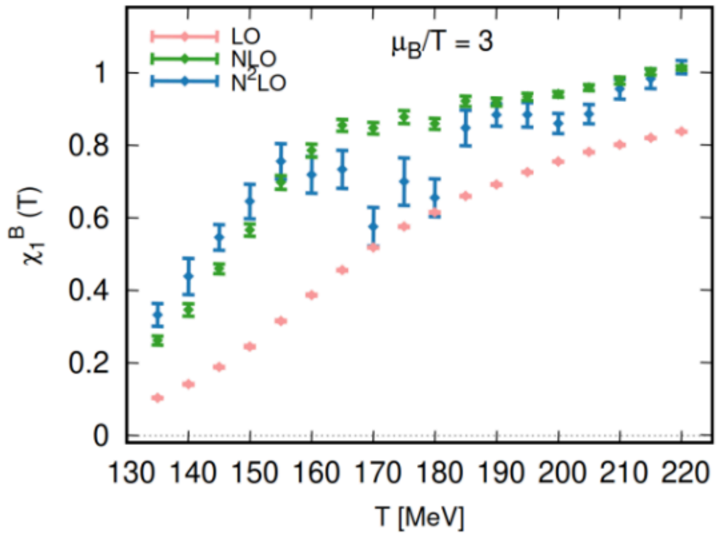
$$\frac{P(T,\hat{\mu}_B,\hat{\mu}_S,\hat{\mu}_Q)}{T^4} = \sum_{i,j,k} \frac{1}{i!j!k!} \chi_{ijk}^{BSQ}(T)\, \hat{\mu}_B^i\, \hat{\mu}_Q^j\, \hat{\mu}_S^k \qquad \left(\text{with } \hat{\mu}_i = \frac{\mu_i}{T}\right)$$

where

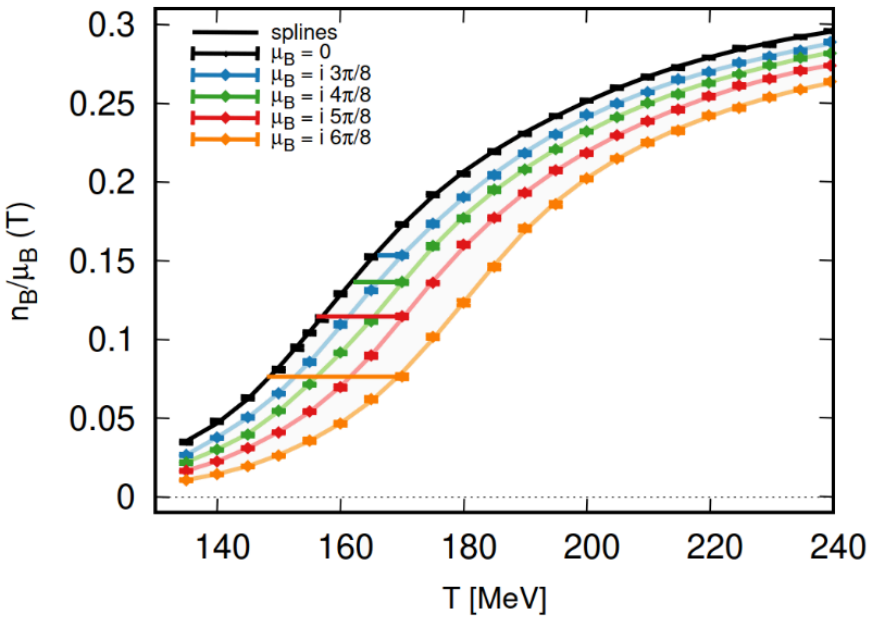
$$\chi_{ijk}^{BQS}(T) = \left.\frac{\partial^{i+j+k}(P/T^4)}{\partial \hat{\mu}_B^i\, \hat{\mu}_S^j\, \hat{\mu}_Q^k}\right|_{\hat{\mu}_B,\hat{\mu}_S,\hat{\mu}_Q=0}$$

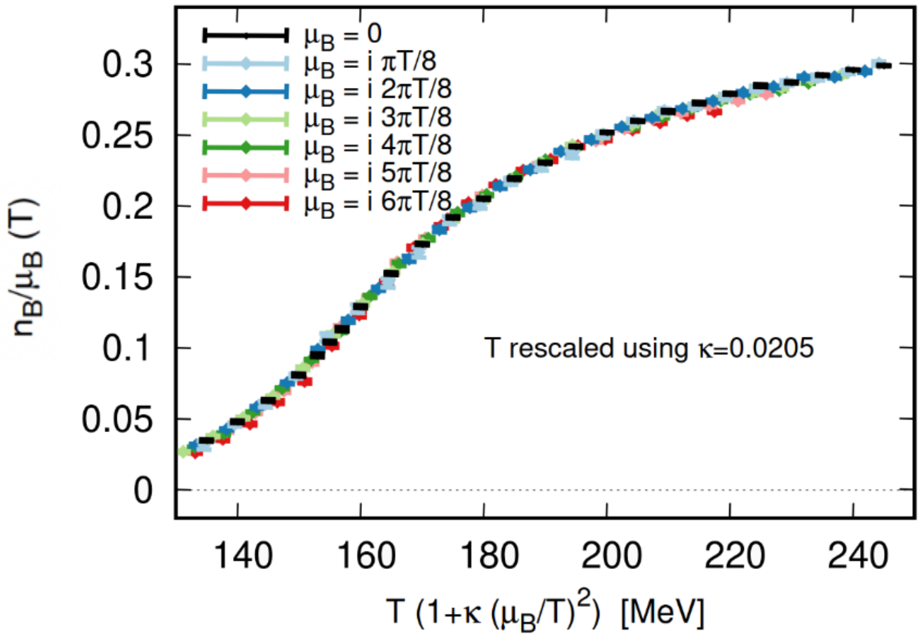
- Problems with Taylor:
  - ➢ Still limited to $\mu_i/T<2.5$
  - ➢ Large errors on higher order terms
  - ➢ Wiggles on Taylor coefficients reflected on observables

S. Borsanyi, C. R. et al., PRL (2021)



A novel expansion scheme at finite μ_B





Simulations at $\mathrm{Im}(\hat{\mu}_B)$: T-dependence of normalised baryon density $\left(\chi_1^B = n_B/T^3\right)$ at finite $\hat{\mu}_B$ appears to be shifted from the value at $\hat{\mu}_B = 0$.

For the 0/0 limit, we have: $\frac{\chi_1^B(T,\hat{\mu}_B)\to 0}{\hat{\mu}_B\to 0} \to \frac{\partial \chi_1^B}{\partial \hat{\mu}_B} = \chi_2^B$

S. Borsanyi, C. R. et al., PRL (2021)

Main identity: $$\frac{\chi_1^B(T,\hat{\mu}_B)}{\hat{\mu}_B} = \chi_2^B(T',0)$$

with $\mathbf{T'(T,\hat{\mu}_B) = T\left(1 + \kappa_2\,.\,\hat{\mu}_B^2 + \kappa_4\,.\,\hat{\mu}_B^4 + \dots\right)}$

captures the finite $\hat{\mu}_B$ dependence of the expansion

T'-Expansion Scheme at finite μ_B

New *TExS EoS* based on coefficients $\kappa_{2/4}^{BB}(T)$ evaluated directly from lattice QCD simulations at $\mu_B = 0$

$$T'(T,\mu_B) = T\left(1 + \kappa_2^{BB}(T)\hat{\mu}_B^2 + \kappa_4^{BB}(T)\hat{\mu}_B^4 \ldots\right)$$

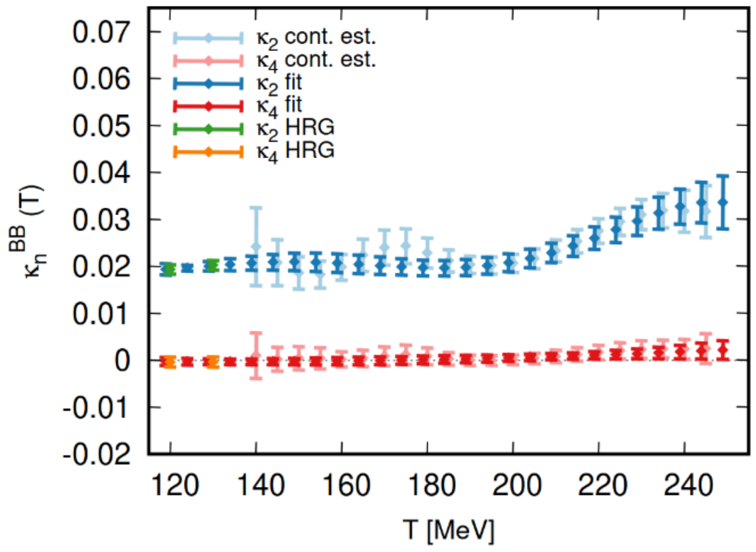
with coefficients $\kappa_i^{BB}(T)$ connected to Taylor coefficients $\chi_i^B(T)$:

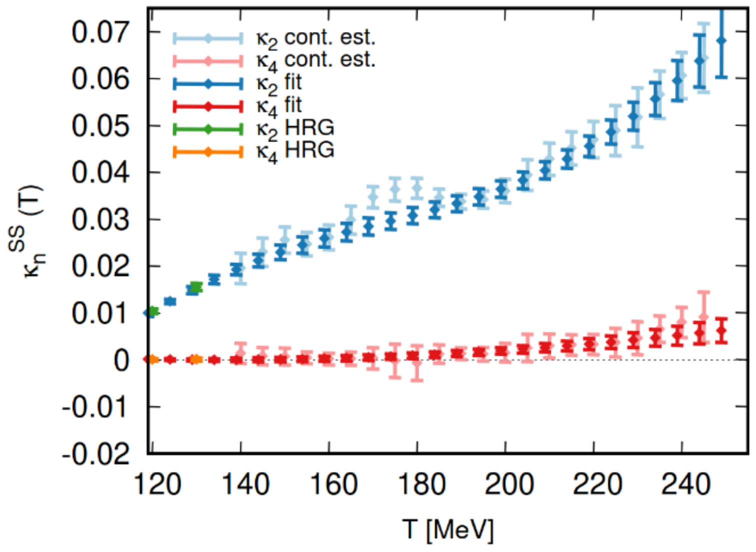
- $\kappa_2^{BB}(T,0) = \frac{1}{6T}\frac{\chi_4^B(T)}{\chi_2'^B(T)}$ with $\chi'(T) = \frac{\partial \chi(T)}{\partial T}$
- $\kappa_4^{BB}(T,0) = \frac{1}{360T \times \chi_2'^B(T)^3}\left(3\chi_2'^B(T) \times \chi_6^B(T) - 5\chi_2''^B(T) \times \chi_4^B(T)^2\right)$

$\Rightarrow$ Clear separation of scales between $\kappa_2(T)$ and $\kappa_4(T)$

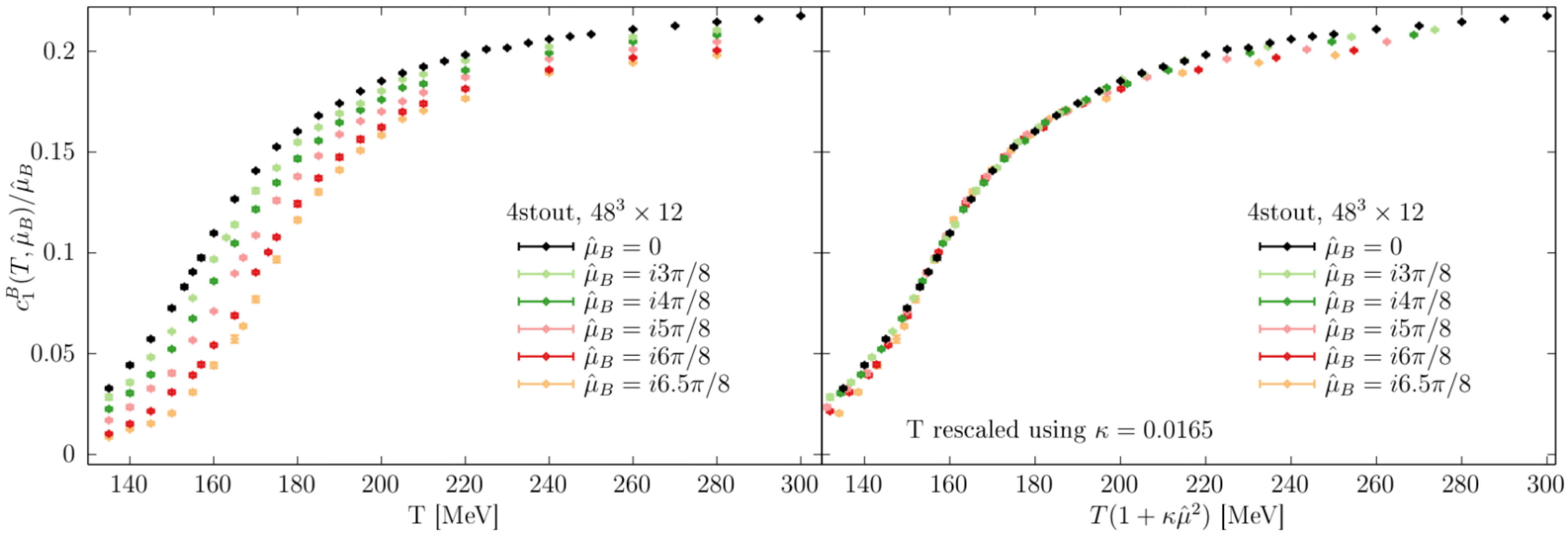
$\Rightarrow$ $\kappa_4(T)$ is almost 0 $\rightarrow$ faster convergence

S. Borsanyi, C. R. et al., PRL (2021)





Smooth connection to $T \to \infty$ limit

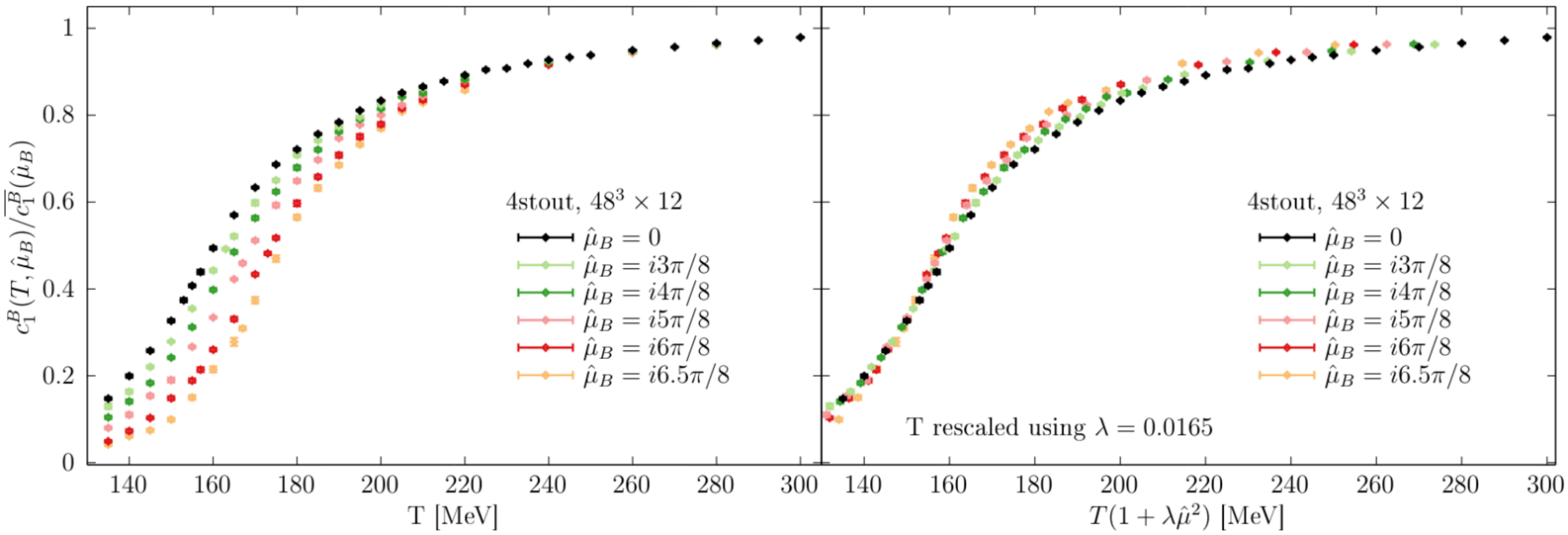


To ensure that our main identity holds when $T \to \infty$,
needs to normalise by Stefan-Boltzmann limits
$\overline{\chi}_1^B(\hat\mu_B)$ and $\overline{\chi}_2^B(0)$:

$$\frac{\chi_1^B(T,\hat\mu_B)}{\overline{\chi}_1^B(\hat\mu_B)} = \frac{\chi_2^B(T'(T,\hat\mu_B),0)}{\overline{\chi}_2^B(0)}$$

S. Borsanyi, C. R. et al., PRD (2022)

Smooth connection to $T \rightarrow \infty$ limit



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This leads to redefine:

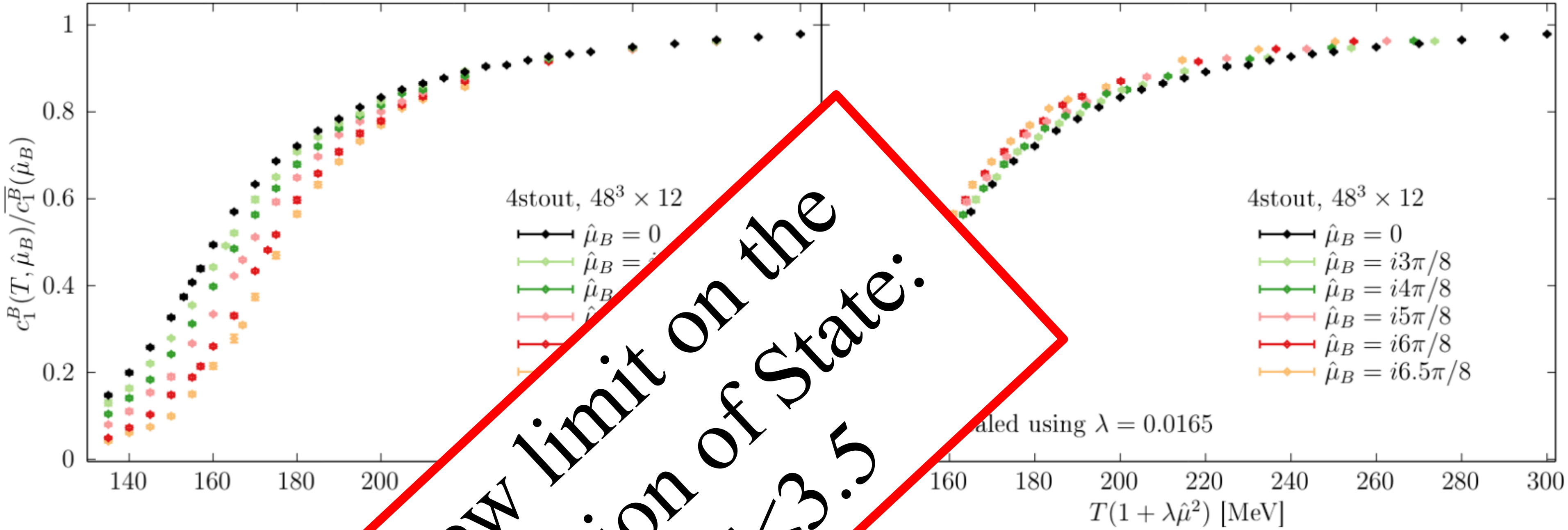
$$T'(T,\mu_B) = T\left(1+\lambda_2^{BB}(T)\hat{\mu}_B^2+\dots\right)$$

with the new expansion coef. embedding the S.B. limit:

$$\lambda_2^{BB}(T) = \frac{1}{6T\chi_2'^B(T)} \times \left(\chi_4^B(T) - \frac{\overline{\chi}_4^B(0)}{\overline{\chi}_2^B(0)}\chi_2^B(T)\right)$$

S. Borsanyi, C. R. et al., PRD (2022)

Smooth connection to $T \to \infty$ limit



New limit on the Equation of State: $\mu_B/T<3.5$

To ensure that our main ide... needs to normalise by Stefa... $\overline{\chi}_1^B(\hat{\mu}_B)$ and $\overline{\chi}_2^B(0)$

$$\frac{\chi_1^B(T,\hat{\mu}_B)}{\overline{\chi}_1^B(\hat{\mu}_B)} = \frac{\chi_2^B(T'(T,\hat{\mu}_B))}{\overline{\chi}_2^B(0)}$$

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S. Borsanyi, C. R. et al., PRD (2022)

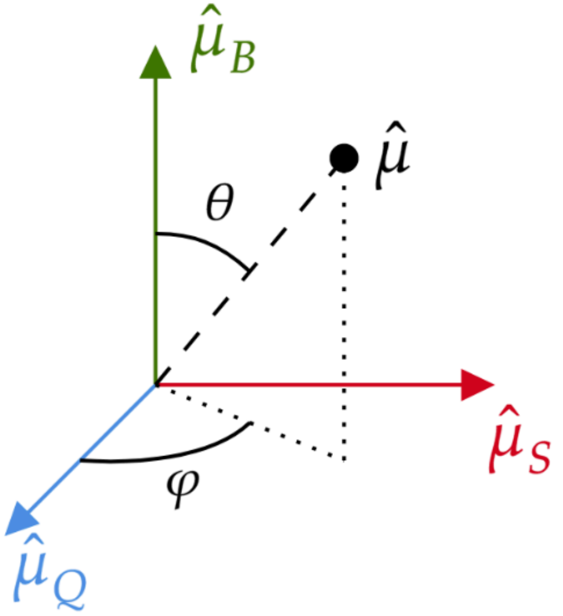
Construction of the new scheme - Basics

One can chose to project the $(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S)$ Cartesian coordinate system into a spherical one using $(\hat{\mu}, \theta, \varphi)$, following the relations:

$$\hat{\mu}_B = \hat{\mu}\,.\cos(\theta) \qquad \hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_Q^2 + \hat{\mu}_S^2}$$

$$\hat{\mu}_Q = \hat{\mu}\,.\sin(\theta)\cos(\varphi) \quad \Longleftrightarrow \quad \varphi = \arccos\left(\frac{\hat{\mu}_Q}{\sqrt{\hat{\mu}_Q^2 + \hat{\mu}_S^2}}\right)$$

$$\hat{\mu}_S = \hat{\mu}\,.\sin(\theta)\sin(\varphi) \qquad \theta = \arccos\left(\frac{\hat{\mu}_B}{\hat{\mu}}\right)$$



Simple way to **reduce** the problem **from 4D to 2D**: a single $\hat{\mu}$ projected along a given direction in the 3D $(\hat{\mu}_B, \hat{\mu}_Q, \hat{\mu}_S)$ space.

→ All previous equations from the 2D-*T*ExS can be used as is!

Redefinition of the susceptibilities

We introduce then X_2, a "generalised 2nd order susceptibility" at $\hat{\mu} = 0$:

$$X_2^{\theta,\varphi}(T) = \left.\frac{\partial^2 P/T^4}{\partial \hat{\mu}^2}\right|_{\hat{\mu}=0}$$

$$= c_\theta^2 \cdot \chi_2^B(T) + s_\theta^2 c_\varphi^2 \cdot \chi_2^Q(T) + s_\theta^2 s_\varphi^2 \cdot \chi_2^S(T)$$

$$+ 2c_\theta s_\theta c_\varphi \cdot \chi_{11}^{BQ}(T) + 2c_\theta s_\theta s_\varphi \cdot \chi_{11}^{BS}(T) + 2s_\theta^2 c_\varphi s_\varphi \cdot \chi_{11}^{QS}(T)$$

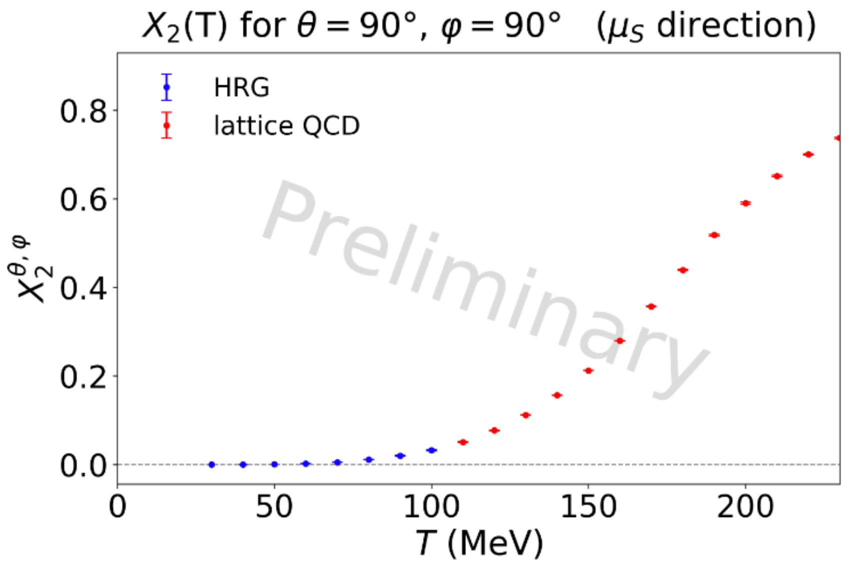
as a combination of the usual susceptibilities $\chi_{11/2}^{BQS}(T)$ at $\hat{\mu}_B = \hat{\mu}_S = \hat{\mu}_Q = 0$ computed from HRG (at low T) + lattice QCD.

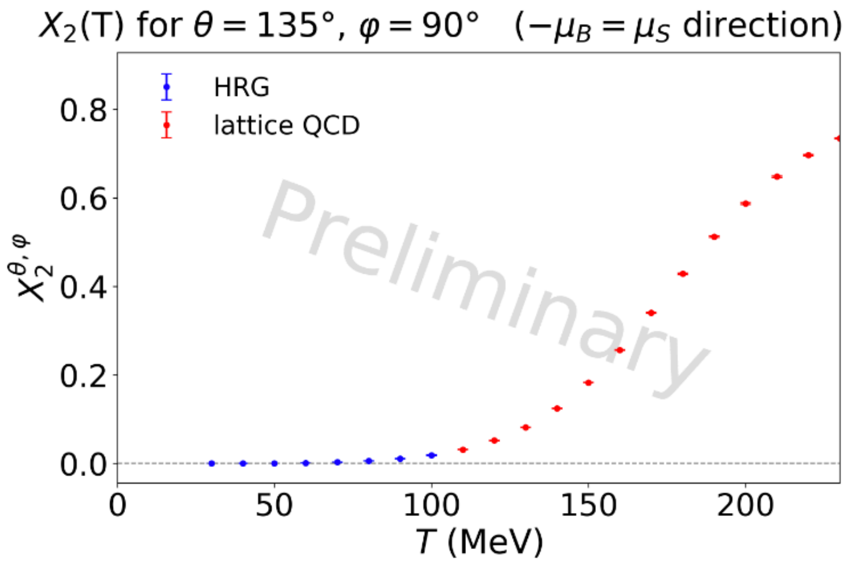
Examples:

- for $(\theta = 90°, \varphi = 90°)$, $\hat{\mu} = \hat{\mu}_S \leftrightarrow X_2 = \chi_2^S$
- for $(\theta = 135°, \varphi = 90°)$, $\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_S^2} \leftrightarrow X_2 = \frac{\chi_2^B}{2} + \frac{\chi_2^S}{2} + \chi_{11}^{BS}$

The same way, one obtains:

$$X_4^{\theta,\varphi}(T) = c_\theta^4 \cdot \chi_4^B(T) + s_\theta^4 c_\varphi^4 \cdot \chi_4^Q(T) + s_\theta^4 s_\varphi^4 \cdot \chi_4^S(T) + \ldots$$





New expansion scheme coefficients

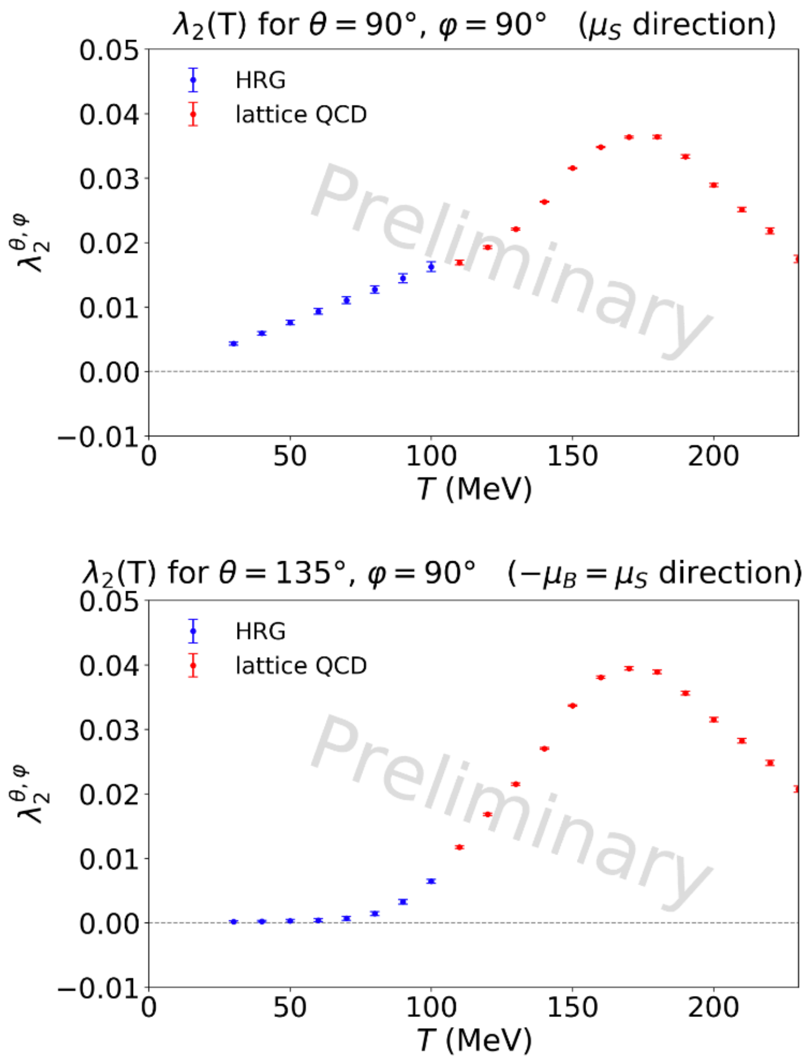
From there, we can build the generalised 2nd order expansion coefficient λ_2:

$$\lambda_2^{\theta,\varphi}(T) = \frac{1}{6T}\frac{1}{X_2'^{\theta,\varphi}(T)} \times \left(X_4^{\theta,\varphi}(T) - \frac{\overline{X}_4^{\theta,\varphi}(0)}{\overline{X}_2^{\theta,\varphi}(0)} X_2^{\theta,\varphi}(T) \right)$$

embedding the S.B. limit correction $\left(\overline{\lambda}_2 = \lim_{T\to\infty}(\lambda_2) = 0\right)$, with $\overline{X}_{2/4}^{\theta,\varphi}(0)$ being the S.B. limits for $X_{2/4}^{\theta,\varphi}(T)$ at $\hat{\mu} = 0$. We employ here the latest $\chi_{2/4}^{BQS}$ data from the WB collaboration.

Examples:

- *for* $(\theta = 90°, \varphi = 90°)$, $\hat{\mu} = \hat{\mu}_S$
- *for* $(\theta = 135°, \varphi = 90°)$, $\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_S^2}$



Generalized charge density X_1

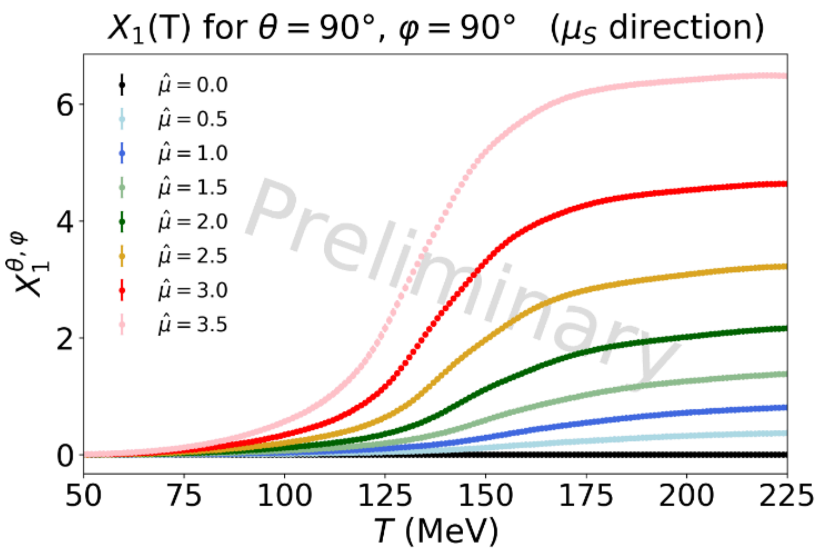
Using the previously obtained expansion coefficient $\lambda_2^{\theta,\varphi}(T)$, one can build the shifted temperature expansion $T'^{\theta,\varphi}(T,\hat{\mu})$:

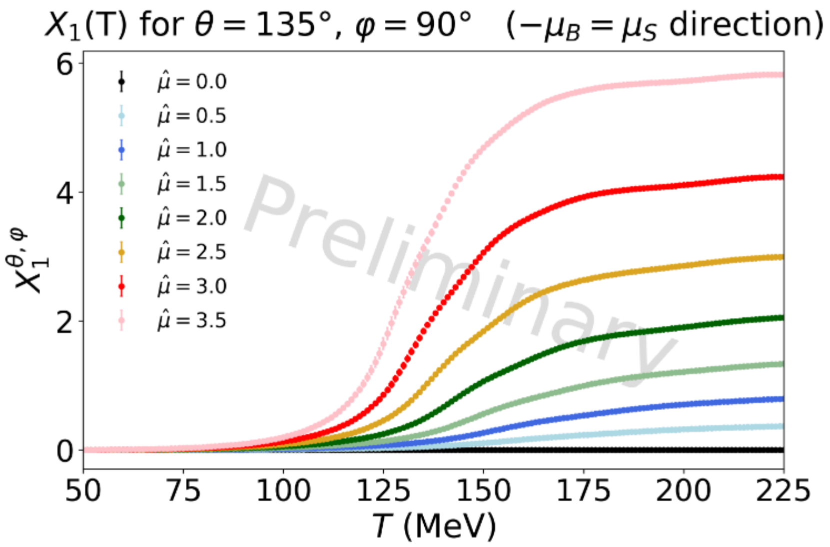
$$T'^{\theta,\varphi}(T,\hat{\mu}) = T\left(1+\lambda_2^{\theta,\varphi}(T)\hat{\mu}_B^2\right)$$

Then, using the TExS main identity, one can express the generalised charge density $X_1^{\theta,\varphi}$ at finite $\hat{\mu}$:

$$X_1^{\theta,\varphi}(T,\hat{\mu}) = \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} \times X_2^{\theta,\varphi}\left(T'^{\theta,\varphi}(T,\hat{\mu}),0\right)$$

where we compute $X_2^{\theta,\varphi}(T',0)$ using χ_2^{BQS} data from the Wuppertal-Budapest collaboration. [a, b, c]





Pressure

We integrate $X_1^{\theta,\varphi}(T,\hat{\mu})$ to compute the pressure:

$$P^{\theta,\varphi}(T,\hat{\mu}) = P(T,0) + \int_0^{\hat{\mu}} X_1^{\theta,\varphi}(T,\hat{\mu}')d\hat{\mu}'$$

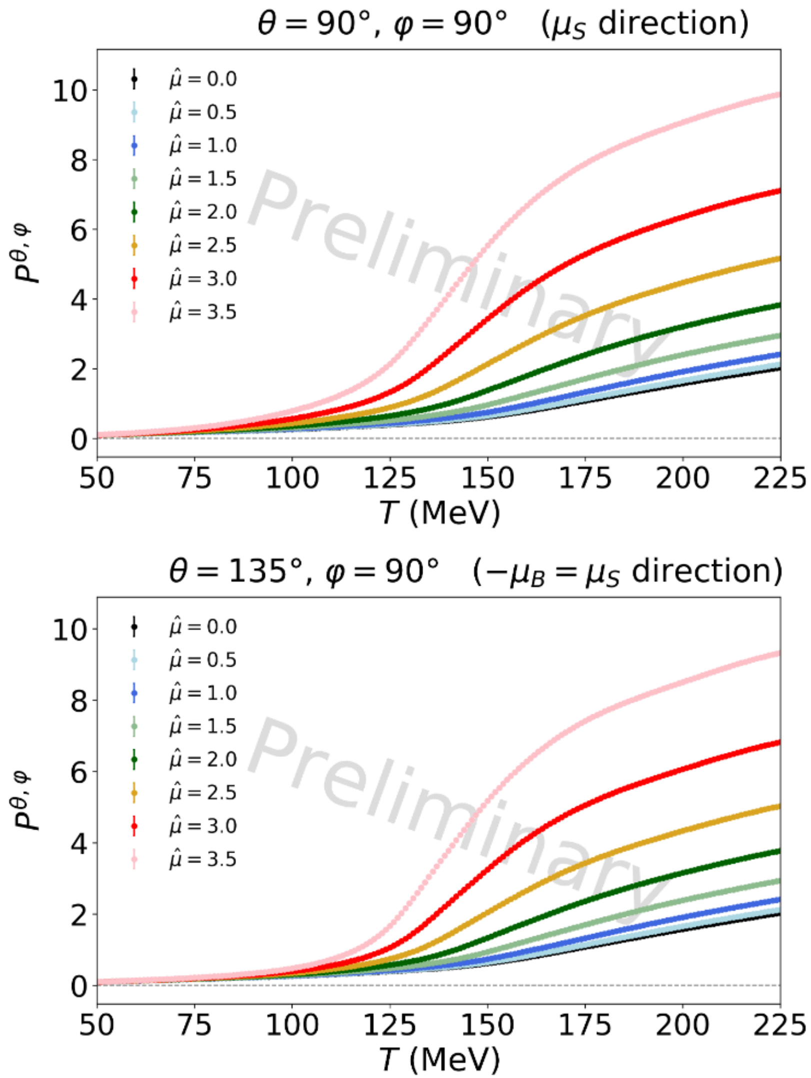
$$= P(T,\hat{\mu}_B,\hat{\mu}_Q,\hat{\mu}_S)$$

using lattice results for $P(T,0)$ with recent precision improvement from the Wuppertal-Budapest collaboration.[a]

Examples:

- for $(\theta = 90°, \varphi = 90°)$, $\hat{\mu} = \hat{\mu}_S$
- for $(\theta = 135°, \varphi = 90°)$, $\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_S^2}$

[a] P. Parotto, talk at QM 2023



Entropy density

The expression for entropy density s is given as:

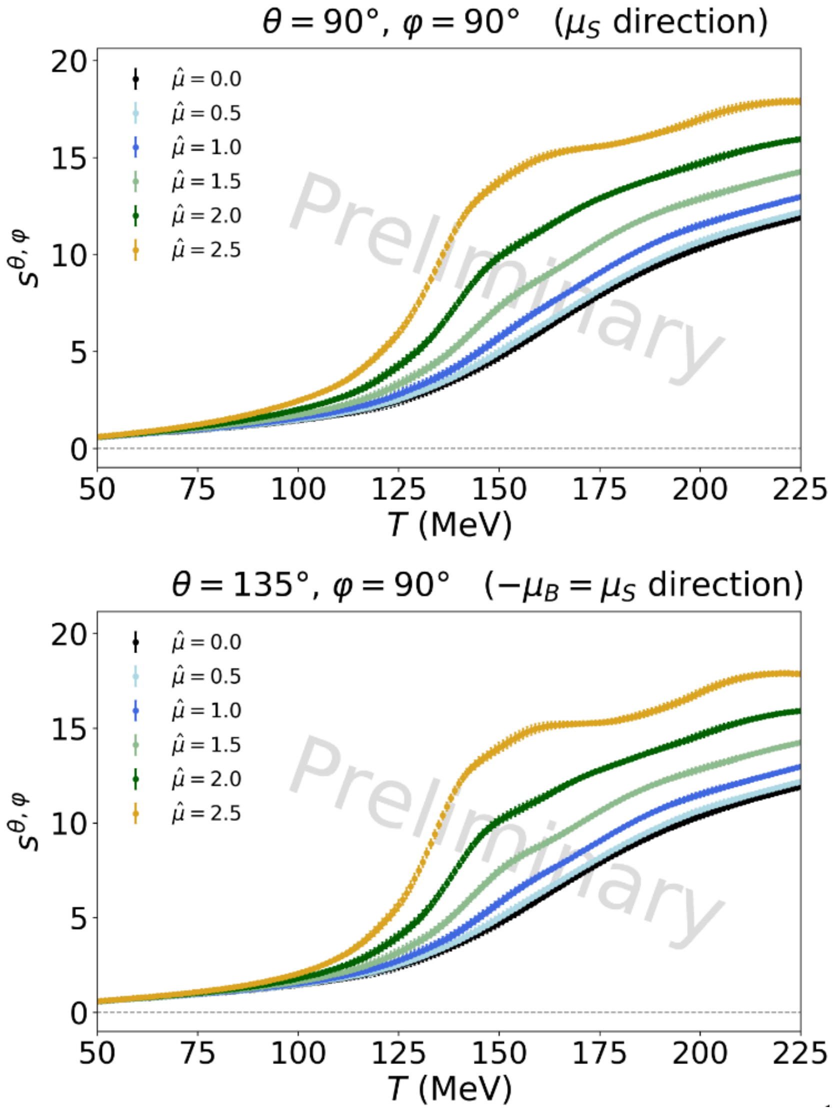
$$s^{\theta,\varphi}(T,\hat{\mu}) = \frac{\partial P}{\partial T}\bigg|_{\mu} = \frac{\partial}{\partial T}\left[\int_0^{\mu'} X_1^{\theta,\varphi}(T,\hat{\mu})d\mu'\right]_{\mu}$$

$$= s(T,0) + \int_0^{\mu'} \frac{\partial}{\partial T}\left[\frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)}\right]_{\mu} \times X_2^{\theta,\varphi}(T',0)d\mu'$$

$$+ \int_0^{\mu'} \frac{\overline{X}_1^{\theta,\varphi}(\hat{\mu})}{\overline{X}_2^{\theta,\varphi}(0)} \times \frac{\partial T'}{\partial T} \times \frac{\partial X_2^{\theta,\varphi}(T',0)}{\partial T'} d\mu'$$

Examples:

- *for* $(\theta = 90°, \varphi = 90°)$, $\hat{\mu} = \hat{\mu}_S$
- *for* $(\theta = 135°, \varphi = 90°)$, $\hat{\mu} = \sqrt{\hat{\mu}_B^2 + \hat{\mu}_S^2}$



Energy density

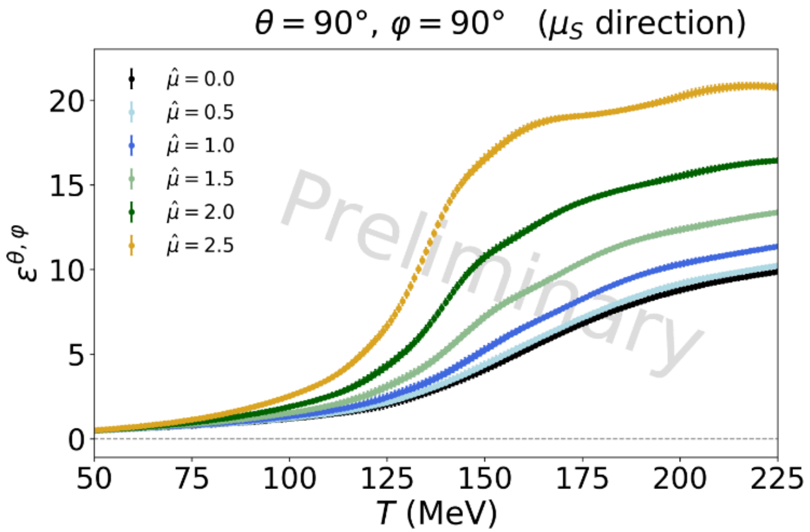
One can then compute energy density ε as:

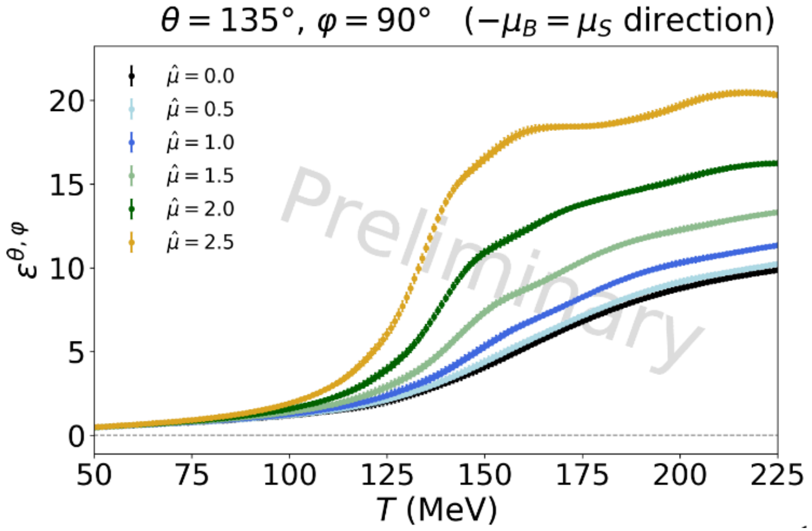
$$\varepsilon^{\theta,\varphi}(T,\hat{\mu}) = s.T - P + \sum_{i=B,Q,S} \mu_i \times n_i$$

$$= s.T - P + \mu_B \times n_B + \mu_Q \times n_Q + \mu_S \times n_S$$

$$= s.T - P + \mu \times (c_\theta \cdot \chi_1^B + s_\theta c_\varphi \cdot \chi_1^Q + s_\theta s_\varphi \cdot \chi_1^S)$$

$$= s.T - P + \mu \times X_1^{\theta,\varphi}$$





Examples:

- for $(\theta = 90°, \varphi = 90°)$, $\varepsilon(T,\mu_B,\mu_S) = s.T - P + \mu_S n_S$
- for $(\theta = 135°, \varphi = 90°)$, $\varepsilon(T,\mu_B,\mu_S) = s.T - P + \frac{\mu_S}{\sqrt{2}} n_S - \frac{\mu_B}{\sqrt{2}} n_B$

Conclusions

muses

We present a new 4D lattice-based EoS construction using the T'-Expansion Scheme to extend the coverage from the 4D Taylor expansion ($\hat{\mu} \lesssim 2.5$) up to $\hat{\mu} \sim 3.5$.

4D-*T*ExS EoS

- We have generalized the T'-Expansion Scheme to 4D by computing $X_{2/4}^{\theta,\varphi}(T,\mu)$ from lattice data at $\hat{\mu} = 0$
  (projecting a generalised $\mu = \sqrt{\mu_B^2 + \mu_Q^2 + \mu_S^2}$ onto spherical coordinates)
- We have shown extension from the (T, μ_B) plane to (T, μ_B, μ_S) and computed thermodynamics
  (pressure P, charge densities $n_{B/Q/S}$, entropy density s, energy density ε)

Currently: we are working on extending to full 4D space (T, μ_B, μ_S, μ_Q)

Disclaimer: error shown in the preliminary results of this talk are underestimated
$\rightarrow$ need to complete the analysis of error consistently