Colliding Gravitational Waves of Different Strengths Revisited

Kamran Qadir Abbasi*
Department of Mamthematics
Faculty of Engineering and Computing (FE&C)
National University of Modern Languages (NUML),
Sector H-9, Islamabad, Pakistan.
email:kamranqadir@numl.edu.pk

Asghar Qadir
Pakistan Academy of Sciences
3-Constitution Avenue Sector G-5/2,
Islamabad, Pakistan.
email: asgharqadir46@gmail.com
Telephone number: +923318554537
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ABSTRACT

The Khan-Penrose and Szekeres solution for colliding impulsive and sandwich plane gravitational waves (GWs) were for equal strengths. We explored how to define the strength of GWs and used that to construct the solution for arbitrary strengths. Penrose had pointed out that a Lorentz transformation would yield different strengths and wondered if we had something more general. We have checked that Lorentz transformation do yield different strengths and shown that we do, indeed have a greater generalization then given by the Lorentz transformation.

I. INTRODUCTION

General Relativity (GR) is a nonlinear theory of gravitation that fundamentally alters our understanding of gravity. Investigating the linear consequences of GR is crucial for fully grasping its implications. The linearization of Einstein's field equations (EFEs) naturally leads to the prediction of GWs [1]. Unlike Newtonian gravity, where gravity is treated as a static force, GR describes gravity as a dynamic field capable of propagating through spacetime as GWs. By definition, GWs are non-static (time-varying) vacuum solutions of the EFEs [2]. For these solutions, the stress-energy tensor is zero, which initially led to debates regarding the existence of GWs [3]. However, this issue was addressed by Weber and Wheeler [4], and later by Bondi and Robinson [5] for cylindrical and plane GWs respectively. The matter was fully resolved after the direct detection of GWs on September 14, 2015 [6].

So far, three types of exact GW solutions have been obtained: exact solution for plane GW [7], cylindrical GWs [8], and those resulting from the collision of plane GWs [9–11]. The main motivation behind this work is to assess the strength of colliding plane GWs. In a previous publication [12], we discussed the collision of GWs with unequal strengths, where the strength depends on the size of the sandwich. Penrose pointed out [13] that unequal strengths would be obtained by applying a Lorentz transformation to the wave amplitudes.

In this study, we adopt an alternative approach to defining the strength of GWs. Originally introduced by Einstein [14], this method involves using the"pseudo-tensor,"which is treated as a source term in vacuum solutions. This approach was developed to describe the conservation of energy and momentum in the context of GR, recognizing that gravitational energy cannot be localized in the same way as other forms of energy due to the equivalence principle. For that purpose, we perturb the metric for a single wave by $h_{\mu\nu}$. In the context of linearized gravity, second and higher-order terms of $h_{\mu\nu}$ are typically neglected. However, for the pseudo-tensor we retain these terms and shift them to the other side of the field equations, where they appear as $\tau_{\mu\nu}$. They represent the effective stress-energy tensor of the gravitational field [15]. Solving the linearized portion of the wave equation by an ansatz, we determine constants related to the amplitude and frequency of the waves.

The paper is organized as follows. In the subsequent section, we provide a brief overview of colliding plane GWs of equal and unequal strengths. In section III, we discuss the strength od the GWs with some examples from the literature. In section IV, we present a detailed analysis and our findings in light of Penrose's suggestion, including the results we obtained by solving the homogeneous part of the sandwich GWs in all three curved regions. We conclude our discussion in section V.

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* Also at School of Natural Sciences (SNS),
National University of Sciences and Technology (NUST),
Sector H-12, Islamabad 44000, Pakistan
email: kamran.qadir@sns.nust.edu.pk
Telephone number: +923465300068

II. A REVIEW OF COLLIDING PLANE GRAVITATIONAL WAVES

The Szekeres line element for colliding sandwich GWs is given by [9],

$$
\begin{aligned}
ds^2 = & T^2 e^{-M(u,v)} du dv \\
& - e^{-U(u,v)} \left[e^{V(u,v)} dx^2 + e^{-V(u,v)} dy^2 \right],
\end{aligned} \tag{1}
$$

where u and v are advanced and retarded times defined as $u = \frac{t-z}{T}$ and $v = \frac{t+z}{T}$. The spacetime is conveniently divided into six regions, as depicted in Fig. 1. The solution for each of these regions is as follows:
Region I ($u < 0, v < 0$): Here, $M = U = V = 0$.
Region II ($0 < u \leq u_0 < 1, v < 0$):

$$
\begin{aligned}
& U = -\ln(1-u^4)\ ,\ V = \sqrt{6}\ \tanh^{-1} u^2, \\
& M = -\frac{1}{4} \ln(1-u^4)\ .
\end{aligned} \tag{2}
$$

Region III ($0 < u, 0 < v \leq v_0 \leq 1$): The expressions for M, U, and V from Region II apply here, but with u replaced by v.
Regions IV ($u > u_0 < 1, v < 0$) and V ($0 < u, v > v_0 < 1$) are flat.
Region VI $\left(u_0 < u, v_0 < v < 1, u^4 + v^4 < 1\right)$ is given by

$$
\begin{aligned}
& U = -\ln\left(1-u^4-v^4\right), \\
& V = \sqrt{6}\left[\tanh^{-1} u^2 \left(1-v^4\right)^{-1/2} + \tanh^{-1} v^2 \left(1-u^4\right)^{-1/2}\right], \\
& M = \ln\left[\left(1-u^4\right)\left(1-u^4\right)\right]^{-3/4} - \ln\left(1-u^4-v^4\right) \\
& + 3\tanh^{-1} u^2 v^2 \left[\left(1-u^4\right)\left(1-v^4\right)\right]^{-1/2}.
\end{aligned} \tag{3}
$$

The metric (3) corresponds to the collision region. As before, T represents the "doomsday" or the "end of time." The question then is: How can we identify T for an actual GW as opposed to just a theoretical solution? What does it represent physically? Since T is related to inverse frequency, it should serve as an inverse measure of the "strength" of the GWs!

A. Colliding Plane Gravitational Waves of Unequal Strengths

What occurs when colliding GWs have different strengths? To answer this, we must determine how their strengths are reflected in the metrics. Notably, the momentum imparted to test particles will no longer cancel out at the moment of collision. This requires us to identify the parameters that define the strength of the waves based on this observation.

We applied the Weber-Wheeler [4] method and the eψN formalism [16–19] to address this issue. It was found that the strength of these waves varies as T^{-4} [4]. To account for differing strengths, we used different "doomsdays" for the two waves, denoted by T_2 and T_3. The resulting vacuum solution metric is then given by:

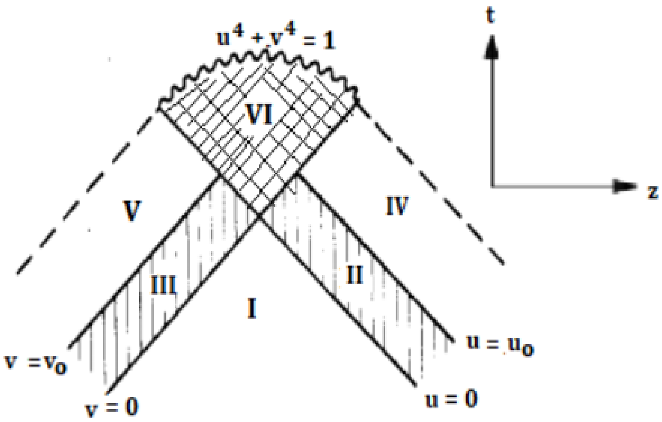

FIG. 1. The colliding sandwich GW spacetime of equal strength.

$$
\begin{aligned}
ds^2 = & T_2 T_3 e^{-M(u,v)} du dv \\
& - e^{-U(u,v)} \left[e^{V(u,v)} dx^2 + e^{-V(u,v)} dy^2 \right],
\end{aligned} \tag{4}
$$

with u, v, U, and V defined for each region, and specifically u_2, v_2, U_2, and V_2 for Region II, and u_3, v_3, U_3, and V_3 for Region III, as illustrated in Fig. 2, the solutions for these regions are given by:

$$
\begin{aligned}
& U_{2,3} = -\ln(1-(\frac{t \mp z}{T_{2,3}})^4),\ V_{2,3} = \sqrt{6}\tanh^{-1}(\frac{t \mp z}{T_{2,3}})^2, \\
& M_{2,3} = -\frac{1}{4}\ln(1-(\frac{t \mp z}{T_{2,3}})^4);
\end{aligned} \tag{5}
$$

and the singularity is at

$$
[(t-z)/T_2]^4 + [(t+z)/T_3]^4 = {u_2}^4 + {v_3}^4 = 1. \tag{6}
$$

Fig. 2, shows the spacetime configuration for $T_3/T_2 = 0.8$. Since a larger T indicates a weaker GW, and given that $T_2 > T_3$, the wave with T_2 is stronger. In Region VI, both u_2 and v_3 are employed.

B. Colliding Impulsive Gravitational Waves

Khan and Penrose (KP) provided a solution for colliding impulsive GW using the Penrose cut-and-paste method [11]. Intuitively, a sandwich GW can be seen as a generalization of the impulsive GW, where the pulse duration is extended to a finite period of time [20, 21].

By taking the limit as the "thicknesses of the sandwiches," u_0 and v_0, approach zero, while increasing the "intensity" such that the doomsday remains unchanged

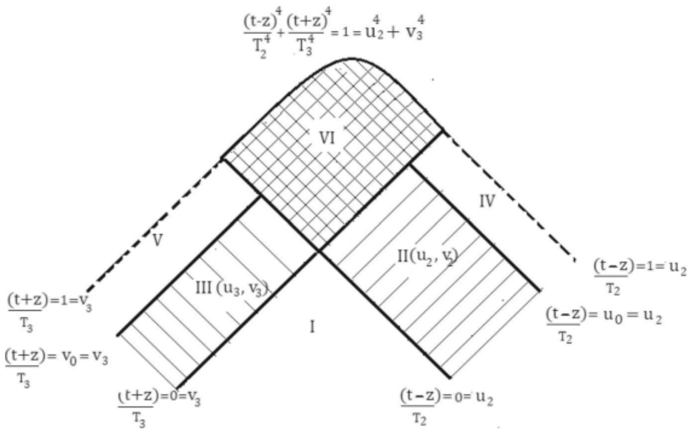

FIG. 2. The colliding sandwich gravitational wave (GW) spacetime has unequal strengths. For illustrative purposes, the ratio of the sandwich wave strengths is taken to be $T_3/T_2 = 0.8$.

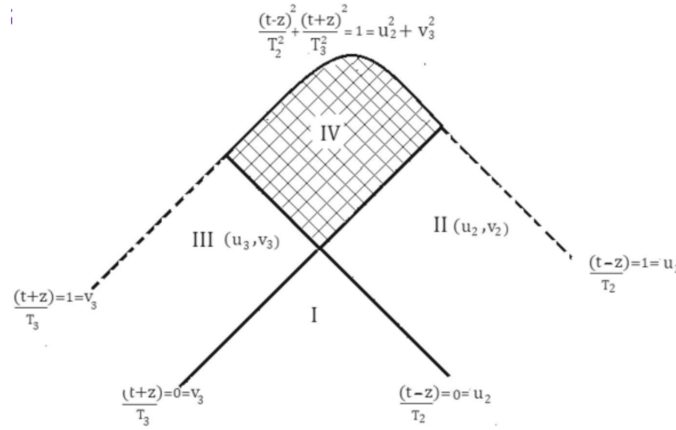

FIG. 3. The colliding impulsive GWs of unequal strengths.

in each case, we obtain the KP GW:

$$ds^2 = T_2 T_3 B\left(u_2, v_3\right)^{3/2} \Big\{ A\left(u_2, v_3\right) du_2 dv_3 - \left[B_+\left(u_2, v_3\right) dx^2 + B_-\left(u_2, v_3\right) dy^2\right] \Big\}, \tag{7}$$

$$\begin{aligned} A &= 1/\left[\sqrt{1-u_2^2-v_3^2}\left(u_2 v_3 + \sqrt{1-u_2^2-v_3^2}\right)^2\right], \\ B &= 1-u_2^2-v_3^2, \\ B_\pm &= \left\{\frac{\sqrt{1-u_2^2} \pm v_3}{\sqrt{1-u_2^2} \mp v_3}\right\}\left\{\frac{\sqrt{1-v_3^2} \pm u_2}{\sqrt{1-v_3^2} \mp u_2}\right\}. \end{aligned} \tag{8}$$

Again, spacetime is depicted for $T_3/T_2 = 0.8$. Again, the left wave is stronger. In region VI, both u_2 and v_3 are used. There are now only four regions.

III. THE STRENGTH OF GRAVITATIONAL WAVES

The existing literature on exact solutions for GWs often employs dimensionless variables, which makes it difficult to determine the physical strength of these waves in concrete terms. In GR, energy is not as straightforwardly defined as in classical mechanics, so the energy content of GW is not directly specified. For a classical analogy, consider a simple pendulum: its energy is clearly related to the amplitude of oscillation, the frequency (f), or the period (T), which in turn depend on parameters like mass (m), string length (l), and gravitational acceleration (g). Weber and Wheeler calculated the momentum transferred to test particles by analyzing the acceleration of these particles as they moved along geodesics in the presence of cylindrical GWs [4]. Later, Ehlers and Kundt applied a similar approach to study plane-fronted GW [5].

The strength of GW is described by their strain, which quantifies the relative change in distance induced by the wave as it travels through space. This is generally extremely small, often around 10^{-21} or even less, due to the minimal interaction of GW with matter. The amplitude is influenced by factors such as the masses of the interacting objects, their separation, and the energy released in the event. For example, the collision of two black holes (BHs) produces GWs with a strain detectable by highly sensitive instruments like LIGO and Virgo, even across vast cosmic distances. Studying these waves' strength provides essential insights into the sources and mechanisms of these cosmic events and helps test the boundaries of our understanding of GR and the nature of spacetime.

IV. INTERPRETATION OF STRENGTH OF COLLIDING PLANE GRAVITATIONAL WAVES

Penrose's observation regarding the Lorentz transformation in the direction of motion suggests that the resulting "strength" corresponds to a frequency. For a frame moving at a speed

$$v = \sqrt{1-\left(\frac{T_3}{T_2}\right)^2}, \tag{9}$$

relative to the second frame, the two waves would appear to have a constant time given by

$$T = \sqrt{\frac{T_2}{T_3}}. \tag{10}$$

We have confirmed this observation. However, this "strength" does not account for the amplitude of the wave. How can this amplitude be determined for GWs?

We solved the homogeneous equation for a single sandwich GW using an ingenious method, yielding the solution

$$h_{\mu\nu} = C\frac{u^2(1+u^4)}{(1-u^4)^2}\eta_{\mu\nu}, \tag{11}$$

which can be applied to Region II by substituting u by u_2, and to Region III by substituting v_3, with corresponding constants C_2 and C_3.

For Region VI, we can use a linear combination as the solution to the homogeneous part of the equation, imposing the boundary condition that for equal strengths, they cancel at the point and time of collision. Thus, one constant can be taken as the negative of the other in that scenario. In general, we can express them as C_2 and $-C_3$.

To account for the "effective energy" resulting from the curvature of spacetime, $\tau_{\mu\nu}$ was defined for linearized gravity, with all the nonlinear terms transposed to the right-hand side as a "source." For GWs this approach leads to a non-homogeneous wave equation with $\tau_{\mu\nu}$, as the source:

$$\Box h_{\mu\nu} = \kappa\tau_{\mu\nu} = f(u,v)\eta_{\mu\nu}, \tag{12}$$

where $\eta_{\mu\nu}$ is the Minkowski metric tensor. For the interaction Region VI, the function $f(u,v)$ is given by

$$f(u_2,v_3) = \frac{u_2^2v_3^2\left(1+u_2^4\right)^2\left(1+v_3^4\right)^2}{(1-u_2^4)(1-v_3^4)(1-u_2^4-v_3^4)}. \tag{13}$$

Thus, the complete solution is:

$$h_{\mu\nu} = \left\{C_2\frac{u_2^2(1+u_2^4)}{(1-u_2^4)^2} - C_3\frac{v_3^2\left(1+v_3^4\right)}{\left(1-v_3^4\right)^2} + \kappa\Box^{-1}f(u_2,v_3)\right\}\eta_{\mu\nu}. \tag{14}$$

Our method does not directly address the last term, but we can obtain an approximate or numerical solution for any given (u_2, v_3). For any values of C_2 and $-C_3$, we can find a (u_2, v_3) such that $h_{\mu\nu} = 0$, where no momentum will be imparted to test particles.

V. CONCLUSION

By using classical wave theory as a guide, we reexamined the physics of GWs in semi-classical terms to understand what Wheeler referred to as "points of principle" for the exact solution of colliding plane GW.

We solved the linearized wave equation for a single sandwich GW, and then for the colliding waves in the region of intersection. When the waves have equal "strengths," a test particle—akin to a mosquito caught between clapping hands—remains stationary.

As explained in more detail elsewhere [22], there are four constants: two inverse "frequencies," T_2 and T_3, and two "amplitudes," C_2 and C_3, which define the "strengths" of the two waves. The amplitudes represent differences, such as those between GW from binary pulsars and those from supernova explosions. With different frequencies but identical amplitudes, the test particle will always be displaced. By varying both the frequencies and amplitudes, the particle may be stationary.

Thus, the framework we envisioned is indeed more general. By taking appropriate limits, we derived the solution for KP waves of unequal strength for impulsive plane waves. When the strengths are equal, we recovered the KP solution, providing us with the first derivation of the KP solution.

We still need to find the particular solution of the ordinary differential equation (ODE) and take the limit as the width of the sandwich approaches zero to explicitly understand the behavior of KP waves.

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