Relativistic turbulence: a covariant approach to LES

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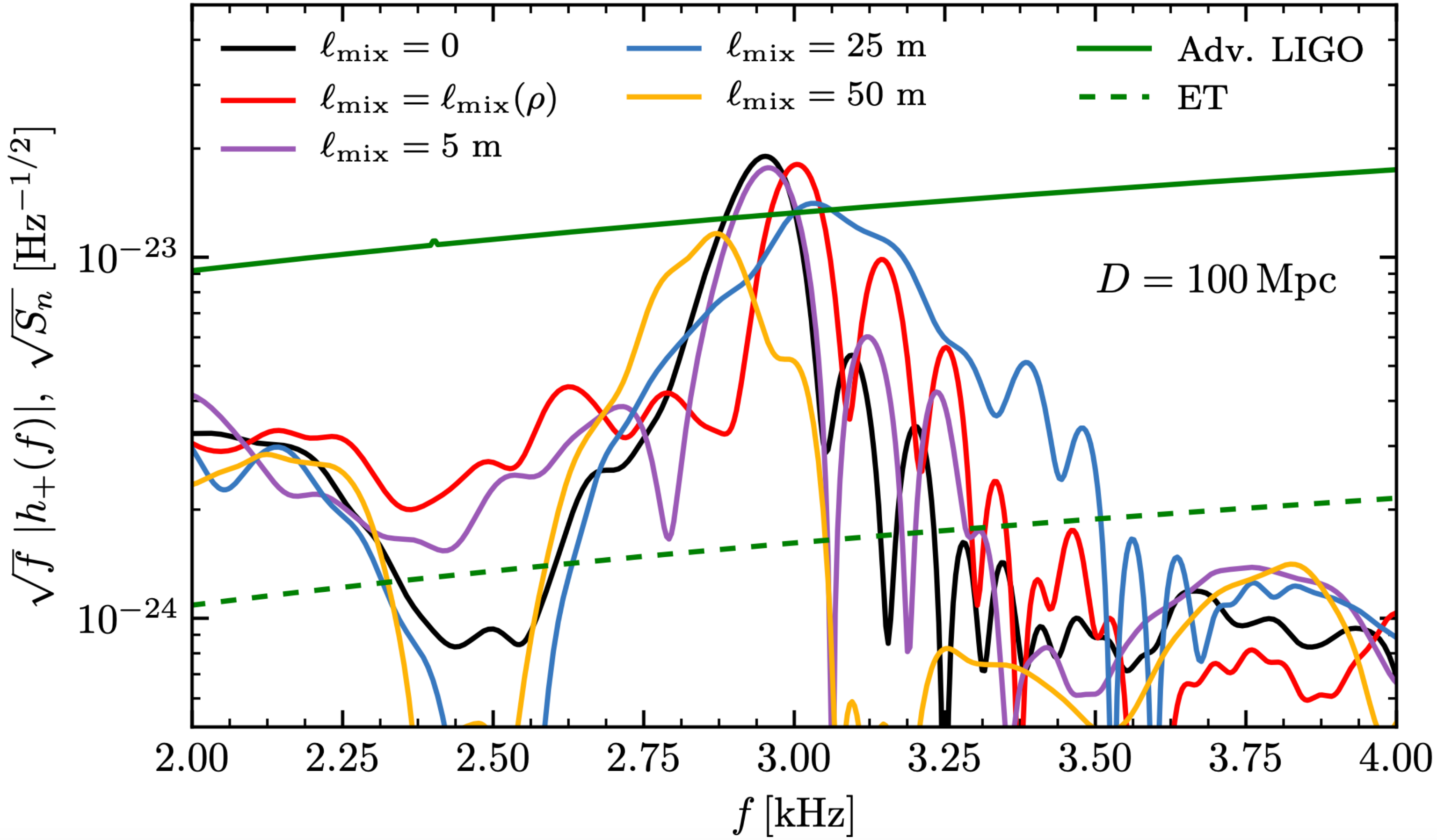
[PRD: 104 084090 , arXiv: 2405.13593]

Turbulence in mergers

Turbulence is driven in mergers, for example, by the Kelvin-Helmholtz instability (KHI) developing at the slip-line between merging NS. Turbulence modelling has a quantitative impact on the *merger dynamics*, the *outflows properties*, the *magnetic field amplification* and the post-merger *gravitational wave spectrum*.



[Radice (2020)]

Turbulence and Large Eddy Simulations

DNS of mergers are not feasible:

- Dissipation scale in mergers ≈ 1 cm ("conservative est.")
- Best resolution in large-scale simulations ≈ 10 m

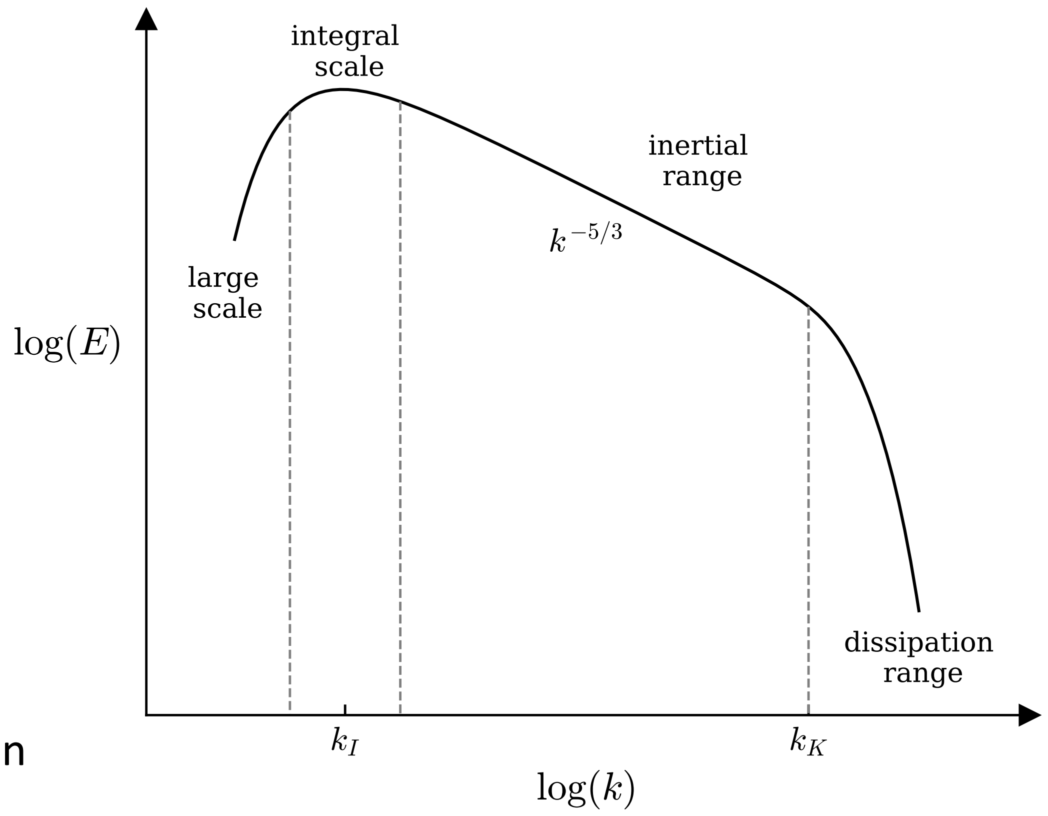
$$\mathrm{Re} \approx 10^6 \div 10^{15}$$

Accounting for the unresolved physics:

- Filtering to separate into resolved and unresolved
- Evolve large-scale dynamics, model the rest

A little bit of context:

- Applications to numerical relativity have shown impressive results, e.g. Aguilera-Miret+(2022)
- All practical implementations so far break covariance, both in filtering and in the closures Radice-Hawke(2024)



General relativistic LES: the issue of covariance

The 3+1 "operational" approach

Filtering as a simple set of rules to apply directly on the 3+1 equations

→ Same as for Newtonian theory

→ Quick derivation of the coarse-grained equations

$$A = \langle A \rangle + \delta A$$
$$\langle c \rangle = c$$
$$\langle A + B \rangle = \langle A \rangle + \langle B \rangle$$
$$\langle \partial_a A \rangle = \partial_a \langle A \rangle$$

This raises a number of "theory" questions...

- Space + time split from a relativistic perspective?
- Covariant derivatives? Metric and EFE?

The issue of covariance: not only theory

"Non covariant choice of closure schemes can induce artificial (coordinate independent) artefacts. For example, one expects turbulent momentum transport to operate only when there is non-zero shear in a Local Lorentz frame, which is guaranteed only for covariant closures."

[Duez+ (2020)]

A fully covariant approach: Lagrangian filtering

Theoretical framework: [PRD: 104 084090]

- covariant: dynamically identify a physically meaningful observer
- Fermi coordinates: geometry sector unaffected

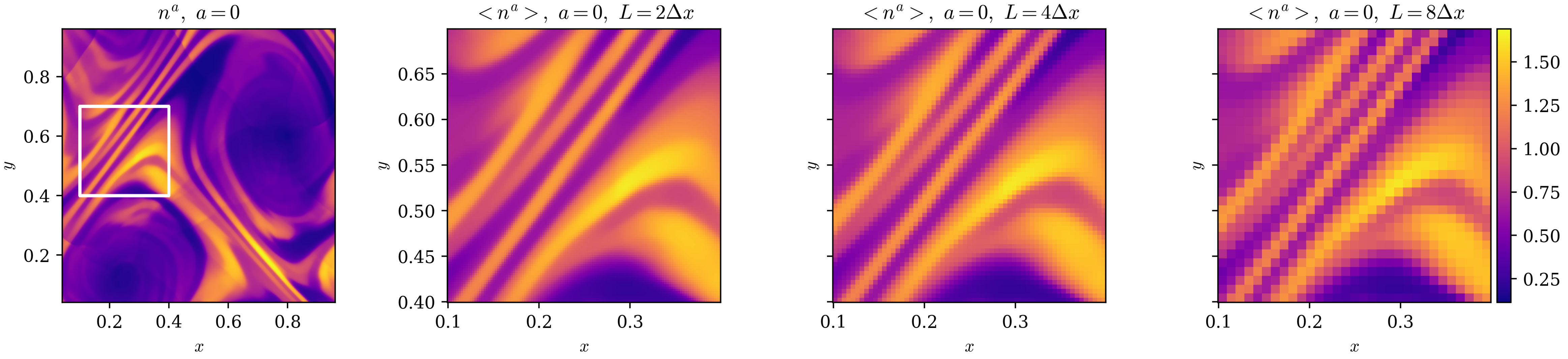
$$E^a_{(1)} = e^a_{(x)} + U^a U_b e^b_{(x)} \;, \quad E^a_{(1)} E^{(1)}_a = 1$$

$$r_{(I)} = \int_{\mathcal{V}_L} E^a_{(I)} n_a \, d\mathcal{V}_L \;, \quad I = 1, 2, 3$$

$$\langle X \rangle = \int_{V_L} X \, dV_L$$

Key steps in practice:

- run SR box simulations of KHI (I used[1] METHOD)
- build filtering observers: minim. average particle drift
- perform the **Lagrangian** filtering (tilted box)



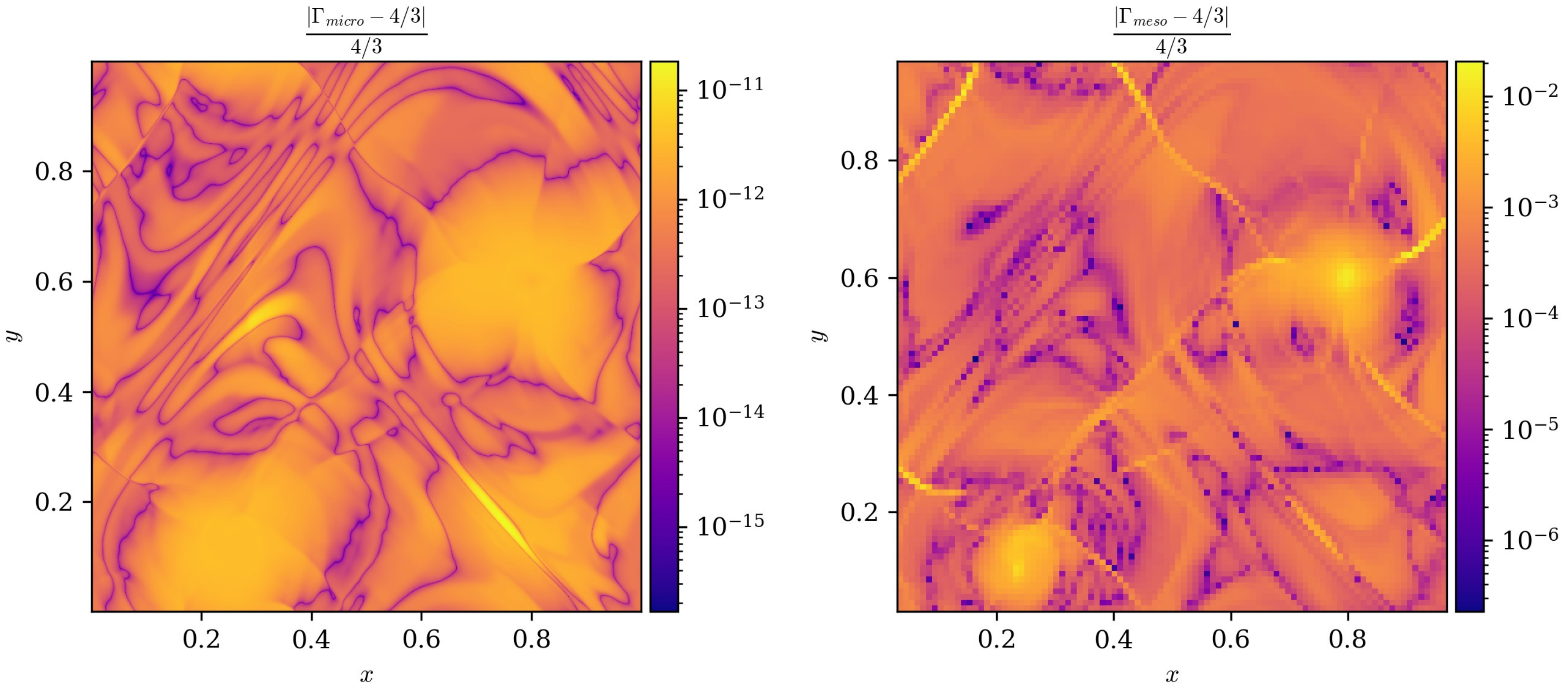
[1] https://github.com/AlexJamesWright/METHOD

Impact on matter sector: thermodynamics

$$\langle p \rangle = -\tilde{\varepsilon} + \tilde{\mu}\tilde{n} + \tilde{T}\tilde{s} + M$$

- Pressure as a non-linear closure in NR: filtering impact?
- Neglected so far.

Testing the null-hypothesis: what if I ignore the non-linearities in the pressure?



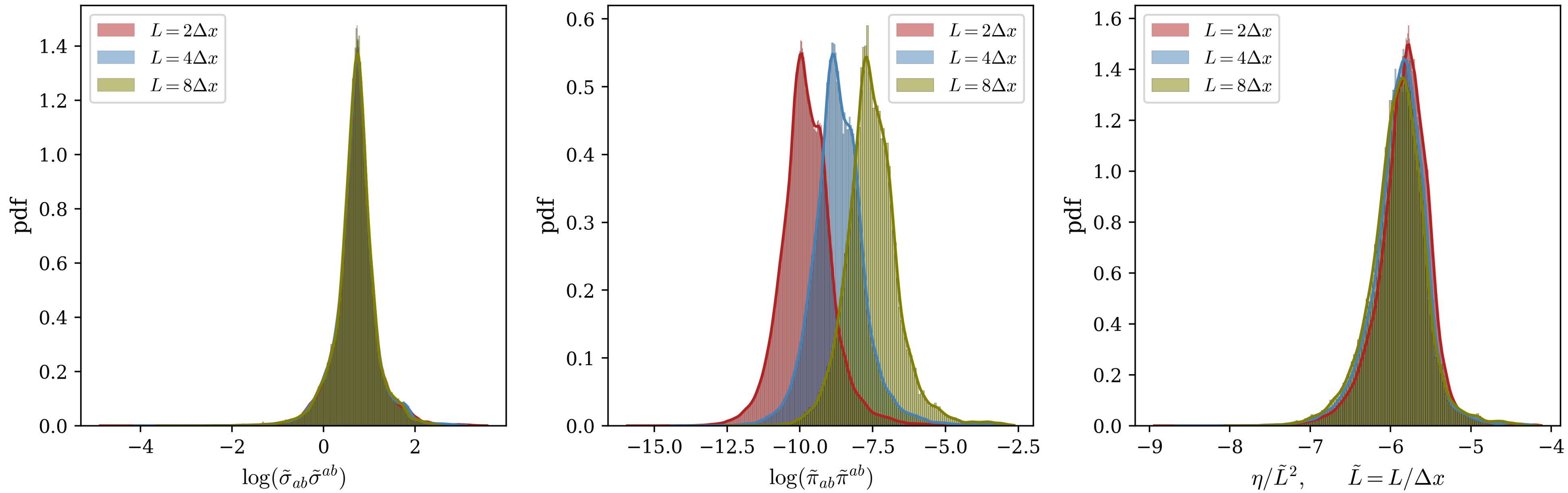
(Caveat: need 3D, realistic EoS, but...)

Impact on matter sector: effective dissipative terms

$$\langle T^{ab}\rangle = \underbrace{(\tilde{\varepsilon} + \langle p\rangle)\,\tilde{u}^a\tilde{u}^b + \langle p\rangle g^{ab}}_{\text{ideal terms}} + \overbrace{2\tilde{u}^{(a}\tilde{q}^{b)} + \tilde{s}^{ab}}^{\text{“dissipative” terms}}$$

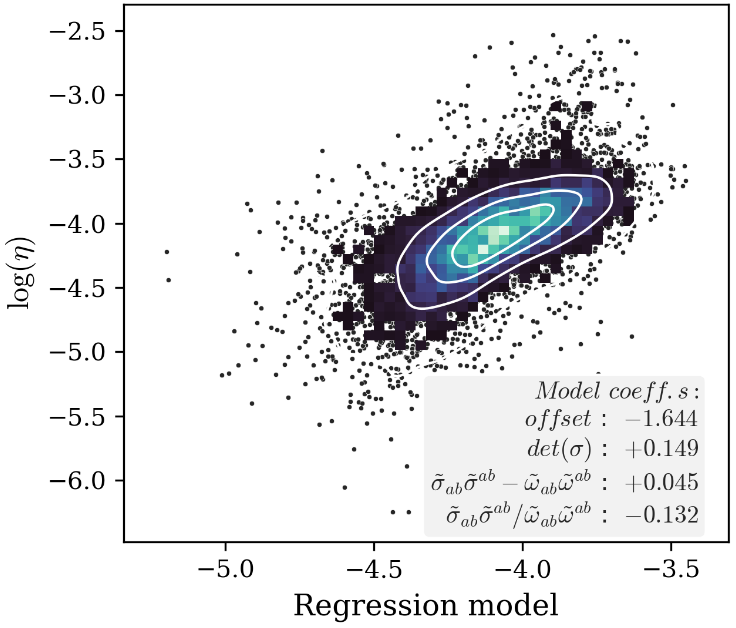
- Residuals need modelling: closures!
- EoM: “effective” dissipative fluid

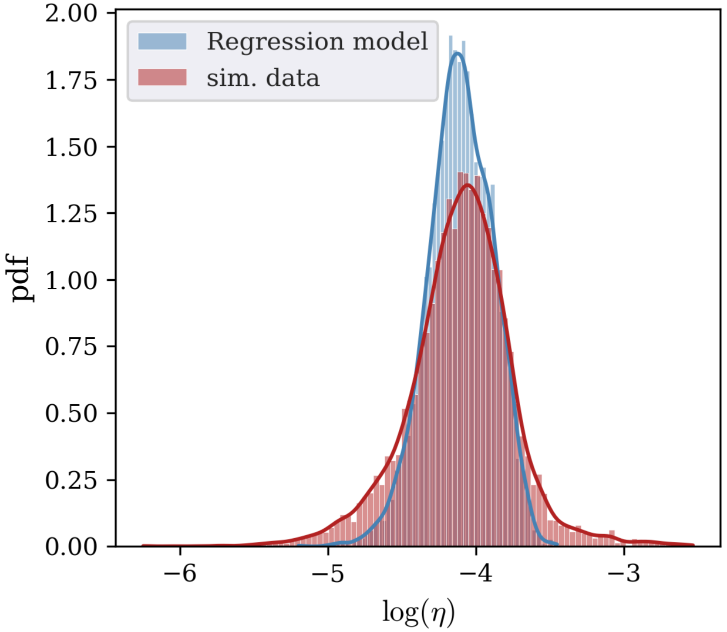
“Residuals” due to non-linearities, capturing the impact of sub-filter fluctuations

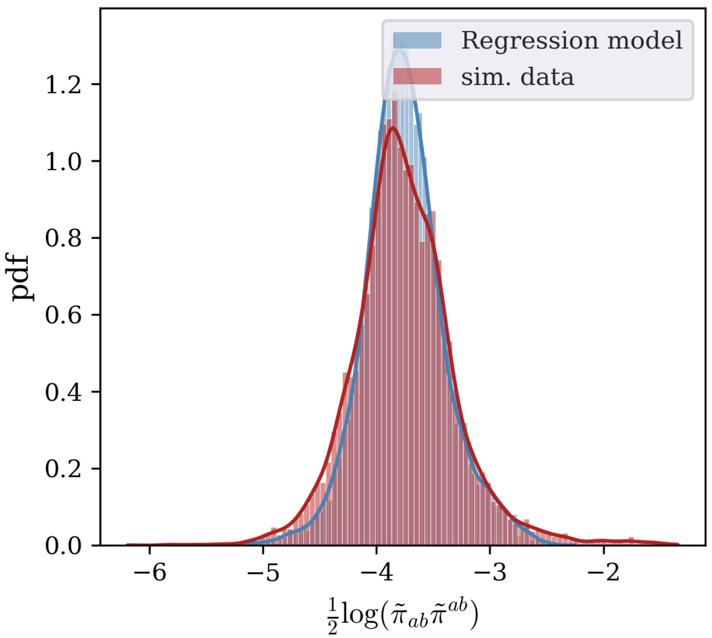


A simple linear regression model

- explanatory vars: $\left\{\tilde{T},\, \tilde{n},\, \tilde{\sigma}_{ab}\tilde{\sigma}^{ab},\, \det(\tilde{\sigma}),\, \tilde{\omega}_{ab}\tilde{\omega}^{ab},\, \tilde{\sigma}_{ab}\tilde{\sigma}^{ab} - \tilde{\omega}_{ab}\tilde{\omega}^{ab},\, \tilde{\sigma}_{ab}\tilde{\sigma}^{ab}/\tilde{\omega}_{ab}\tilde{\omega}^{ab}\right\}$
- "Quality factor": $W_1(X,Y) = \sum_i ||X_{(i)} - Y_{(i)}||$







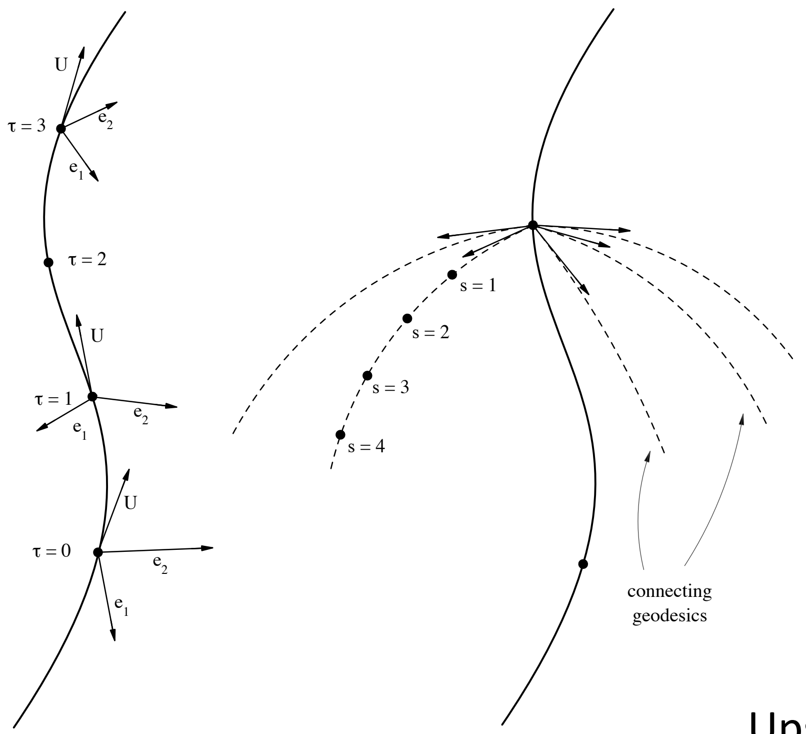
Recap/conclusions:

- Turbulence develops in mergers, with a quantitative impact on dynamics
- It is imperative to model it to fully realize the potential of MM NS astrophysics
- Modelling turbulence requires LES-type strategies (DNS not feasible)
- Proposed a covariant framework to do so in general relativistic settings
- Now presented a first practical implementation of the strategy
    - first positive results on a **"a priori"** calibration
    - tool for investigating a number of open issues in relativistic large-eddy modelling

Thank you for listening!

Back-up slides

Fibration framework and Fermi coordinates



Key ideas:

- Rel. Fluid dynamics: natural fibration of space-time
- Fermi coordinates: meaningful space-time split associated with a local observer

Advantages:

- Filtering explicitly defined: *metric unaffected can be shown, rather than assumed*
- Filtering operation is covariant: from SR to any spacetime

<u>Upshot: "geometry side" is untouched</u>

$$\langle \mathbf{G}\left(g, \partial g, \partial^2 g\right)\rangle = \mathbf{G}\left(\langle g\rangle, \partial\langle g\rangle, \partial^2\langle g\rangle\right) = \mathbf{G}\left(g, \partial g, \partial^2 g\right)$$

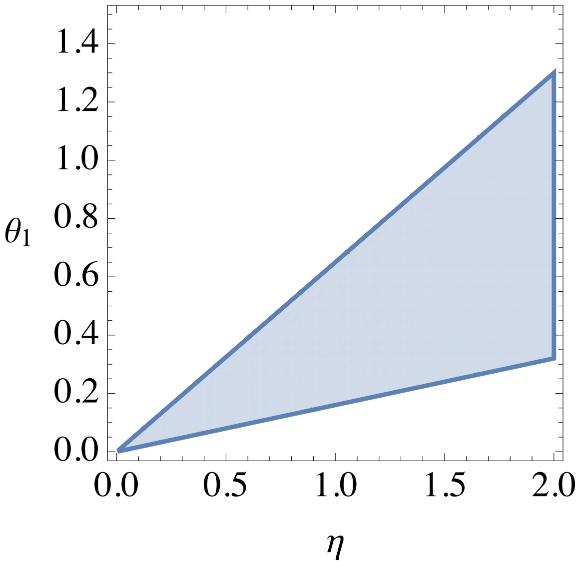
Impact of filtering: covariant stability

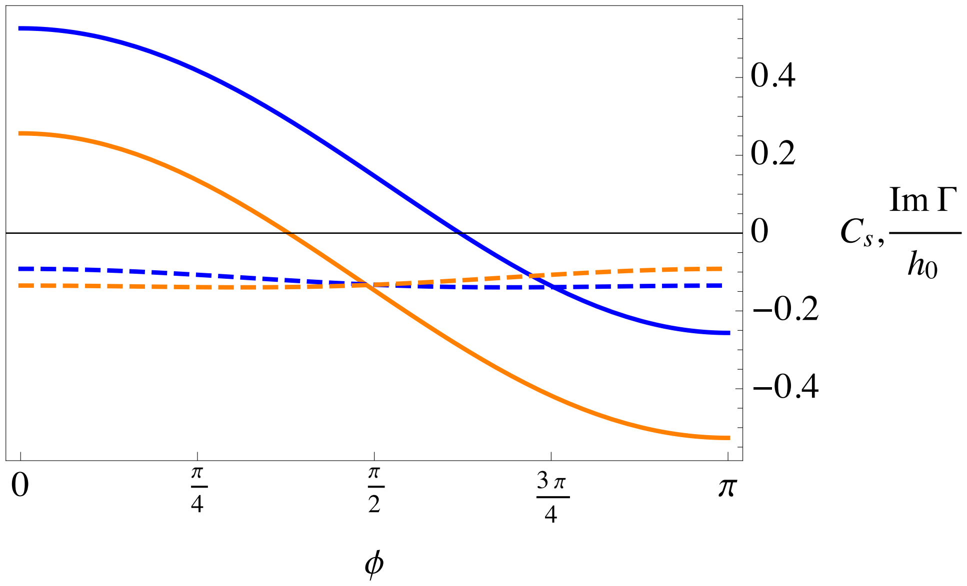
According to the Boussinesq hp, turbulent stresses are *dissipative in the mean*. The "classic" Smagorinsky model is based on this, and we are generalizing it to ensure covariance.

$$\nabla_a \tilde{n} = \perp_a^b \nabla_b \tilde{n} - \tilde{u}_a \dot{\tilde{n}}$$
$$\nabla_a \tilde{\varepsilon} = \perp_a^b \nabla_b \tilde{\varepsilon} - \tilde{u}_a \dot{\tilde{\varepsilon}}$$
$$\nabla_a \tilde{u}_b = -\tilde{u}_a \tilde{a}_b + \tilde{\omega}_{ab} + \tilde{\sigma}_{ab} + \frac{1}{3}\tilde{\theta} \perp_{ab}$$

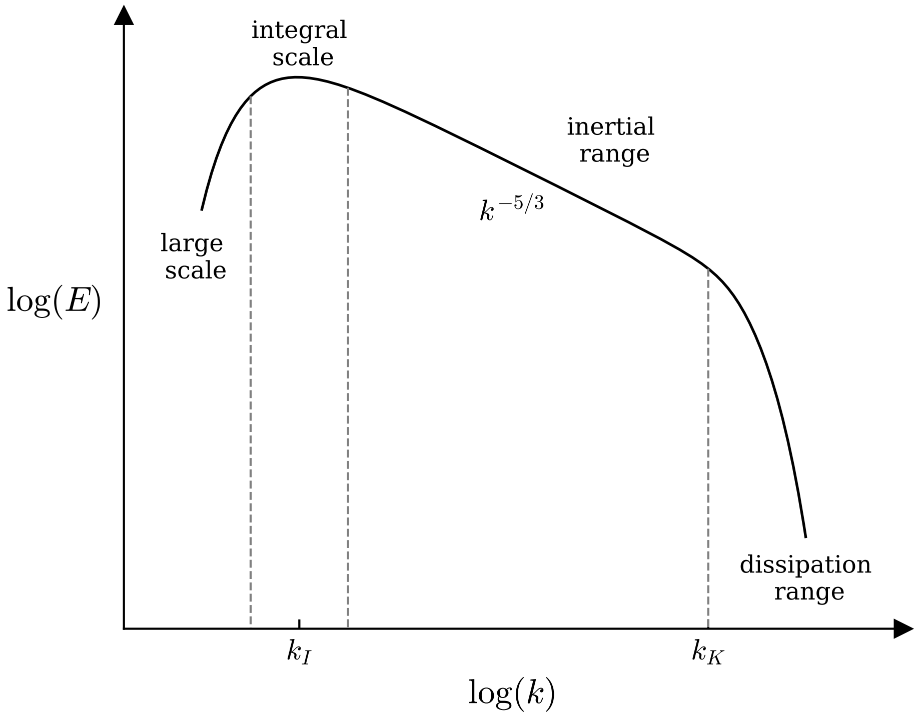
Inspired by BDNK

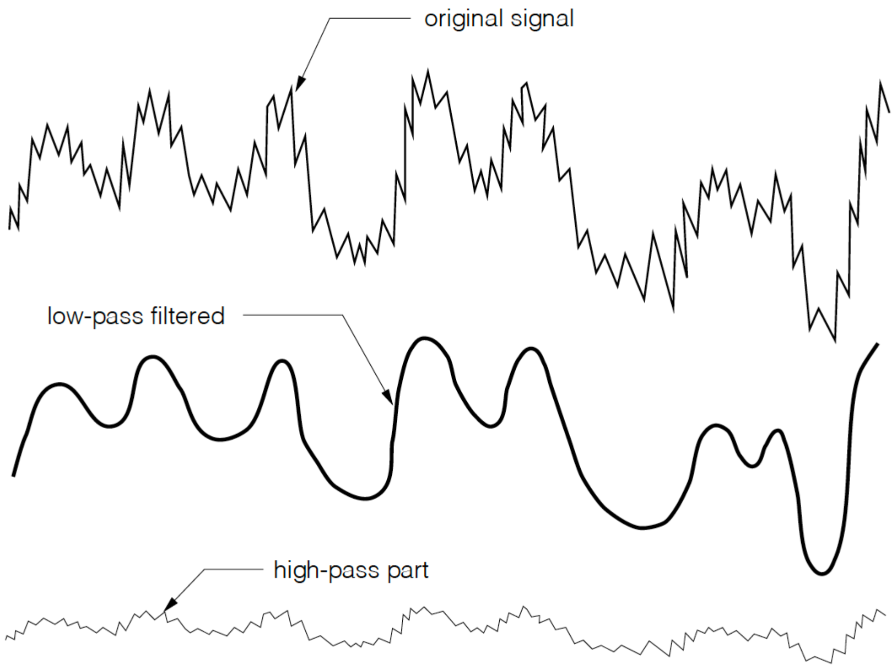
$$\tilde{\pi}^{ab} = \eta \tilde{\sigma}^{ab}$$
$$\tilde{\Pi} = \pi_1 \tilde{\theta} + \pi_2 \dot{\tilde{n}} + \pi_3 \dot{\tilde{\varepsilon}}$$
$$\tilde{q}^a = \theta_1 \tilde{a}^a + \theta_2 \perp^{ab} \nabla_b \tilde{n} + \theta_3 \perp^{ab} \nabla_b \tilde{\varepsilon}$$
$$\tilde{M} = \chi_1 \tilde{\theta} + \chi_2 \dot{\tilde{n}} + \chi_3 \dot{\tilde{\varepsilon}}$$





LES as a "low-pass filter"





$$A(\mathbf{x},t) = \sum_{\mathbf{k}} a_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

$$\delta A(\mathbf{x},t) = \sum_{|\mathbf{k}|\geq k_c} a_{\mathbf{k}}(t) e^{i\mathbf{k}\cdot\mathbf{x}}$$

Resolving (or not) the UV limit: bulk viscous case

Writing the equations in non-dimensional form we see that the reaction timescale is decoupled from the rest:

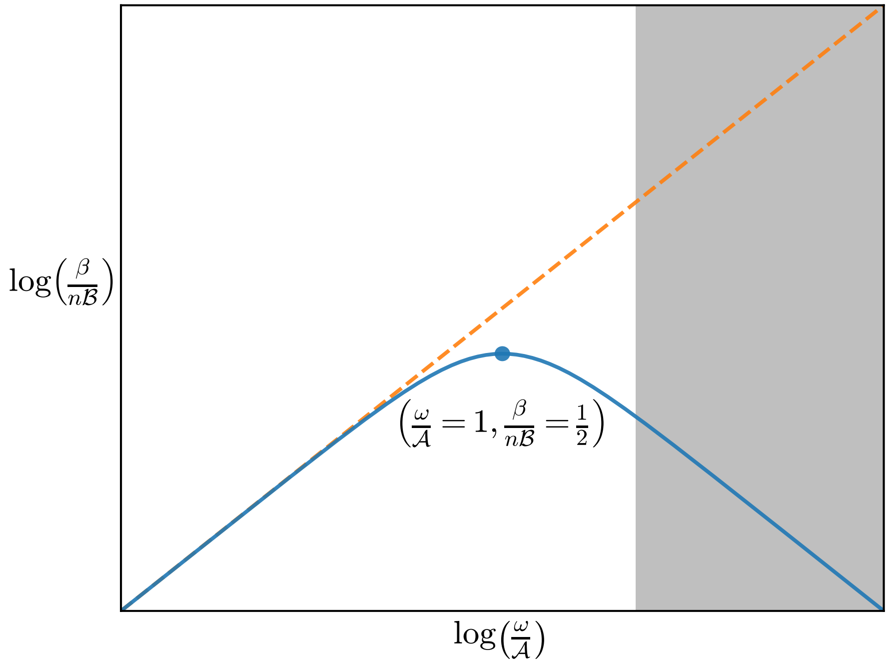
$$\frac{d\varepsilon}{dt} = -\frac{1}{\epsilon_{St}}(\varepsilon + c_r^2 p)\theta$$

$$a_b = -\frac{1}{\epsilon_{St}}\frac{1}{\epsilon_{Ma}^2}\frac{1}{\varepsilon + c_r^2 p} \perp_b^c \nabla_c p$$

$$\frac{dn}{dt} = -\frac{1}{\epsilon_{St}} n\theta$$

$$\frac{dY_e}{dt} = -\frac{1}{\epsilon_A}(Y_e - Y_e^{eq})$$

Integrating out the electron fraction via multi-scale methods, we obtain a NS-type bulk-viscous pressure:



Fast with respect to what? Resolving vs not-resolving the UV limit.